# Financial Applications of Copula Functions

Jean-Frédéric Jouanin, Gaël Riboulet and Thierry Roncalli
Groupe de Recherche Opérationnelle, Crédit Lyonnais, France

**Abstract**

Copula functions have been introduced recently in finance. They are a general tool to construct multivariate distributions and to investigate dependence structure between random variables. In this paper, we show that copula functions may be extensively used to solve many financial problems. As examples we show how to them to monitor the market risk of basket products, to measure the credit risk of a large pool of loans and to compute capital requirements for operational risk.

**JEL Classification:** G10

**Key words:** copula, risk management, market risk, credit risk, operational risk

## 1 Introduction

One of the main issues in risk management is the aggregation of individual risks. This problem can be easily solved or at least discarded by assuming that the random variables modelling individual risks are independent or are only dependent by means of a common factor, so that we can reduce to the case of independent variables, by eventually adding one new variable (describing the common factor) to the set of the risks.

The problem becomes much more involved when one wants to model fully dependent random variables or, when one does not know what the exact joint distribution is, to produce a large set of dependence between the given individual risks. Again a classic solution is to assume a Gaussian behavior of the vector of risks (with some given covariance matrix); however, all risks are not likely to be well described by a Gaussian distribution – think of the distribution of the loss of a pool of credit which is highly asymmetric – or the Normal joint distribution may fail to catch some exotic key features of the dependence of the studied risks – e.g. a tail dependence. Besides, we stress that we do not often have as much information on the joint distribution as that on each separate risk.

A powerful and user-friendly tool to describe dependent risks is provided by the concept of *copula*. A copula is nothing else but the joint distribution of a vector of uniform random variables. Since it is always possible to map any vector of r.v's into a vector of r.v's with uniform margins, we are able to split the margins of that vector and a digest of the dependence, which is the copula (that is roughly the meaning of Sklar's theorem). The concept of copula was introduced by Sklar [1959] and studied by many authors such as Deheuvels [1979], Genest and MacKay [1986]. Details and complete references are provided by the books of Nelsen [1999], Joe [1997] and also Hutchinson and Lai [1990].

With the concept of copula we are given plenty of families of functions (Student, Gumbel, Clayton, etc.) that enable us to produce a richer dependence than the usual Gaussian assumption; and Sklar's lemma tells us how to impose one form of dependence between random variables with given margins. But we stress that this does not give any hint on how to choose the dependence to input between the random variables, which may be seen as a severe shortcoming of the copula approach. While the assumption of Gaussian dependence can be considered as quite natural, we need now to motive any choice of a particular family of copulas, in cases where it is not easy to decide because of the frequent lack of information on the dependence. Yet, in the practice of banks until now, the choice of the copula – when not purely arbitrary – is justified rather by convenience and tractability arguments (Student and 'Fraitly' families are often easy to simulate in a Monte-Carlo methodology) than a thorough analysis of the risks to be modelled.

The goal of this survey paper is to provide some simple applications of copulas for risk management from an industrial point of view. First, we remind some basics about copulas. Then, some applications of copulas for market risk, credit risk and operational risk are given. A special attention will be paid to credit risk modelling since we think the applications of copulas have been the most successful in this field until now. We will not lay the emphasis on the various concepts of dependence and correlations which were clarified in a sparkling manner by Embrechts, McNeil and Straumann [2002].

# 2 Copula Functions

We will not provide a full mathematical treatment of the subject here and we refer interested readers to Nelsen [1999], from which the following results are drawn. For short, we also choose to stick to the bivariate case – but it is easy to generalize definitions to the $n$-dimensional case. Mathematically speaking, a function $\mathbf{C} : [0,1]^2 \rightarrow [0,1]$ is a *copula function* if it fulfills the following properties:

1. $\mathbf{C}$ is 'increasing' in the sense that for $0 \leq u_1 \leq u_2 \leq 1$ and $0 \leq v_1 \leq v_2 \leq 1$ we have
$$\mathbf{C}\left([u_1, v_1] \times [u_2, v_2]\right) \equiv \mathbf{C}(u_2, v_2) - \mathbf{C}(u_2, v_1) - \mathbf{C}(u_1, v_2) + \mathbf{C}(u_1, v_1) \geq 0;$$
2. $\mathbf{C}$ satisfies $\mathbf{C}\left(1, u\right) = \mathbf{C}\left(u, 1\right) = u$ and $\mathbf{C}\left(0, u\right) = \mathbf{C}\left(u, 0\right) = 0$ for all $u \in [0,1]$.

Thus, we may interpret a copula function as a cumulative probability function of two uniform random variables $(U_1, U_2)$. Now let $F_1$ and $F_2$ be any two univariate distributions. It is obvious to show that
$$F\left(x_1, x_2\right) \equiv \mathbf{C}\left(F_1\left(x_1\right), F_2\left(x_2\right)\right)$$
is a probability distribution, the margins of which are exactly $F_1$ and $F_2$. We say that it is a distribution with fixed (or given) marginals. Conversely, Sklar proved in 1959 that any bivariate distribution $F$ admits such a representation and that the copula $\mathbf{C}$ is unique provided the margins are continuous.

To illustrate the idea behind the copula function, we can think about the bivariate Gaussian distribution. To state that a random vector $(X_1, X_2)$ is Gaussian is equivalent to state that:

1. the univariate margins $F_1$ and $F_2$ are Gaussian;
2. these margins are 'linked' by a unique copula function $\mathbf{C}$ (called the Normal copula) such that:
$$\begin{aligned}\mathbf{C}\left(u_1, u_2; \rho\right) &= \mathbf{\Phi}\left(\phi^{-1}\left(u_1\right), \phi^{-1}\left(u_2\right); \rho\right) \\ &= \int_0^{u_1} \phi\left(\frac{\phi^{-1}\left(u_2\right) - \rho\phi^{-1}\left(u\right)}{\sqrt{1-\rho^2}}\right) \mathrm{d}u\end{aligned}$$
with $\phi$ the standard univariate Gaussian cumulative distribution and $\mathbf{\Phi}\left(x_1, x_2; \rho\right)$ the bivariate Normal cumulative function with correlation $\rho$.

A very important special case of Normal copula family is obtained with $\rho = 0$: it is the *product copula* $\mathbf{C}^{\perp}\left(u_1, u_2\right) \equiv u_1 u_2$. Of course, two variables are independent if and only if their copula is the product copula.

There exists many other copula families: most of them are obtained like the Normal family by inversion from well known bivariate distributions – an example which is important in risk management is the Student copula, that is the copula function of the bivariate Student$-t$ distribution –, but we also know direct methods to construct new copulas: the most famous family of this kind is no doubt the *Archimedean copulas*
$$\mathbf{C}_{\psi}(u_1, u_2) \equiv \psi^{-1}\left(\psi(u_1) + \psi(u_2)\right)$$

with $\psi : [0,1] \to \mathbb{R}^+$ a convex, continuous, decreasing function such that $\psi(1) = 0$. Again an important example is the Gumbel copula where we set $\psi(t) \equiv (-\log t)^\theta$ with $\theta \geq 1$.

The copula decomposition may be used to extend univariate models to multivariate models. Let us consider a simple example with two random variables. $X_1$ and $X_2$ are distributed according to an Inverse Gaussian distribution and a Beta distribution. If we consider classic handbooks on multivariate distributions ([26] for example), it is difficult to find a bivariate distribution with such margins. Figure 1 corresponds to a bivariate distribution with a Normal copula[1]. This example might be an illustration of a credit risk

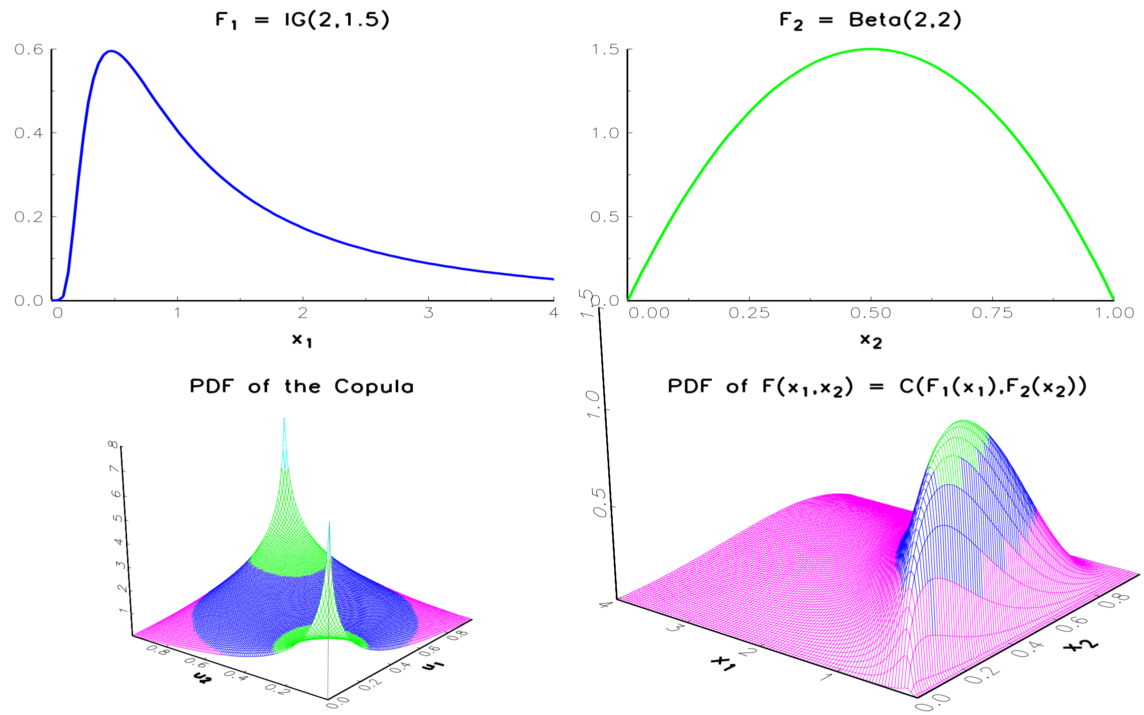


Figure 1: An example of bivariate distribution with given marginals

modelling problem. $X_1$ may be a default time and $X_2$ a recovery rate. The dependence between the two random variables are introduced through a copula function. The copula method is also very powerful from an industrial point of view because

1. the risk can be split into two parts: the individual risks and the dependence structure between them,
2. and the dependence structure may be defined without reference to the modelling specification of individual risks.

Because of this *sequential* approach, the copula framework enables in this way to solve some calibration issues.

In the paper [13], the authors discuss many concepts of dependence and correlation. We remind here the most important results. Let $\mathbf{X} = (X_1, X_2)$ be a bivariate random vector. We define the copula $\mathbf{C}\langle X_1, X_2\rangle$ of the random variables $X_1$ and $X_2$ as the copula of the corresponding bivariate distribution $\mathbf{F}$. Schweizer and Wolff [1981] show that the copula function is invariant under strictly increasing transformations. It means that if the functions $h_1$ and $h_2$ are strictly increasing, the copula of $(h_1(X_1), h_2(X_2))$ is the same as the copula of $(X_1, X_2)$. Indeed, the copula $\mathbf{C}\langle X_1, X_2\rangle$ may be viewed as the **canonical measure of the dependence** between $X_1$ and $X_2$ in the sense that it is the minimal statistic of the dependence between $X_1$ and $X_2$.

[1] $\rho$ is equal to 0.70%.

One of the most useful dependence ordering is the concordance order. We say that the copula $\mathbf{C}_1$ is *smaller* than the copula $\mathbf{C}_2$ and we note $\mathbf{C}_1 \prec \mathbf{C}_2$ if we verify that $\mathbf{C}_1(u_1, u_2) \leq \mathbf{C}_2(u_1, u_2)$ for all $(u_1, u_2) \in [0,1]^2$. We can show that any copula function satisfy:

$$\max(u_1 + u_2 - 1, 0) \equiv \mathbf{C}^-(u_1, u_2) \leq \mathbf{C}(u_1, u_2) \leq \mathbf{C}^+(u_1, u_2) \equiv \min(u_1, u_2)$$

where the two bounds are respectively called the lower and upper Fréchet bound copulas. The bound copulas correspond to the case where the random variables $X_1$ and $X_2$ are such that $X_2$ is a monotonic function $f$ of $X_1$. This function $f$ is increasing if the copula is $\mathbf{C}^+$ and decreasing if the copula is $\mathbf{C}^-$. We notice that the Normal copula corresponds to the copula functions $\mathbf{C}^-$, $\mathbf{C}^\perp$ and $\mathbf{C}^+$ when the parameter $\rho$ takes the respective values $-1$, 0 and 1 — the copula family is called *comprehensive* when it contains those three copula functions. In some sense, copula functions are extensions of the linear dependence notion in the Gaussian framework. That is why $\mathbf{C}^-$ and $\mathbf{C}^+$ may be viewed as the most negative and positive dependence structure. By analogy with the linear correlation, a copula function $\mathbf{C}$ is a negative dependence structure if $\mathbf{C}^- \prec \mathbf{C} \prec \mathbf{C}^\perp$ and a positive dependence structure if $\mathbf{C}^\perp \prec \mathbf{C} \prec \mathbf{C}^+$ — the random variables $X_1$ and $X_2$ are said respectively NQD or PQD (negative and positive quadrant dependent)[2].

# 3 Market Risk Management

The copula methodology can be applied both to compute value-at-risk ($VaR$) and to perform stress testing. The two approaches are explained in the following sections. Moreover, we consider the problem of managing the risk of dependence for exotic basket equity or credit derivatives.

## 3.1 Non-Gaussian Value-at-risk

As noted by Embrechts, McNeil and Straumann [2002], the correlation is a special case through all measures that are available to understand the relationships between all the risks. If we assume a Normal copula[3], the empirical correlation is a good measure of the dependence only if the margins are Gaussian. To illustrate this point, we can construct two estimators:

1. The empirical correlation $\hat{\rho}$;
2. The *canonical* correlation $\hat{\rho}_{\mathsf{om}}$ obtained as follows: the data are mapped to empirical uniforms and transformed with the inverse function of the Gaussian distribution. The correlation is then computed for the transformed data[4].

The advantage of the canonical measure is that **no distribution is assumed for individual risks**. Indeed, it can be shown that a misspecification about the marginal distributions (for example to assume Gaussian margins if they are not) leads to a biased estimator of the correlation matrix (see [11] for numerical experiments). This is illustrated by the following example for asset returns. The database of the London Metal Exchange[5] is used and the spot prices of the commodities Aluminium Alloy (AL), Copper (CU), Nickel (NI), Lead (PB) and the 15 months forward prices of Aluminium Alloy (AL-15), dating back to January 1988, are considered. The two correlation measures of asset returns are reported in Tables 1 and 2. We observe some significant differences between the two correlation matrices[6].

The previous algorithm to compute $\hat{\rho}_{\mathsf{om}}$ is derived from the maximization of the log-likelihood function with empirical margins. If the copula function is Student with $\nu$ degrees of freedom, the maximization of the log-likelihood function implies to solve a non-linear matrix equation. This can be done by means of Jacobi iterations (see [5]):

[2] Note however that $\prec$ is a partial ordering. So, there are copulas which may not be comparable to the product copula.

[3] In the remaining of the article, we use the term 'correlation matrix' to refer to the matrix of the parameters of the Normal or Student copula even if this is a misuse.

[4] $\hat{\rho}_{\mathsf{om}}$ is also called the 'omnibus estimator'. It is consistent and asymptotically normally distributed (Genest, Ghoudi and Rivest [1995]).

[5] The database is available on the web site of the LME http://www.lme.co.uk.

[6] The standard errors are not reported here. However, the correlations of Table 2 and Table 3 are in italics if they are significantly different from Table 1 at 5% confidence level.

|  | AL | AL-15 | CU | NI | PB |
|---|---|---|---|---|---|
| AL | 1.00 | 0.8193 | 0.4435 | 0.3628 | 0.3312 |
| AL-15 |  | 1.00 | 0.3893 | 0.3358 | 0.2975 |
| CU |  |  | 1.00 | 0.3693 | 0.3121 |
| NI |  |  |  | 1.00 | 0.3089 |
| PB |  |  |  |  | 1.00 |

Table 1: Correlation matrix $\hat{\rho}$ of the LME data

|  | AL | AL-15 | CU | NI | PB |
|---|---|---|---|---|---|
| AL | 1 | 0.8418 | *0.4850* | 0.3790 | 0.3525 |
| AL-15 |  | 1 | *0.4262* | 0.3390 | 0.3133 |
| CU |  |  | 1 | *0.4037* | *0.3524* |
| NI |  |  |  | 1 | *0.3292* |
| PB |  |  |  |  | 1 |

Table 2: Correlation matrix $\hat{\rho}_{\mathsf{om}}$ of the LME data

1. Let $\hat{\rho}_0$ be the estimate of the $\rho$ matrix for the Normal copula. We note $n$ the dimension of the copula and $m$ the number of observations. $\hat{u}_{j,i}$ is the value taken by the $i^{th}$ empirical margin for the observation $j$.

2. $\hat{\rho}_{k+1}$ is obtained using the following equation:

$$\hat{\rho}_{k+1} = \frac{1}{m}\left(\frac{\nu+n}{\nu}\right)\sum_{j=1}^{m}\frac{\varsigma_j\varsigma_j^\top}{1+\varsigma_j^\top\hat{\rho}_k^{-1}\varsigma_j/\nu}$$

where $\varsigma_j = \left[\begin{array}{ccc} \mathbf{t}_\nu^{-1}\left(\hat{u}_{j,1}\right) & \cdots & \mathbf{t}_\nu^{-1}\left(\hat{u}_{j,n}\right)\end{array}\right]^\top$.

3. Repeat the second step until convergence — $\hat{\rho}_{k+1} = \hat{\rho}_k \ (:= \hat{\rho}_\infty)$.

Even if we assume that the margins are Gaussian, we will show that the choice of the dependence structure has a big impact on the *VaR* computation of a portfolio. If we consider that the dependence of the LME data is a Student copula with one degree of freedom, we obtain the matrix of parameters given in Table 3.

The value-at-risk with a confidence level equal to $\alpha$ corresponds to the quantile of the loss distribution. We have $\mathrm{VaR}(\alpha) = -\inf\{x : \Pr\{\mathrm{PnL}(t;h) \leq x\} \geq 1-\alpha\}$. Let us consider a linear equity portfolio with a fixed allocation $\theta = \left[\begin{array}{ccc}\theta_1 & \cdots & \theta_n\end{array}\right]^\top$. The PnL of the portfolio for a given holding period $h$ is $\mathrm{PnL}(t;h) = \sum_{i=1}^{n}\theta_i P_i(t)\, r_i(t;h)$ where $P_i(t)$ and $r_i(t;h)$ are respectively the price at time $t$ and the return between $t$ and $t+h$ of the asset $i$. For linear equity portfolio, we may assume that the risk factors are the return of the assets. The computation of the value-at-risk will then be sensible to the statistical modelling of the random variables $r_i(t;h)$. We consider three different portfolios[7]:

|  | AL | AL-15 | CU | NI | PB |
|---|---|---|---|---|---|
| $\mathsf{P}_1$ | 1 | 1 | 1 | 1 | 1 |
| $\mathsf{P}_2$ | −1 | −1 | −1 | 1 | 1 |
| $\mathsf{P}_3$ | 2 | 1 | −3 | 4 | 5 |

For these three portfolios, we assume that the margins are Gaussian and compare the value-at-risk under the assumption of Normal copula and Student copula with $\nu = 1$. For low confidence levels (90% and 95%), the Student copula gives a smaller *VaR* than the Normal copula (see Tables 4 and 5). For high confidence levels, we obtain opposite results. Now, if we use a Normal copula but with Student-$t_4$ margins, we obtain Table 6. These three tables show the importance of modelling of the margins and the copula function.

[7] A negative number corresponds to a short position.

| | AL | AL-15 | CU | NI | PB |
|---|---|---|---|---|---|
| AL | 1 | 0.8183 | *0.3256* | *0.2538* | *0.1888* |
| AL-15 | | 1 | *0.2706* | *0.2225* | *0.1634* |
| CU | | | 1 | *0.2668* | *0.2184* |
| NI | | | | 1 | *0.1944* |
| PB | | | | | 1 |

Table 3: Correlation matrix $\hat{\rho}_{\mathsf{om}}$ with Student copula ($\nu = 1$) of the LME data

| | 90% | 95% | 99% | 99.5% | 99.9% |
|---|---|---|---|---|---|
| $P_1$ | 7.340 | 9.402 | 13.20 | 14.65 | 17.54 |
| $P_2$ | 4.047 | 5.176 | 7.318 | 8.107 | 9.753 |
| $P_3$ | 13.96 | 17.90 | 25.21 | 27.92 | 33.54 |

Table 4: Gaussian margins and Normal copula

If we compare these values at risk with the historical *VaR* (which is reported in Table 7), we notice that the Gaussian *VaR* underestimates systematically the risk for a confidence level greater than 95%. That explains why that parametric (or Gaussian) *VaR* are not very often used by international banks. Nevertheless, the copula framework is likely to lead in the near future to more realistic parametric or semi-parametric[8] *VaR* which will be adopted by practitioners.

**Remark 1** *No analytical formula is generally available for the VaR computation with a copula framework. The computation is also carried out by simulations. So, we can treat non linear portfolios without any further difficulties.*

## 3.2 Stress Testing

To compute regulatory capital for market risks, a bank may use an internal model. But in this case, it must satisfy some criteria and it must be validated by the supervisory authority. One of these criteria concerns stress testing which are defined in the following way in the document [1] of the Basel Committee on Banking Supervision:

> [...] *Banks' stress scenarios need to cover a range of factors that can create extraordinary losses or gains in trading portfolios, or make the control of risk in those portfolios very difficult. These factors include low-probability events in all major types of risks, including the various components of market, credit, and operational risks.*

Generally speaking, a problem for the bank is to justify stress testing programs to their home regulator. That is why banks use very often historical scenarios. Some banks have developed more sophisticated stress scenarios based on extreme value theory. Since it is now familiar to practitioners, we give here a very brief description. Let $X$ be a risk factor with probability distribution $\mathbf{F}$. We would like to build risk scales for extreme events. Suppose that the portfolio is short with respect to $X$. The underlying idea is then to find a scenario $x$ such that $\mathbb{P}\{X > x\}$ is very low. One of the issue is to define what is a low probability. In this case, risk managers prefer to use the concept of 'return time'. Gumbel [1958] defines it in the following way:

> [...] *Consider first a discontinuous variate as in the case of the classical die, from which so many problems of the calculus of probability arose. For an unbiased die, the probability that a certain point occurs is $\frac{1}{6}$. Therefore, we may expect to get this point, in the long run and on the average, once in $T = 6$ trials. For a continuous variate, there is no probability for a certain value $x$, but only a density of probability. However, there is a probability $1 - \mathbf{F}(x)$ of a value to be equalled*

[8]In this case, the dependence of the risks corresponds to a parametric copula function whereas the distribution of the risks are the empirical distributions.

| | 90% | 95% | 99% | 99.5% | 99.9% |
|---|---|---|---|---|---|
| $\mathsf{P}_1$ | 5.797 | 8.112 | 13.36 | 15.65 | 20.15 |
| $\mathsf{P}_2$ | 3.812 | 5.531 | 9.801 | 11.65 | 16.34 |
| $\mathsf{P}_3$ | 13.45 | 19.32 | 34.15 | 40.77 | 54.95 |

Table 5: Gaussian margins and Student-$\mathbf{t}_1$ copula

| | 90% | 95% | 99% | 99.5% | 99.9% |
|---|---|---|---|---|---|
| $\mathsf{P}_1$ | 6.593 | 8.885 | 14.33 | 16.98 | 23.96 |
| $\mathsf{P}_2$ | 3.778 | 5.021 | 7.928 | 9.351 | 13.42 |
| $\mathsf{P}_3$ | 12.80 | 17.13 | 27.52 | 32.89 | 49.09 |

Table 6: Student-$\mathbf{t}_4$ margins and Normal copula

*or exceeded by $x$. Its reciprocal*

$$\mathbf{t}(x) = \frac{1}{1-\mathbf{F}(x)}$$

*is called the* ***return period****. It is the number of observations such that, on the average, there is one observation equalling or exceeding $x$. Obviously, the return period increases with the variate.*

In order to find the scenario $x$ related to a given return time $\mathbf{t}$, we have to define the distribution $\mathbf{F}$ and more particularly to estimate the tail of the distribution. We prefer then to work with the extreme order statistic $X_{n:n}$. The return time associated to the event $\{X_{n:n} > x\}$ is $\mathbf{t} = n\left(1-\mathbf{F}_{n:n}(x)\right)^{-1}$. We deduce that $x = \mathbf{F}_{n:n}^{-1}\left(1-n\mathbf{t}^{-1}\right)$. One generally uses the *Generalized Extreme Value* distribution with a block-componentwise method to estimate $\mathbf{F}_{n:n}$. Let us consider the example of the risk factors CAC40 and DowJones. Using daily returns and $n$ fixed to 22 trading days, we obtain the results in Table 8.

How to build now multivariate stress scenarios in the same way ? A rough solution is to collect univariate stress scenarios. For example, the return times of a fall of 10% of the CAC40 and a fall of 10% of the DowJones are respectively 5.090 and 4.959 years. It comes out that the return time of the simultaneous fall of 10% of the CAC40 and 10% of the DowJones is 5.09 years if the risk factors are perfectly correlated **but 304.12 years if they are independent**. Collecting univariate stress scenarios with the same return time has no sense if the copula function is not $\mathbf{C}^{+}$.

The extension to the multivariate case may be a difficult issue if we do not consider a copula framework. Indeed, any copula function $\mathbf{C}_{\star}$ such that $\mathbf{C}_{\star}\left(u_1^t,u_2^t\right) = \mathbf{C}_{\star}^t\left(u_1,u_2\right)$ for all $t>0$ can be used to construct a bivariate extreme value distribution (Deheuvels [1978]). Such copula functions are called *extreme value copulae* (Joe [1997]). For example, the Gumbel copula of the first section is an extreme value copula.

Let $X$ and $Y$ be two risk factors. Suppose that the portfolio is short both in $X$ and $Y$. We have

$$\begin{aligned}\mathbf{t}(x,y) &= \frac{n}{\Pr\left(X_{n:n}>x,Y_{n:n}>y\right)} \\ &= \frac{n}{1-\mathbf{F}_{n:n}(x)-\mathbf{G}_{n:n}(y)+\mathbf{C}_{\star}\left(\mathbf{F}_{n:n}(x),\mathbf{G}_{n:n}(y)\right)}\end{aligned}$$

There is now a set $\mathcal{S}$ of solutions to achieve a given return time. If we denote $x_{\mathrm{u}}$ and $y_{\mathrm{u}}$ the corresponding one-dimensional stress scenarios, the set of bivariate stress scenarios verifies that $\mathcal{S}(x,y;\mathbf{t}) \subset \{(x,y): x\le x_{\mathrm{u}}, y\le y_{\mathrm{u}}\}$. If the copula is $\mathbf{C}^{+}$, we have

$$\mathcal{S}^{+}(x,y;\mathbf{t}) = \{(x,y_{\mathrm{u}}): x\le x_{\mathrm{u}}\} \cup \{(x_{\mathrm{u}},y): y\le y_{\mathrm{u}}\}.$$

However if the copula is $\mathbf{C}^{\perp}$, the set is

$$\mathcal{S}^{\perp}(x,y;\mathbf{t}) = \left\{(x,y): \mathbb{P}\left(X_{n:n}>x\right)\mathbb{P}\left(Y_{n:n}>y\right) = \mathbb{P}\left(X_{n:n}>x_{\mathrm{u}}\right) = \mathbb{P}\left(Y_{n:n}>y_{\mathrm{u}}\right)\right\}.$$

| | 90% | 95% | 99% | 99.5% | 99.9% |
|---|---|---|---|---|---|
| $P_1$ | 6.627 | 9.023 | 14.43 | 16.52 | 29.56 |
| $P_2$ | 3.434 | 5.008 | 8.946 | 11.28 | 16.24 |
| $P_3$ | 12.24 | 17.32 | 31.77 | 36.09 | 50.02 |

Table 7: Historical value at risk

| Risk scales (daily variation) | | | | |
|---|---|---|---|---|
| | for short positions (in %) | | for long positions (in %) | |
| Return time (in years) | CAC40 | DowJones | CAC40 | DowJones |
| 5 | 9.95 | 6.13 | −9.93 | −10.04 |
| 10 | 13.06 | 7.38 | −12.85 | −13.91 |
| 25 | 18.66 | 9.36 | −17.95 | −21.35 |
| 50 | 24.41 | 11.13 | −23.01 | −29.48 |
| 75 | 28.55 | 12.30 | −26.59 | −35.59 |
| 100 | 31.91 | 13.19 | −29.44 | −40.67 |

Table 8: Univariate stress scenarios

Because extreme value copulas satisfy $\mathbf{C}^{\perp} \prec \mathbf{C} \prec \mathbf{C}^{+}$, bivariate stress scenarios are between these two sets for the general case. For example, using the previous example for CAC40 and DowJones, the set of bivariate stress scenarios for a 5-year return time is reported in Figure 2. We use a Gumbel copula, the parameter of which is estimated by a maximum-likelihood method.

We can extend the bivariate case to the multivariate case. However, it is very difficult to define explicitly the set $\mathcal{S}(x_1, \ldots, x_n; \mathbf{t})$. Nevertheless, the framework may be used to evaluate the severity of a given stress scenario (given by home economists for example). In this case, it is a good tool to compare the strength of stress scenarios and of course to quantify them.

## 3.3 Monitoring the Risk of the Dependence in Basket Derivatives

The pricing of derivatives generally involves some subjective parameters. This is the case for basket equity or credit derivatives, which require to define the dependence function between the risk factors. To illustrate that, let us consider a two-asset option in the equity market. We note $S_1$ and $S_2$ the asset prices, and $G(S_1, S_2)$ the payoff function of the *European* option. In the classic Black-Scholes model[9], the price of this option depends on the correlation parameter $\rho$ between the asset returns, so that the trader has to set a numerical value for $\rho$ in the pricer. Nevertheless, the trader does not generally know what is **exactly** the right value to input in the pricer, even if he performs some sophisticated statistics on historical data. From a risk management point of view, we face to **a parameter risk**, which is slighty different from model risk (Derman [2001], Rebonato [2001]):

> *It means that even if the model is the right model, there might be a mispricing risk due to the fact that we do not use the correct value of the parameter.*

In the case of *European* basket options and within the Black-Scholes model, Rapuch and Roncalli [2001] have proved that if "*$\partial^2_{1,2} G$ is a nonpositive (resp. nonnegative) measure then the option price is nonincreasing (resp. nondecreasing) with respect to $\rho$*". The risk manager may then use this proposition to define a *conservative* value for $\rho$. If the model is not Black-Scholes any longer, the dependence parameter is not $\rho$, but the copula function of the risk-neutral distribution. In this case, the previous result may be generalized in the following way: "*if the distribution $\partial^2_{1,2} G$ is a nonpositive (resp. nonnegative) measure then*

[9] In this model, the asset prices are correlated geometric Brownian motions

$$\left\{ \begin{array}{l} \mathrm{d}S_1(t) = \mu_1 S_1(t)\,\mathrm{d}t + \sigma_1 S_1(t)\,\mathrm{d}W_1(t) \\ \mathrm{d}S_2(t) = \mu_2 S_2(t)\,\mathrm{d}t + \sigma_2 S_2(t)\,\mathrm{d}W_2(t) \end{array} \right.$$

where $\mathbb{E}[W_1(t) W_2(t)] = \rho t$.

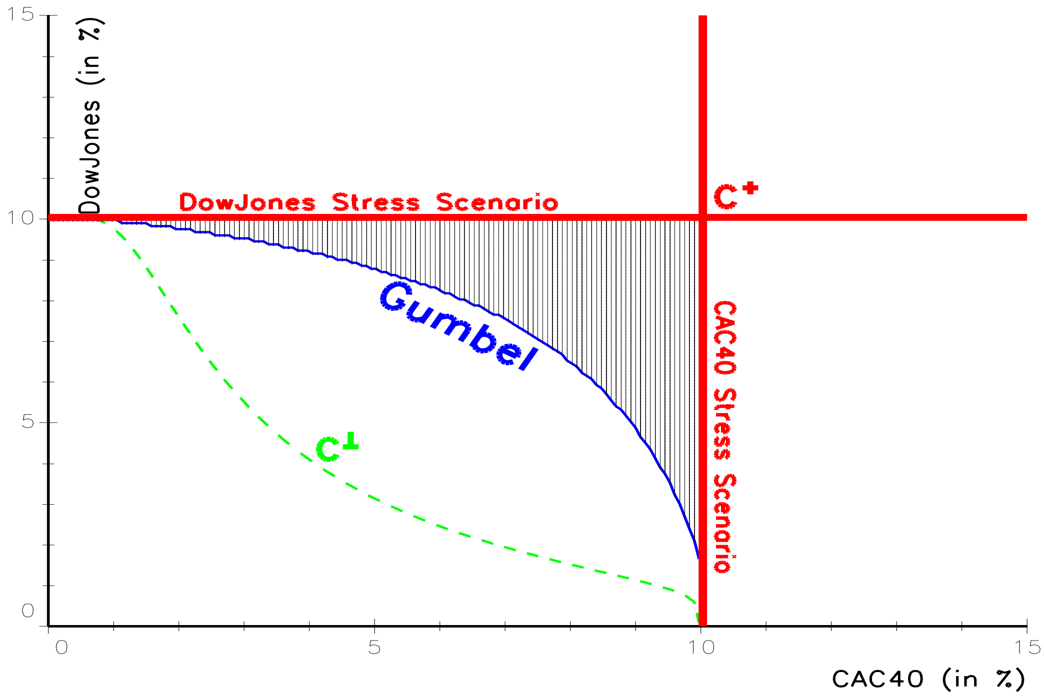


Figure 2: Bivariate stress scenarios

*the option price is nonincreasing (resp. nondecreasing) with respect to the concordance order* $\succ$". The result obtained in the Black-Scholes framework is then a particular case of this general result when the dependence function is the Normal copula with parameter $\rho$.

What about more complicated basket derivatives ? Rapuch and Roncalli [2001] present an example to show that previous results may not be easily generalized to multi-asset options, even if the model is Black-Scholes. And the generalization may be more difficult if the product is path-dependent. It is then very important to have tools to monitor the risk of the dependence function[10]. Nevertheless, we need to be very careful with empirical evidence, in particular for high-dimensionnally basket derivatives.

We give now an example to illustrate the difficulty to manage this type of problem. We suppose that the dependence function between the risk factors is the Normal copula with a correlation matrix $\Sigma$. Sometimes, traders prefer to replace $\Sigma$ by a *constant-correlation matrix*[11] $C(\rho)$, either because they have little information of the dependence between risk factors (for example in credit derivatives), or because estimated correlations are not stable in time (for example in multi-asset options). Moreover, especially for high-dimensionnally basket derivatives, using a constant-correlation matrix $C(\rho)$ and not a *full-correlation* matrix $\Sigma$ enables to reduce the number of parameters to be controlled. Nevertheless, we must avoid some false ideas:

**Fallacy 1** *To get a conservative price, it is enough to find the maximum (or minimum) price in the set of all $C(\rho)$ matrices.*

**Fallacy 2** *If the price is an increasing (resp. decreasing) function with respect to the parameter $\rho$, the price is an increasing (resp. decreasing) function with respect to the concordance order. We may then take the copula function $\mathbf{C}^{\perp}$ or $\mathbf{C}^{+}$ to get the conservative price*[12].

[10] like simulating correlation matrices (Davis and Higham [2000]).

[11] $C(\rho)$ is a constant-correlation matrix or an $A$-correlation matrix if all non-diagonal elements is equal to $\rho$ and if $\rho \geq -1/(k-1)$ where $k$ is the dimension of the matrix $C(\rho)$ (Tiit and Helemäe [1997]).

[12] Because the maximal element of the Normal copula for the set of all $C(\rho)$ matrices is the Normal copula with the correlation

To illustrate these two fallacies, we consider a 'tranching' of a credit risk product, that is a partition of the total loss $L$. We consider $J$ tranches defined in the following way :

$$\begin{aligned} L &= \min(L, K_1) + \min\left((L-K_1)^+, K_2\right) + \\ &\quad \min\left((L-K_1-K_2)^+, K_3\right) + \ldots + \left(L - \sum_{j=1}^{J-1} K_j\right)^+ \end{aligned}$$

And we suppose that the total loss $L$ is defined by $L = L_1 \cdot 1\left\{L_2 > \bar{L}_2\right\} + L_3 \cdot 1\left\{L_4 > \bar{L}_4\right\}$. The individual losses $L_i$ are distributed according to the distribution $\mathcal{LN}(\mu_i, \sigma_i)$, the dependence function between the individual losses is a Normal copula with a correlation matrix $\Sigma$ and the numerical value of the barriers $\bar{L}_2$ and $\bar{L}_4$ is 105. In order to limit the effect of the marginals, we take the same values for the parameters: $\mu_i = 5$ and $\sigma_i = 0.5$. First, we set $\Sigma$ to the constant-correlation matrix $C(\rho)$. In Figure 3, we report the net present value[13] of several tranches with respect to the parameter $\rho$. The relationship between the NPV and the parameter $\rho$ depends on the tranche. For an 'equity-like' tranche, it is decreasing whereas it is increasing for a 'senior-like' tranche. Then, we calculate the net present value for the following full correlation matrix

$$\Sigma = \Sigma_1 = \begin{pmatrix} 1 & & & \\ 0.8 & 1 & & \\ 0.1 & 0.5 & 1 & \\ 0.5 & 0.8 & 0.8 & 1 \end{pmatrix}$$

and we indicate the mean correlation in $\Sigma_1$ — we note it $\rho_m(\Sigma_1)$. We point out that for the tranche $\min\left((L-200)^+, 25\right)$, the NPV with the $\Sigma_1$ matrix is larger than the NPV with a $C(\rho)$ matrix when $\rho \leq 95\%$, but the biggest correlation value in $\Sigma_1$ is 80% and the mean correlation $\rho_m(\Sigma_1)$ is 58.3%! Nevertheless, the NPV calculated with the copula $\mathbf{C}^+$ is even larger than $\mathrm{NPV}(\Sigma_1)$. If now we change slightly the marginals, for example if $\sigma_4$ is not equal to 0.5 but 1, we have a surprising result: $\mathrm{NPV}(\mathbf{C}^+)$ is smaller than $\mathrm{NPV}(\Sigma_1)$! Now, we change the loss expression $L = L_1 \cdot 1\left\{L_2 > \bar{L}_2\right\} + L_3 \cdot 1\left\{L_4 < \bar{L}_4\right\}$, but we keep the same parameters values. For a 'senior-like' tranche $\min\left((L-200)^+, 50\right)$, the net present value is not a monotonic function with respect to $\rho$ — see Figure 4. However, for a reasonable range, say $\rho \in [0, 60\%]$, the net present value is almost constant. We may think that the effect of dependence is small for this tranche. Taking the following full-correlation matrix

$$\Sigma = \Sigma_2 = \begin{pmatrix} 1 & & & \\ 0.5 & 1 & & \\ 0.1 & 0.2 & 1 & \\ 0.5 & 0.0 & 0.1 & 1 \end{pmatrix}$$

and we find out that this not true.

Of course, it is a toy example. But, you can imagine that we may encounter same problems in some exotic multi-asset options or high-dimensionally basket credit derivatives. Monitoring the dependence function is then a challenge for the risk management (and the front office too).

# 4 Credit Risk Management

No doubt, one of the most popular application of copula functions in finance is credit risk modelling. From the work of Li [29] to the important paper of Schönbucher and Schubert [37], there is now a big literature on this subject, both from academics and professionals and dealing with risk management and pricing. Here we first show that one can achieve in practice to compute the measure of the risk of a large pool of loans in a copula framework, and then turn to the pricing of basket credit derivatives, which, like equity basket exotic products, are very sensitive to the dependence between (the defaults of) the firms.

matrix parameters $C(1)$ or the copula function $\mathbf{C}^+$. And the minimal element tends to $\mathbf{C}^\perp$ with the number of factors tends towards infinite.

[13] Without loss of generality, we assume that the interest rates are zero.

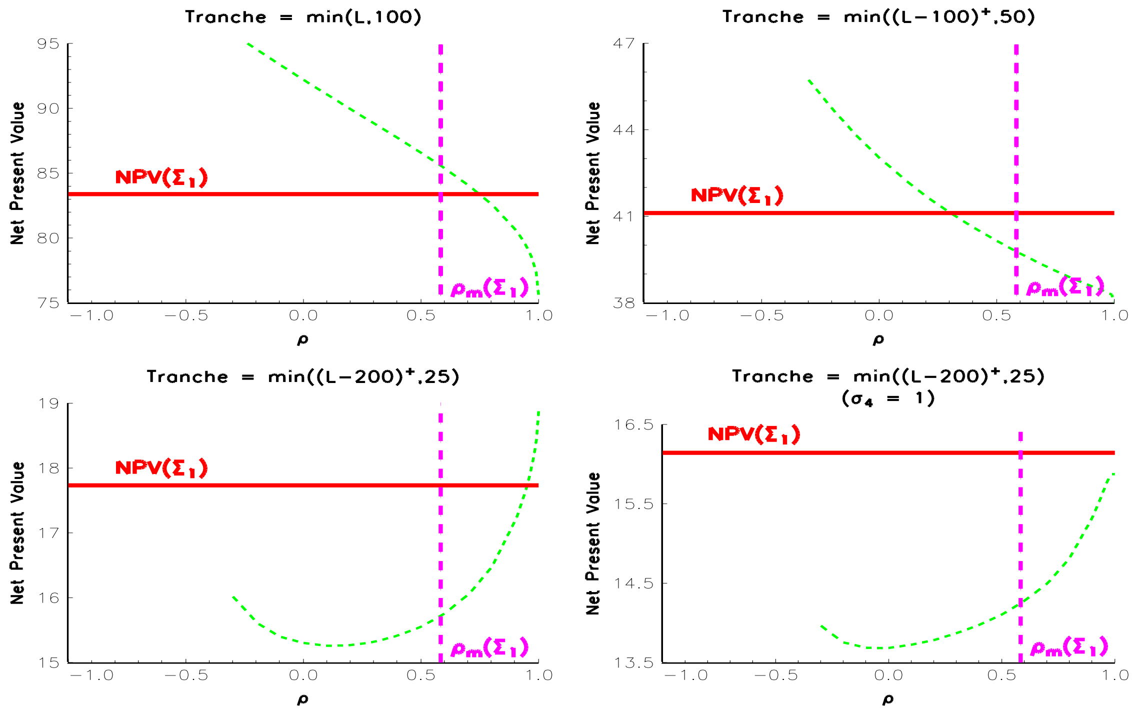


Figure 3: Tranching products and dependence function (I)

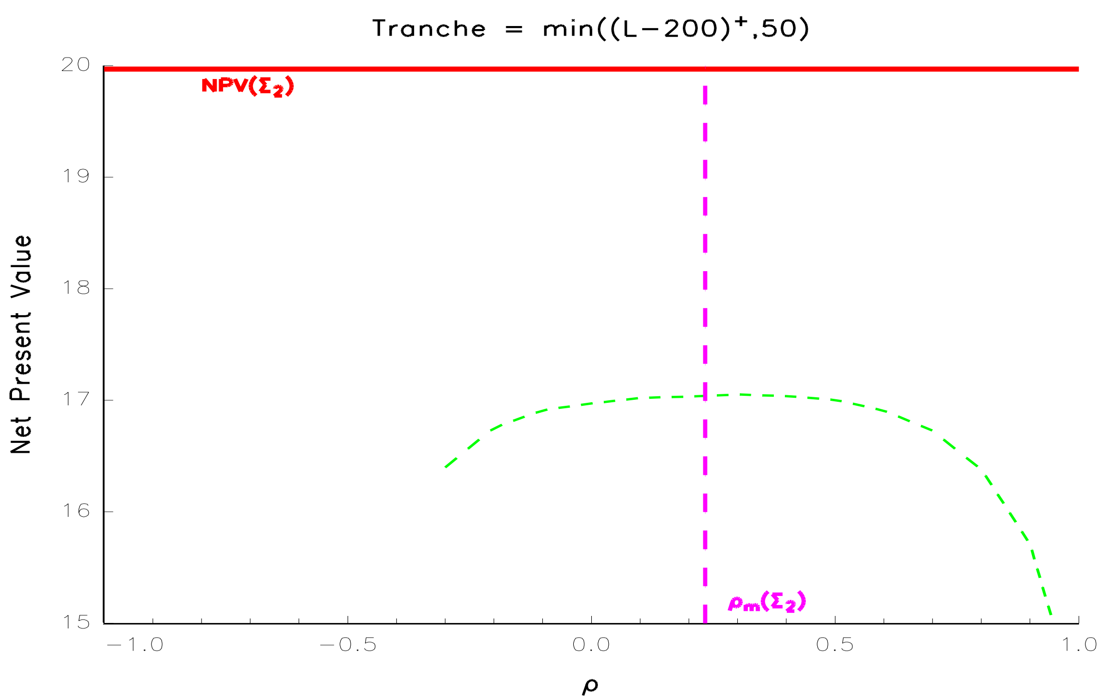


Figure 4: Tranching products and dependence function (II)

## 4.1 Measuring the Risk of a Credit Portfolio

In order to simplify the presentation, we consider the loss expression of a credit portfolio as for the IRB model of Basel II:

$$L(t) = \sum_{i=1}^{I} L_i(t) = \sum_{i=1}^{I} x_i \cdot (1 - R_i) \cdot \mathbf{1}\{\tau_i \leq t_i\}$$

where $L_i(t)$ is the loss of the $i^{th}$ name, $x_i$ is the exposure at default, $R_i$ is the recovery rate, $\tau_i$ is the default time and $t_i$ is the maturity horizon. We make the following assumptions:

(A1) $R_i$ is a stationary random variable with mean $\mu_i(R)$ and standard deviation $\sigma_i(R)$ independent of the random variables $R_j$ ($j \neq i$) and all the default times (including $\tau_i$) ;

(A2) the joint survival function $\mathbf{S}$ of the default times $(\tau_1, \ldots, \tau_I)$ is defined with a copula representation

$$\mathbf{S}(t_1, \ldots, t_I) = \breve{\mathbf{C}}(\mathbf{S}_1(t_1), \ldots, \mathbf{S}_I(t_I))$$

where $\breve{\mathbf{C}}$ is the survival copula and $\mathbf{S}_i$ is the survival functions of the default time $\tau_i$.

The copula framework is a natural method to 'correlate' default times because of **internal credit rating systems**, which are the core of managing credit risk in banks. An internal credit rating system enables to assign internal ratings to credits and to define for each rating its default rate with respect to a maturity. It means that the survival functions $\mathbf{S}_i$ are given. To build the joint survival function using the internal credit rating system, the copula approach is then the more easy to implement.

The loss distribution is generally computed by a Monte-Carlo method. From a professional point of view, one big issue is the computational time of the loss simulation, because credit portfolios are often very large. In [34], we propose to use a Normal or Student copula with a *sector-correlation matrix* (or *a factor-correlation matrix*). Let $j = 1, \ldots, J$ ($J \ll I$) denotes the $j^{th}$ sector. Let $\rho^\star$ be the symmetric matrix with $\rho^\star(j, j)$ the intra-sector correlations and $\rho^\star(j_1, j_2)$ the inter-sector correlations. $j = m(i)$ is the mapping function between the loan $i$ and its sector $j$. The sector-correlation matrix $\Sigma$ is then defined as follows[14]:

$$\Sigma = m(\rho^\star) = \begin{pmatrix} 1 & \rho^\star(m(1), m(2)) & \cdots & \rho^\star(m(1), m(I)) \\ & 1 & & \vdots \\ & & & \rho^\star(m(I-1), m(I)) \\ & & & 1 \end{pmatrix} \tag{1}$$

We can simulate the random variates $(u_1, \ldots, u_I)$ from the Normal copula with the correlation matrix $\Sigma$ in two different ways:

[14]Consider an example with 4 sectors

| Sector | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| 1 | **30%** | *20%* | *10%* | *0%* |
| 2 | | **40%** | *30%* | *20%* |
| 3 | | | **50%** | *10%* |
| 4 | | | | **60%** |

and 7 loans

| $i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 |
|---|---|---|---|---|---|---|---|
| $j = m(i)$ | 1 | 1 | 2 | 3 | 3 | 3 | 4 |

The sector-correlation matrix $\Sigma$ is then

$$\Sigma = \begin{pmatrix} 1.00 & 0.30 & 0.20 & 0.10 & 0.10 & 0.10 & 0.00 \\ & 1.00 & 0.20 & 0.10 & 0.10 & 0.10 & 0.00 \\ & & 1.00 & 0.30 & 0.30 & 0.30 & 0.20 \\ & & & 1.00 & 0.50 & 0.50 & 0.10 \\ & & & & 1.00 & 0.50 & 0.10 \\ & & & & & 1.00 & 0.10 \\ & & & & & & 1.00 \end{pmatrix}$$

1. The [CHOL] algorithm is $u_i = \Phi(z_i)$ where $\mathbf{z} = \mathrm{chol}(\Sigma)\,\varepsilon$ and $\varepsilon$ is a vector of independent gaussian random numbers $\mathcal{N}(0,1)$.

2. The [Sloane] algorithm proposed by Riboulet and Roncalli [2002] is the following:

$$\begin{aligned} \rho^\star &= V^\star \Lambda^\star V^{\star\top} \text{ (eigensystem)} \\ A^\star &= V^\star_\bullet (\Lambda^\star)^{\frac{1}{2}} \ (V^\star_\bullet \text{ is the } L_2\text{-normalized matrix of } V^\star) \\ z_i &= \sum_{j=1}^{J} A^\star_{m(i),j} x_j + \sqrt{1-\rho^\star(m(i),m(i))}\varepsilon_i \\ u_i &= \Phi(z_i) \end{aligned}$$

   where $(x_1,\ldots,x_J,\varepsilon_1,\ldots,\varepsilon_I)$ is a vector of independent gaussian random numbers $\mathcal{N}(0,1)$.

The algorithm order of [CHOL] is $I^2$, but the algorithm order of [Sloane] is $I$ (because $J$ is fixed). Here are some other characteristics of the two algorithms:

| | Dimension of the matrix | Number of random variates | Number of + operations | Number of × operations |
|---|---|---|---|---|
| [CHOL] | $I \times I$ | $I$ | $I \times (I-1)$ | $I \times I$ |
| [Sloane] | $J \times J$ | $I + J$ | $I \times J$ | $I \times J$ |
| | 10000 loans + 20 sectors | | | |
| [CHOL] | $10^8$ | 10000 | $\simeq 10^8$ | $10^8$ |
| [Sloane] | 400 | 10020 | $2 \times 10^5$ | $2 \times 10^5$ |

This explains why the [Sloane] algorithm is very fast (and less memory-consuming) compared to the [CHOL] algorithm (in average, the computational time is divided by a ratio of $I/J$). Moreover, the [Sloane] algorithm yields the following proposition: if the eigenvalues $\lambda^\star_j$ are positive, then the matrix $\Sigma$ obtained by the construction (1) is a correlation matrix.

The [Sloane] algorithm is related to the IRB model of Basel II. This model corresponds to the Vasicek/Merton default model with one factor (Finger [1999]). We assume that the value of the assets $Z_i$ are driven by a common, standard normally distributed factor $X$ and an idiosyncratic normally distributed component $\varepsilon_i$ such that $Z_i = \sqrt{\rho}X + \sqrt{1-\rho}\varepsilon_i$. It comes out that $(Z_1,\ldots,Z_I)$ is a standard Gaussian vector with a constant-correlation matrix $C(\rho)$. In this model, the default occurs when $Z_i$ falls below a barrier $B_i$. Let us denote the default probability at time $t$ by $P_i(t)$. We remark that

$$\begin{aligned} \mathbf{S}(t_1,\ldots,t_I) &= \Pr\{\tau_1 > t_1,\ldots,\tau_I > t_I\} \\ &= \Pr\{Z_1 > B_1,\ldots,Z_I > B_I\} \\ &= \Pr\left\{Z_1 > \Phi^{-1}(P_1(t_1)),\ldots,Z_I > \Phi^{-1}(P_I(t_I))\right\} \\ &= \breve{\mathbf{C}}(1-P_1(t_1),\ldots,1-P_I(t_I);C(\rho)) \\ &= \breve{\mathbf{C}}(\mathbf{S}_1(t_1),\ldots,\mathbf{S}_I(t_I);C(\rho)) \end{aligned}$$

where $\breve{\mathbf{C}}$ is the survival Normal copula and the corresponding correlation matrix is $C(\rho)$. To simulate this model, we may use the so-called [$\sqrt{\rho}$] algorithm, which is defined as follows: $u_i = \Phi(z_i)$ where $z_i = \sqrt{\rho}x + \sqrt{1-\rho}\varepsilon_i$ and $(x,\varepsilon_1,\ldots,\varepsilon_I)$ is a vector of independent gaussian random numbers $\mathcal{N}(0,1)$. It is in fact a special case of the [Sloane] algorithm when $J = 1$.

Let us take the example given in [36]. The portfolio is composed of 500 loans with same maturity 5 years and same exposure at default 1000 euros. The repartition by rating is the following:

| Rating | AAA | AA | A | BBB | BB | B | CCC |
|---|---|---|---|---|---|---|---|
| Number of loans | 5% | 15% | 20% | 30% | 15% | 10% | 5% |

For each rating, we calibrate an exponential survival function. We then assume that the dependence between default times is a Normal copula with a factor-correlation matrix as in Footnote 14. The repartion by sector is the following:

| Sector | 1 | 2 | 3 | 4 |
|---|---|---|---|---|
| Number of loans | 20% | 30% | 10% | 40% |

For the recovery rate, we calibrate a Beta distribution with $\mu_i(R) = 50\%$ and $\sigma_i(R) = 20\%$. Figure 5 presents the density of the loss distribution for the Normal copula with the factor-correlation matrix, and under other assumptions on the dependence function.

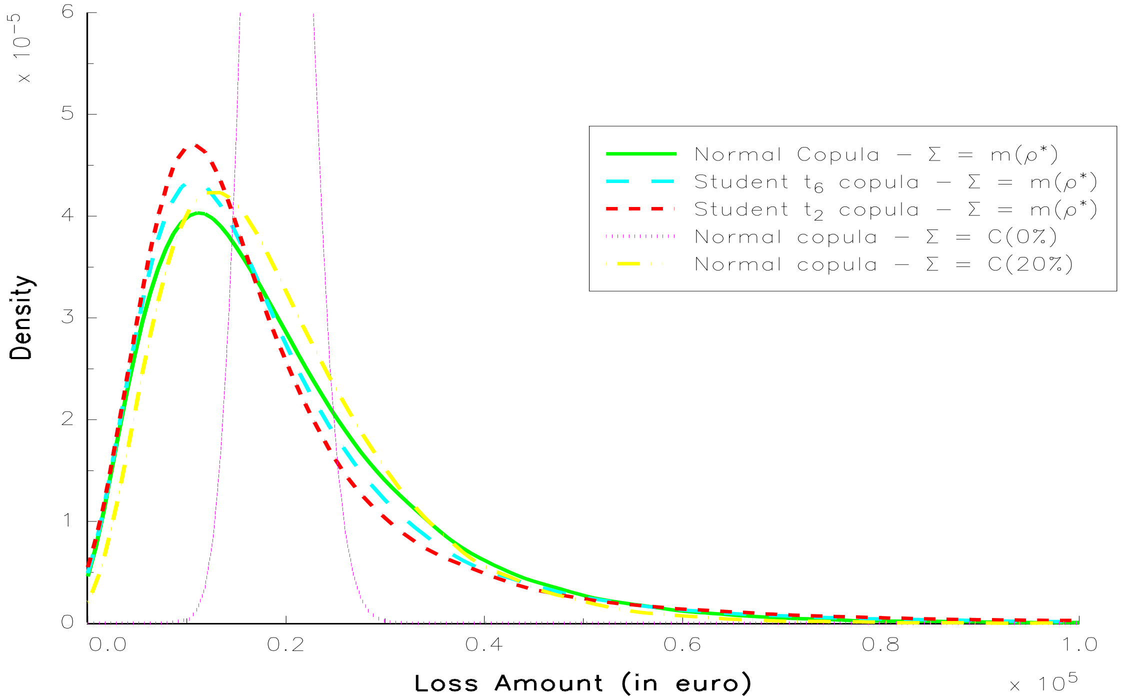


Figure 5: Impact of the copula function on the loss distribution of the credit portfolio

Now, we turn to the computation of the risk of the credit portfolio, which is generally computed with a *VaR* measure[15] defined as the percentile $\alpha$ of the loss $L(t)$ – $\mathrm{CreditVaR}(\alpha) = \inf\{L : \mathbb{P}\{L(t) \leq L\} \geq \alpha\}$ – and its sensitivities. In Figure 6, we report the CreditVaR measure for the different copula functions. We remark that the dependence modelling assumption has a big impact on the risk measure. We may wonder if we have the same conclusion with the recovery rate modelling assumption. Indeed, because we have an *homogeneous portfolio*[16] and because the number of loans is large, we may considered that it is almost *an infinitely fine-grained portfolio* (Gordy [2001]). So, as in the Basel II model, the recovery rate modelling assumption has no impact on the CreditVaR measure, only the values taken by $\mu_i(R)$ play a role[17]. To manage the risk of the credit portfolio, we have to compute the *derivatives at Risk* defined as $\mathrm{DR}(i) = \partial\,\mathrm{CreditVaR}(\alpha)/\partial x_i$. For that, we use the result of Gouriéroux, Laurent and Scaillet [2000] and Tasche [2000]:

**Theorem 1** *Let $(\varepsilon_1, \ldots, \varepsilon_I)$ be a random vector and $(x_1, \ldots, x_I)$ a vector in $\mathbb{R}^I$. We consider the loss $L$ defined by $L = \sum_{i=1}^{I} x_i \cdot \varepsilon_i$ and we set $Q(L;\alpha)$ the percentile $\alpha$ of the random variable $L$. Then, we have*

$$\frac{\partial\, Q(L;\alpha)}{\partial\, x_i} = \mathbb{E}[\varepsilon_i \mid L = Q(L;\alpha)]$$

[15] We do not here discuss the choice of the risk measure (see Szegö [2002] for this type of problem). We just remark that the *VaR* measure is the most used risk measure in banks.

[16] We have the same exposure at default for all loans.

[17] It is confirmed by a Monte Carlo simulation (see [36]).

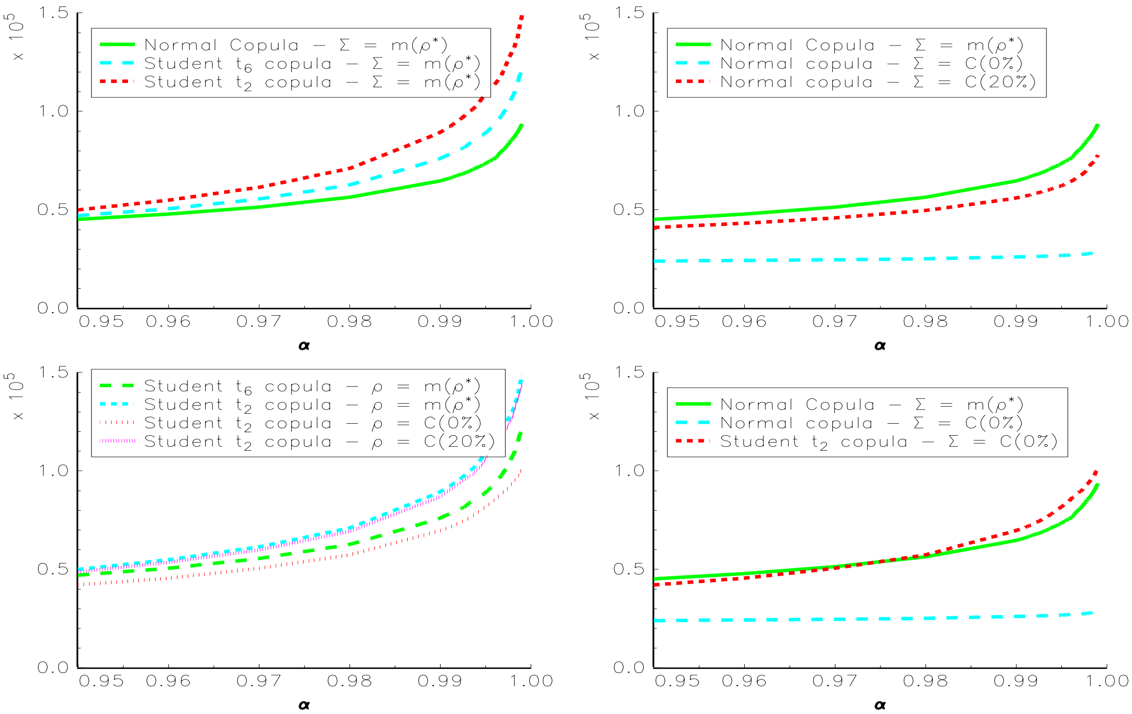


Figure 6: Impact of the copula function on the CreditVaR

This result holds even in the non-Gaussian case. If we apply this result to our loss $L(t)$, we have

$$\mathrm{DR}(i) = \frac{\partial \ \mathrm{CreditVaR}(\alpha)}{\partial \, x_i} = \mathbb{E}\left[(1-R_i)\cdot \mathbf{1}\{\tau_i \le t_i\} \mid L = \mathrm{CreditVaR}(\alpha)\right] \tag{2}$$

The issue is now to compute this conditional expectation with the Monte-Carlo method. Let $(L_1,\ldots,L_n)$ be the sample of the simulated losses. Computing the CreditVaR by a Monte-Carlo method is equivalent to search the value taken by the order statistic $L_{n\alpha:n}$ of the simulated losses. A 'natural' estimator of the conditional expectation (2) is then $\widehat{\mathrm{DR}}(i) = L_{i,m} = (1-R_{i,m})\cdot\mathbf{1}\{\tau_{i,m}\le t_i\}$ where $R_{i,m}$ et $\tau_{i,m}$ are the values taken by $R_i$ and $\tau_i$ for the simulation $m$ corresponding to the order statistic $L_{n\alpha:n}$. The problem is that the variance of this estimator is very large, because the conditional expectation is estimated by using only one point. So, in order to reduce the variance, we use a localization method. We suppose that $\mathrm{CreditVaR}(\alpha) = \sum_{m\in\mathcal{M}} p_m L_m$ with $\sum_{m\in\mathcal{M}} p_m = 1$. We may interpret this value-at-risk as a mathematical expectation under the probability measure $\{p_m, m\in\mathcal{M}\}$. In this case, we have $\frac{\partial \ \mathrm{CreditVaR}(\alpha)}{\partial \, x_i} = \sum_{m\in\mathcal{M}} p_m \frac{L_{i,m}}{x_i}$. For the choice of the values of $p_m$, we may consider a kernel method (like uniform or triangular). In order to show the interest of the localization method, we take the example of a Gaussian loss $L=\sum_{i=1}^{2} x_i\cdot\varepsilon_i$ with $\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}0\\0\end{pmatrix},\begin{pmatrix}1 & 0.5\\ 0.5 & 1\end{pmatrix}\right)$ and $x_1 = 100$ and $x_2 = 50$. An analytical computation gives $\mathrm{CreditVaR}(99\%) = 307.747$, $\mathrm{DR}(1) = 2.198$ and $\mathrm{DR}(2) = 1.759$. With $n$ equal to 5,000 simulations, we notice in Figure 7 that the localization method[18] gives a better result than the one-point method for the estimation of the sensitivities[19]. If we now apply this method to our example, we obtain the derivatives at risk of Figure 8. Besides, the localization method has another advantage. It enables to retrieve the Euler principle:

$$\mathrm{CreditVaR}(\alpha) = \sum_{i=1}^{I} \underbrace{x_i \cdot \frac{\partial \ \mathrm{CreditVaR}(\alpha)}{\partial \, x_i}}_{\text{continuous marginal contribution}} \tag{3}$$

[18] $h$ is the window of the kernel.

[19] Note that the localization method does not improve the estimate of the VaR in this example

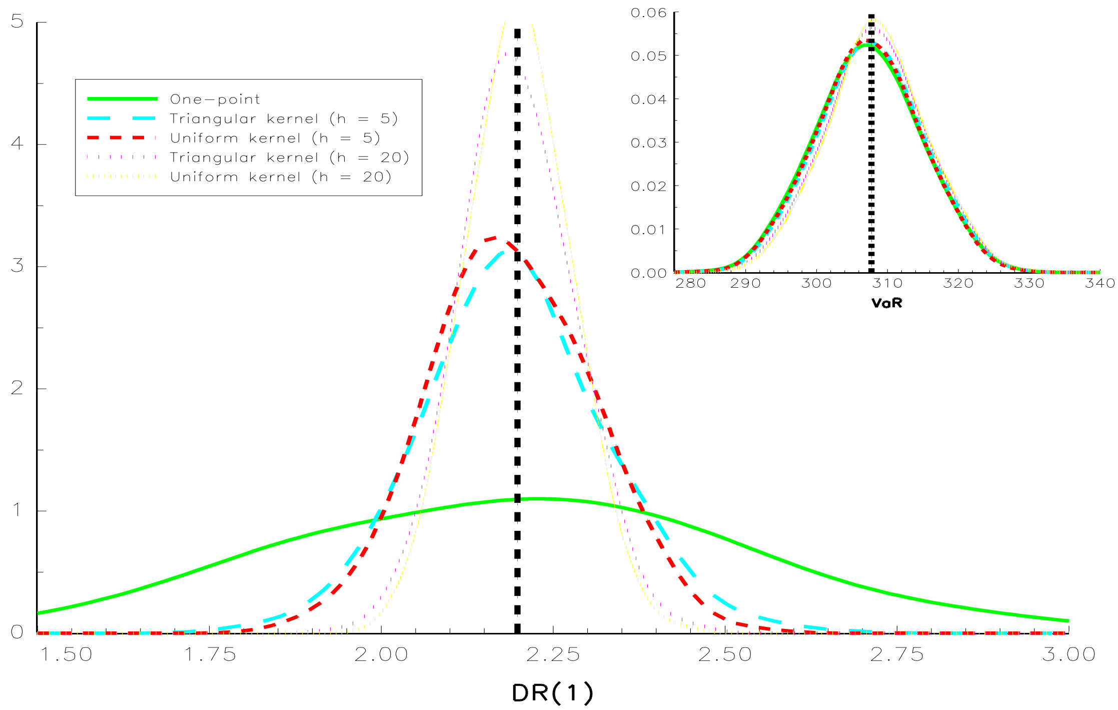


Figure 7: Density of the estimator $\widehat{\mathrm{DR}}$ with the localization method

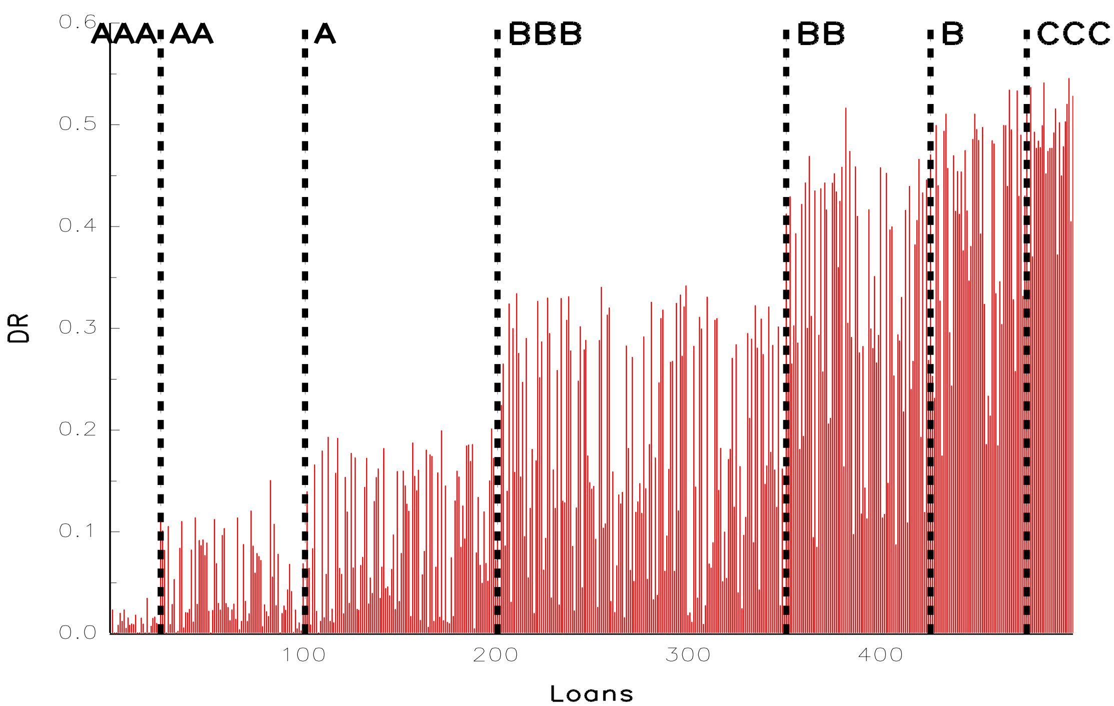


Figure 8: Derivative at risk of the credit portfolio

$x_i \cdot \mathrm{DR}(i)$ is called the *continuous marginal contribution* (Koyluoglu and Stocker [2002]). With the localization method, the CreditVaR appears to be the sum of all the continuous marginal contributions. We may then decompose it by sector, rating, etc. We verify this decomposition property in our example:

| rating / sector | 1 | 2 | 3 | 4 | Total by rating |
|---|---|---|---|---|---|
| AAA | 0 | 81 | 13 | 170 | 264 |
| AA | 40 | 725 | 137 | 2752 | 3654 |
| A | 328 | 1849 | 199 | 7061 | 9437 |
| BBB | 1308 | 6718 | 1430 | 16661 | 26117 |
| BB | 1362 | 6988 | 1592 | 13488 | 23430 |
| B | 2275 | 4211 | 3019 | 10323 | 19827 |
| CCC | 1502 | 4983 | 902 | 4561 | 11948 |
| Total by sector | 6816 | 25554 | 7291 | 55015 | 94676 = CreditVaR |

The previous property is very important for capital allocation and for portfolio management. Let us consider the problem of efficient risk/return frontiers for credit risk. We have to solve the following optimization problem:

$$\begin{array}{ll} \min & \mathrm{CreditVaR}(x_1, \ldots, x_I) \\ \text{u.c.} & \mathrm{ExReturn}(x_1, \ldots, x_I) \geq C \end{array}$$

with $C \in \mathbb{R}_+$ and $\mathbf{x} \in \Omega$ where $\Omega$ is a set of constraints (trading constraints, strategic constraints, etc.). We face a non-linear optimization problem which is very difficult to solve numerically with real-life credit portfolios. That is why some authors have proposed to replace the objective function by a proxy function (Mausser and Rosen [1999], Rockafellar and Uryasev [2002]). For example, if we use the conditional value-at-risk measure in place of the value-at-risk measure, the problem may be solved by using a linear programming algorithm (Rockafellar and Uryasev [2002]). The decomposition (3) yields another method. Let $\mathbf{x}^0$ be a starting allocation vector. We may write locally the problem as

$$\begin{array}{ll} \min & \sum_{i=1}^{I} x_i \cdot \mathrm{DR}\left(i; \mathbf{x}^0\right) \\ \text{u.c.} & \mathrm{ExReturn}(x_1, \ldots, x_I) \geq C \\ & \mathbf{x} \in \Omega \end{array}$$

with $\mathrm{DR}\left(i; \mathbf{x}^0\right) = \partial\, \mathrm{CreditVaR}(\alpha) / \partial x_i^0$. The problem becomes locally linear, and so is easier to be solved. Let us consider an illustration with the previous example. We remind that the original portfolio is defined as $x_i^0 = 1000$ for all loans. We would like to know how to find the efficient risk / return frontier by changing only the 25 first components of the portfolio. We report this frontier in Figure 9. With the localized method, starting from the original portfolio, we much improve the risk / return, and the resulting portfolios are very close to the frontier. To get an even better solution, we just need to run again the procedure starting from these resulting portfolios.

## 4.2 Modelling Basket Credit Derivatives

When dealing with basket credit derivatives, one is quickly compelled to cope with the joint default probability for some counterparts. Until a quite recent time, the *structural approach* could be considered as the predominant methodology. Within this framework, the default occurs at the first hitting time of a barrier by the value process. Thus, it looks quite legitimate to model the joint default of two firms in the same way as KMV and CreditMetrics, that is when both value processes cross their own default barriers. As a consequence of such an approach, a Gaussian (or log-normal) assumption on the value processes induces a Gaussian dependence between the defaults.

The main alternative to this *structural approach* tries to take into account the unpredictability of the default random time. Then, when dealing with a single reference credit, one has to pay attention to the way one is going to model the instantaneous default probability. In that approach, which is commonly called the *intensity based approach* (or *reduced-form*), a default event is said to occur when an intensity process exceeds

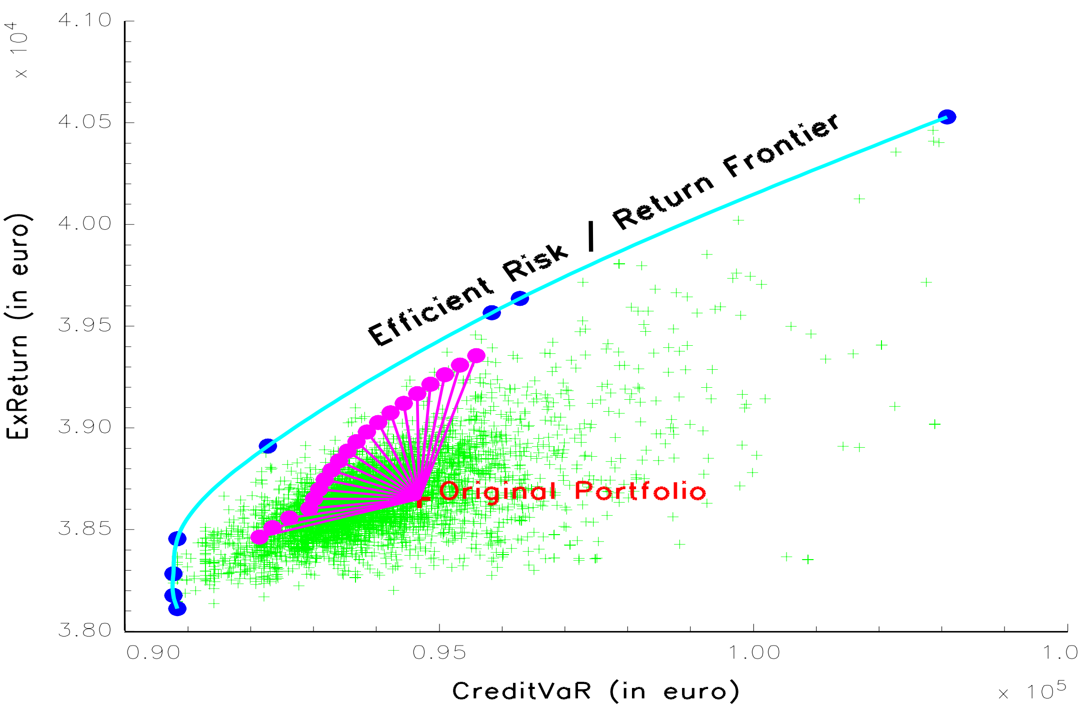


Figure 9: Optimized portfolio with the localized method

an unobserved threshold random variable. For one-firm derivatives, it provides a quite flexible framework and can be easily fitted to actual term structure of credit spreads.

This section tackles the problem of modelling correlated default events within the intensity framework. The problem is trivial when default times are assumed to be (conditionally) independent. But it is well known among market practitioners that such a hypothesis does not allow to fit the observed default correlations. We will recall on a toy example this idea that correlating even perfectly the firms' spreads will not guarantee an implied high dependence between default times. But when one wants to weaken this assumption the problem becomes much more involved. Our method is first to estimate the marginal probability distribution of each individual default and then to use copulas to model the joint distribution, following the recent works of Li [2000], Giesecke [2001] and Schönbucher and Schubert [2001].

Throughout this whole section, we will denote by $(\mathcal{F}_t)$ the filtration (i.e the information) generated by all state variables (economic variables, interest rates, currencies, etc.). Furthermore, we assume the existence of a risk-neutral probability measure $\mathbb{P}$.

### 4.2.1 Conditionally Independent Defaults

We begin to recall the basic ideas on (one then two firms) intensity models. A huge literature is available on this subject. Not to give a too long list of references, we send the reader to Lando [1998] and Bielecki and Rutkowski [2001], where a very thorough account of the intensity framework can be found (and also some elements about the case of dependent defaults). In the case of conditionally independent defaults, once the intensity processes are fixed (those are the latent variables of the model), the residual sources of randomness which may affect the firms' default times are assumed independent. Thus, all possible dependences between defaults will result from the correlations between intensities. But we show that high default correlations cannot be achieved in this way. In particular, it is worthless to compute historical spread correlations to derive an accurate value of default correlations.

When there is only one defaultable firm, the basic elements of the intensity model are an $(\mathcal{F}_t)$-adapted, non-negative and continuous process $(\lambda_t^1)$ (firm 1's *intensity*), and $\theta_1$ (the *threshold*), an exponential random variable of parameter 1 independent from $\mathcal{F}_\infty$. For example, when using a (multi-) factor interest rate model, we can use the factor(s) for also driving the intensity process $\lambda$ in order to provide correlations between interest rates and the default process. Then firm 1's default time is defined by (provided the firm 1 has not defaulted yet)

$$\tau_1 := \inf\left\{t : \int_0^t \lambda_s^1 \,\mathrm{d}s \geq \theta_1\right\} \tag{4}$$

Then the defaultable zero-coupon price of firm 1 is given by

$$B_1(t,T) = \mathbf{1}_{\{\tau_1 > t\}} \mathbb{E}\left[e^{-\int_t^T (r_s + \lambda_s^1)\,\mathrm{d}s} | \mathcal{F}_t\right], \tag{5}$$

which allows to identify the intensity process as firm 1's spread.

This preceding framework is readily generalized to the case of $I$ defaultable firms. For the sake of simplicity, we consider only the case of two firms. Therefore, the default times of two firms 1 and 2 are defined through (4) where we have this time two intensity processes $\lambda^i$, and two random thresholds $\theta_i$ which are still assumed to be independent from $\mathcal{F}_\infty$ and **mutually independent**. This last assumption is usually made in practice for its tractability.

We choose quadratic intensities $\lambda_t^i = \sigma_i (W_t^i)^2$ where $\mathbf{W} = \left(W^1, W^2\right)$ is a vector of two correlated $(\mathcal{F}_t)$ Brownian motions — we shall note $\rho$ for the correlation. The parameters $\sigma_i$ are fixed such that we match the cumulated default probabilities using

$$\mathbb{P}(\tau_i > t) = \frac{1}{\sqrt{\cosh\left(\sigma_i t \sqrt{2}\right)}}$$

In the following numerical application, we take $\sigma_i = 0.04$ which induces cumulated default probabilities quite close to historical data relative to BBB rated firms.

Since an explicit form of the joint distribution of the default times $\tau = (\tau_1, \tau_2)$ can be explicitly derived, one may be able to compute efficiently any correlation measures. The first one is the *discrete default correlation* which corresponds to $\mathrm{cor}\left(\mathbf{1}_{\{\tau_1 < t\}}, \mathbf{1}_{\{\tau_2 < t\}}\right)$ whereas the second one is the correlation between the two random survival times $\mathrm{cor}(\tau_1, \tau_2)$, called the *survival time correlation* by Li [2000]. We remark in Figure 10 that this simple model does not suffice to produce significant correlations between defaults (moreover, the correlations are only positive).

#### 4.2.2 How to Get More Dependence ?

We no longer assume that the random thresholds are independent variables. By coupling the dependence of the thresholds together with the correlation between the intensity processes, we achieve to produce more realistic correlations between defaults. As the triggers are not market data, we shed light on the relationship between the input distribution on the thresholds and the output distribution of the defaults. Many ideas developed here were first explored by Li [2000], Giesecke [2001], and Schönbucher and Schubert [2001].

We still consider two defaultable firms, the default times of which are modelled as in (4). Here, we propose to model the dependence between default process in a two-step algorithm. As is the case with conditionally independent defaults, the first step consists in correlating the intensity processes. Then the second step deals with the choice of the copula $\breve{\mathbf{C}}^\theta$ of the thresholds $(\theta_1, \theta_2)$ which are assumed to be independent from $\mathcal{F}_\infty$.

Within the threshold framework, we can derive a new pricing formula for firm 1's zero-coupon alternative to (5) on $\{\tau_1 > t, \tau_2 > t\}$:

$$B_1(t,T) = \mathbb{E}\left[e^{-\int_t^T r_s\,\mathrm{d}s} \frac{\breve{\mathbf{C}}^\theta\left(e^{-\int_0^T \lambda_s^1\,\mathrm{d}s}, e^{-\int_0^t \lambda_s^2\,\mathrm{d}s}\right)}{\breve{\mathbf{C}}^\theta\left(e^{-\int_0^t \lambda_s^1\,\mathrm{d}s}, e^{-\int_0^t \lambda_s^2\,\mathrm{d}s}\right)} \mid \mathcal{F}_t\right].$$

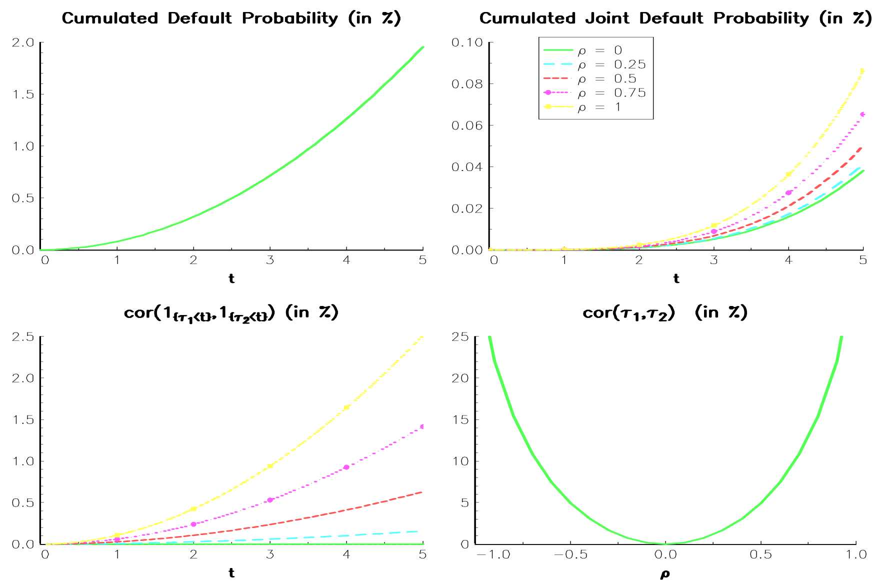


Figure 10: Influence of the correlation parameter on the first default time

What is striking in this formula is the role of firm 2's intensity in the valuation of a claim depending *a priori* on firm 1's default only. In the case of the independent copula $\breve{\mathbf{C}}(u_1, u_2) = u_1 u_2$, we retrieve the usual formula (5) (corresponding to the case of conditionally independent defaults). Of course, a similar valuation formula still holds for more general contingent claims depending on both firms defaults.

As the random triggers are not observable variables, it may be useful to derive a relationship between their copula and the implied dependence of default times. This will enable us to measure the spectrum of dependence between defaults allowed by our method and thus what we gained compared to the assumption of conditionally independent defaults.

Denoting $\breve{\mathbf{C}}^\tau$ for defaults' survival copula, we get from Giesecke [2001]

$$\breve{\mathbf{C}}^\tau \left(S_1^\tau(t_1), S_2^\tau(t_2)\right) = \mathbb{E}\left[\breve{\mathbf{C}}^\theta \left(e^{-\int_0^{t_1} \lambda_s^1 \,\mathrm{d}s}, e^{-\int_0^{t_2} \lambda_s^2 \,\mathrm{d}s}\right)\right].$$

A special case of this formula is worth noticing: if intensities are deterministic, both survival copulas are equal. In particular, this shows that we can achieve high dependence between defaults, as we announced. This also suggests an alternative computational method, which consists in directly imposing the copula $\breve{\mathbf{C}}^\tau$ on default times. This technique is Li's original survival method, which we do not develop here because it requires heavy Monte-Carlo simulations for computing the price of contingent claims.

Having a look at Figure 11, we see that the threshold approach enables to reach a wide range of correlation for the multivariate default times distribution[20]. So, one could be optimistic when calibrating the parameters. That means we can hope that the real correlation could be attainable within the above described framework.

[20] $\rho^W$ is the correlation between the two Wiener processes, whereas $\rho^\theta$ is the parameter of the Normal copula between the thresholds.

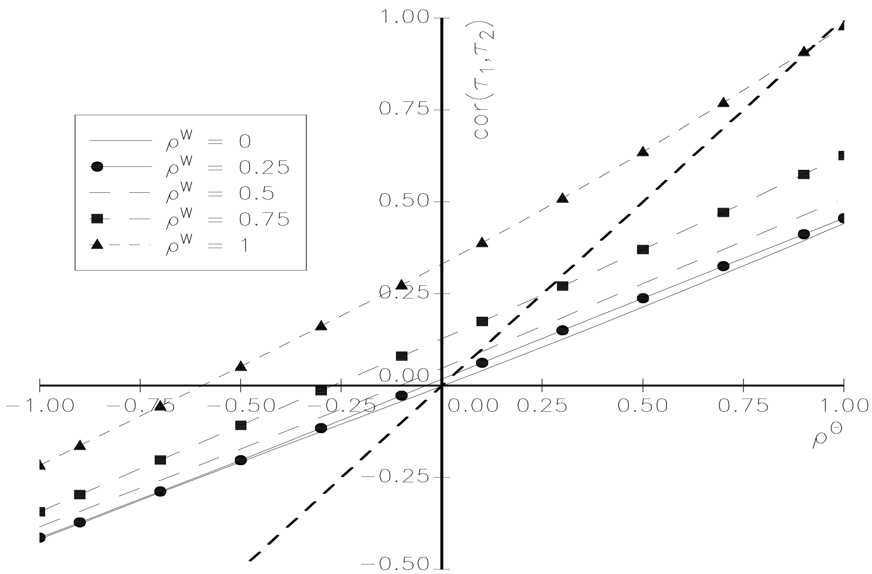


Figure 11: Influence of the dependence function on the correlation

# 5 Operational Risk Management

According to the last proposals by the Basel Committee [3], banks are allowed to use the Advanced Measurement Approaches (AMA) option for the computation of their capital charge covering operational risks. Among these methods, the Loss Distribution Approach (LDA) is the most sophisticated one. It is also expected to be the most risk sensitive as long as internal data are used in the calibration process and then LDA is more closely related to the actual riskiness of each bank.

## 5.1 The Loss Distribution Approach

This method is defined in the supporting document [2] to the New Basel Capital Accord:

> *Under the Loss Distribution Approach, the bank estimates, for each business line/risk type cell, the probability distribution functions of the single event impact and the event frequency for the next (one) year using its internal data, and computes the probability distribution function of the cumulative operational loss.*

Let $i$ and $j$ denote a business line and a loss type. $\zeta(i,j)$ is the random variable which represents the **amount of one loss event** for the business line $i$ and the event type $j$. The **loss severity distribution** of $\zeta(i,j)$ is denoted by $\mathbf{F}_{i,j}$. We assume that the number of events for the next year is random. The corresponding variable $N(i,j)$ has a probability function $p_{i,j}$ and the **loss frequency distribution** $\mathbf{P}_{i,j}$ corresponds to $\mathbf{P}_{i,j}(n) = \sum_{k=0}^{n} p_{i,j}(k)$. The loss for the business line $i$ and the loss type $j$ for the next year is the random sum $\vartheta(i,j) = \sum_{n=0}^{N(i,j)} \zeta_n(i,j)$. Let $\mathbf{G}_{i,j}$ be the distribution of $\vartheta(i,j)$. $\mathbf{G}_{i,j}$ is then a compound distribution

$$\mathbf{G}_{i,j}(x) = \begin{cases} \sum\limits_{n=1}^{\infty} p_{i,j}(n)\,\mathbf{F}_{i,j}^{n\star}(x) & x > 0 \\ p_{i,j}(0) & x = 0 \end{cases}$$

where $\star$ is the *convolution* operator on distribution functions and $\mathbf{F}^{n\star}$ is the $n$-fold convolution of $\mathbf{F}$ with itself.

An analytical expression of $\mathbf{G}$ is known only for a few probability distributions $\mathbf{F}$ and $\mathbf{P}$ (Klugman, Panjer and Willmot [1998]). In other cases, we need a numerical algorithm to estimate $\mathbf{G}$, for example the Monte-Carlo method. In Figure 12, we consider a loss type with $N(i,j)$ and $\zeta(i,j)$ distributed according to the Poisson distribution $\mathcal{P}(50)$ and the log-normal distribution $\mathcal{LN}(8, 2.2)$.

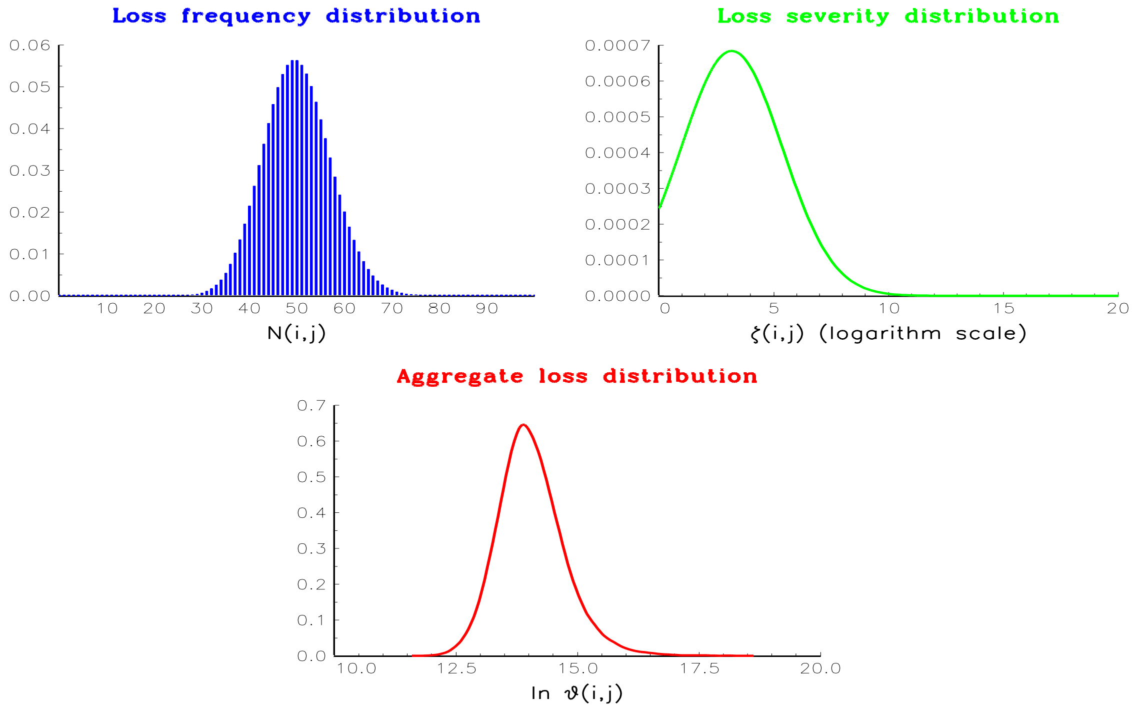


Figure 12: The compound loss distribution

The capital-at-risk denoted CaR for the business line $i$ and the loss type $j$ is then the percentile of the compound loss distribution with a confidence level $\alpha$ fixed at 99.9%.

## 5.2 The Diversification Effect

In the first version of the *New Basel Capital Accord*, "*the capital charge [for the bank as a whole] is based on the simple sum of the operational risk VaR for each business line/risk type cell. Correlation effects across the cells are not considered in this approach*". Let us consider a bank with two operational risks. We denote $\vartheta_1$ and $\vartheta_2$ the corresponding losses with continuous distributions $\mathbf{G}_1$ and $\mathbf{G}_2$. The loss of the bank is $\vartheta = \vartheta_1 + \vartheta_2$. The *Basel II* aggregation method is then $\mathrm{CaR}(\alpha) = \mathrm{CaR}_1(\alpha) + \mathrm{CaR}_2(\alpha)$. We may show that this aggregation rule implies that the dependence function between $\vartheta_1$ and $\vartheta_2$ is $\mathbf{C}^+$. Indeed, we have $\vartheta_2 = \mathbf{G}_2^{-1}(\mathbf{G}_1(\vartheta_1))$. Let us denote $\varpi$ the function $x \mapsto x + \mathbf{G}_2^{(-1)}(\mathbf{G}_1(x))$. it comes that $\alpha = \mathbb{P}\{\vartheta_1 + \vartheta_2 \leq \mathrm{CaR}(\alpha)\} = \mathbb{E}\left[\mathbf{1}_{[\varpi(\vartheta_1) \leq \mathrm{CaR}(\alpha)]}\right] = \mathbf{G}_1\left(\varpi^{-1}(\mathrm{CaR}(\alpha))\right)$ and so $\mathrm{CaR}(\alpha) = \varpi\left(\mathbf{G}_1^{(-1)}(\alpha)\right) = \mathbf{G}_1^{(-1)}(\alpha) + \mathbf{G}_2^{(-1)}\left(\mathbf{G}_1\left(\mathbf{G}_1^{(-1)}(\alpha)\right)\right) = \mathrm{CaR}_1(\alpha) + \mathrm{CaR}_2(\alpha)$.

The simple sum of capital-at-risk corresponds then to the case of the Upper Fréchet bound copula. It implies that we do not have $I \times J$ sources of randomness, but only one. Indeed, if we know the value of $\vartheta(i,j)$ for one pair $(i,j)$, we know the values for all the other pairs. Another problem with such a method is related to how the bank divides its activities into business lines and loss types. With this aggregation method, the capital-at-risk of the bank generally increases with the number of business lines and loss types. Banks will not have an incentive to adopt a sophisticated categorization for business lines and loss types.

The Basel Committee has changed his opinion since that time. In [3], it says:

*The overall capital charge may be based on the simple sum of the operational risk "VaR" for each business line/risk type combination – which implicitly assumes perfect correlation of losses across these cells – or by using other aggregation methods that recognise the risk-reducing impact of less-than-full correlation. [...] The bank will be permitted to recognise empirical correlations in operational risk losses across business lines and event types, provided that it can demonstrate that its systems for measuring correlations are sound and implemented with integrity. In the absence of specific, valid correlation estimates, risk measures for different business lines and/or event types must be added for purposes of calculating the regulatory minimum capital requirement.*

In [5], we propose to put the dependence function on the frequencies. However, we may suppose that both frequencies and severities are correlated. That is why FRACHOT et al. [2001] suggest to put the copula directly between the $\vartheta$'s random variables. The difficulty is then to define (numerically) the total loss of the bank $\vartheta = \sum_{i=1}^{I} \sum_{j=1}^{J} \vartheta(i,j)$ and the corresponding capital-at-risk $\mathrm{CaR}(\alpha) = \inf\{x : \mathbb{P}\{\vartheta \geq x\} \geq \alpha\}$. Indeed, the distribution of $\vartheta(i,j)$ is a compound distribution which has in most case no analytical form, and so may be very difficult to simulate the random variates. However, using a Glivenko-Cantelli argument, it is possible to simulate them by using the empirical quantiles method. Let $\vartheta_1$ and $\vartheta_2$ be two random variables with distribution $\mathbf{G}_1$ and $\mathbf{G}_2$ and a copula function $\mathbf{C}$. The algorithm is the following:

1. Simulate first $m$ random numbers of $\vartheta_1$ and $\vartheta_2$. We denote by $\hat{\mathbf{G}}_{1,m}$ and $\hat{\mathbf{G}}_{2,m}$ the empirical distributions.

2. Simulate a random vector $(u_1, u_2)$ from the copula distribution $\mathbf{C}$.

3. Simulate the random vector $(x_1, x_2)$ by the inverse method and $x_1 = \hat{\mathbf{G}}_{1,m}^{-1}(u_1)$ and $x_2 = \hat{\mathbf{G}}_{2,m}^{-1}(u_2)$. Set $x = x_1 + x_2 = \hat{\mathbf{G}}_{1,m}^{-1}(u_1) + \hat{\mathbf{G}}_{2,m}^{-1}(u_2)$.

4. Repeat the steps 2 and 3 $n$ times.

For example, with $N_1 \sim \mathcal{P}(10)$, $\zeta_1 \sim \mathcal{LN}(1,1)$, $N_2 \sim \mathcal{P}(12)$ and $\zeta_2 \sim \mathcal{LN}(1.25, 0.5)$, and a Normal copula with parameter $\rho$, we obtain Figure 13. In Figure 14, we have reported the capital-at-risk. Note that the (almost) linear relationship between the capital-at-risk and the parameter $\rho$ comes from the example and the Normal copula family.

# References


[1] Basel Committee on Banking Supervision, Amendment to the capital accord to incorporate market risks, Basel Committee Publications No. 24, January 1996

[2] Basel Committee on Banking Supervision, Operational Risk — Consultative Document, Supporting document to the New Basel Capital Accord, January 2001

[3] Basel Committee on Banking Supervision, Working Paper on the Regulatory Treatment of Operational Risk, September 2001

[4] BIELECKI, T. and M. RUTKOWSKI [2001], Credit Risk: Modelling, Valuation and Hedging, Springer

[5] BOUYÉ, E., V. DURRLEMAN, A. NIKEGHBALI, G. RIBOULET and T. RONCALLI [2000], Copulas for finance – a reading guide and some applications, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[6] DAVIES, P.I. and N.J. HIGHAM [2000], Numerically stable generation of correlation matrices and their factors, *BIT*, **40(4)**, 640-651

[7] DAVIS, M. and V. LO [2001], Modelling default correlation in bond portfolios, in Mastering Risk Volume 2: Applications, ed. Carol Alexander, Financial Times Prentice Hall

[8] DEHEUVELS, P. [1978], Caractérisation complète des lois extrêmes multivariées et de la convergence des types extrêmes, *Publications de l'Institut de Statistique de l'Université de Paris*, **23**, 1-36

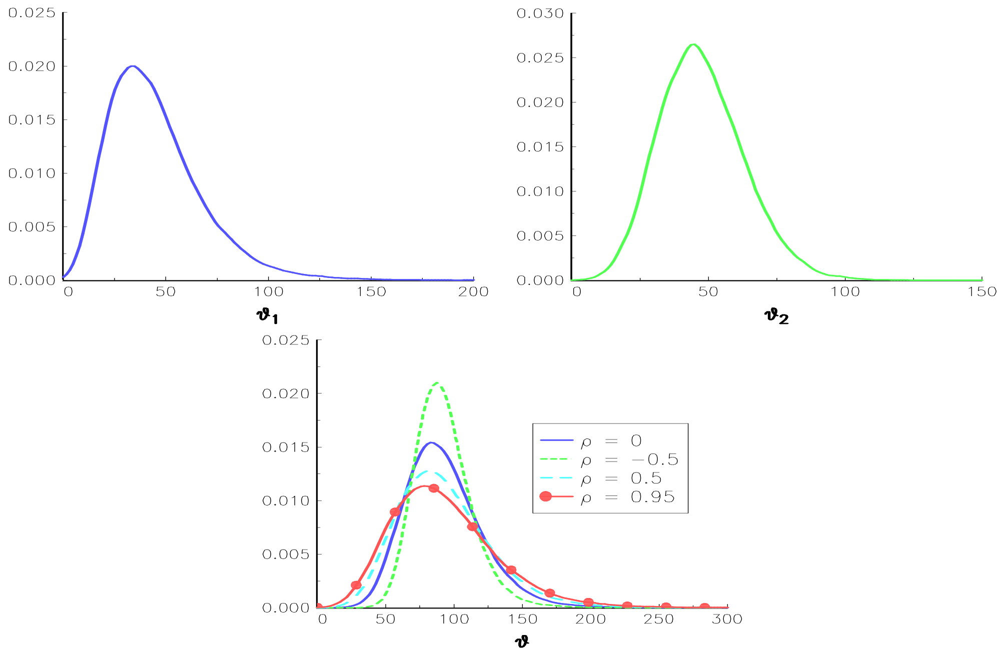


Figure 13: Impact of the copula function on the total loss distribution

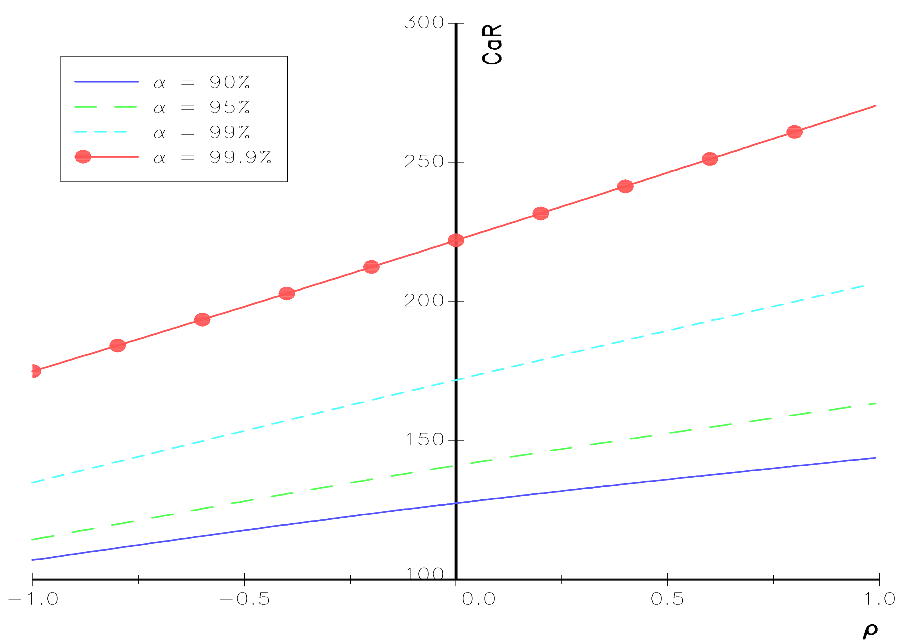


Figure 14: Impact of the copula function on the capital-at-risk

[9] Deheuvels, P. [1979], La fonction de dépendance empirique et ses propriétés — Un test non paramétrique d'indépendance, *Académie Royale de Belgique – Bulletin de la Classe des Sciences – 5e Série*, **65**, 274-292

[10] Derman, E. [2001], Markets and model, *Risk*, **14(7)**, 48-50

[11] Durrleman, V., A. Nikeghbali and T. Roncalli [2000], Which copula is the right one?, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[12] Embrechts, P., Klüppelberg, C. and T. Mikosch [1997], Modelling Extremal Events for Insurance and Finance, Springer, Berlin

[13] Embrechts, P., McNeil, A.J. and D. Straumann [2002], Correlation and dependency in risk management: properties and pitfalls, in Risk Management: Value at Risk and Beyond, ed. M.A.H. Dempster, Cambridge University Press, Cambridge

[14] Finger, C.C. [1999], Conditional approaches for CreditMetrics portfolio distributions, *CreditMetrics Monitor*, **April**, 14-33

[15] Frachot, A., P. Georges and T. Roncalli [2001], Loss distribution approach for operational risk, Crédit Lyonnais, Groupe de Recherche Opérationnelle, *Working Paper*

[16] Genest, C., K. Ghoudi and L-P. Rivest [1995], A semiparametric estimation procedure for dependence parameters in multivariate families of distributions, *Biometrika*, **82(3)**, 543-552

[17] Genest, C. and J. MacKay [1986], The joy of copulas: Bivariate distributions with uniform marginals, *American Statistician*, **40**, 280-283

[18] Giesecke, K. [2001], Structural modeling of correlated defaults with incomplete information, Humboldt-Universität zu Berlin, *Working Paper*

[19] Gordy, M.B. [2001], A risk-factor model foundation for ratings-based bank capital rules, Board of Governors of the Federal Reserve Systems, *Working Paper*

[20] Gouriéroux, C., J-P. Laurent and O. Scaillet [2000], Sensitivity analysis of values at risk, *Journal of Empirical Finance*, **7**, 225-245

[21] Gumbel, E.J. [1958], Statistics of Extremes, Columbia University Press, Columbia

[22] Hutchinson, T. P. and C.D. Lai [1990], Continuous Bivariate Distributions, Emphasising Applications, Rumbsy Scientific Publishing, Adelaide

[23] Joe, H. [1997], Multivariate Models and Dependence Concepts, *Monographs on Statistics and Applied Probability*, **73**, Chapmann & Hall, London

[24] Jouanin, J.-F., G. Rapuch, G. Riboulet and T. Roncalli [2001], Pricing credit derivatives with copulas, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[25] Klugman, S.A., H.H. Panjer and G.E. Willmot [1998], Loss Models: From Data to Decisions, *Wiley Series in Probability and Mathematical Statistics*, John Wiley & Sons, New York

[26] Kotz, S., N. Balakrishnan and N.L. Johnson [2000], Continuous Multivariate Distributions – Volume 1: Models and Applications, John Wiley & Sons, New York

[27] Koyluoglu, H.U. and J. Stocker [2002], Honour your contribution, *Risk*, **15(4)**, 90-94

[28] Lando, D. [1998], On Cox processes and credit risky securities, *Review of Derivatives Research*, **2**, 99-120

[29] Li, D.X. [2000], On default correlation: a copula function approach, *Journal of Fixed Income*, **9(4)**, 43-54

[30] Mausser, H. and D. Rosen [1999], Efficient risk/return frontiers for credit risk, *Algo Research Quarterly*, **2(4)**, 35-47

[31] Nelsen, R.B. [1999], An Introduction to Copulas, *Lectures Notes in Statistics*, **139**, Springer Verlag, New York

[32] Rapuch, G. and T. Roncalli [2001], Some remarks on two-asset options pricing and stochastic dependence of asset prices, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Working Paper*

[33] Rebonato, R. [2001], Model risk: new challenges, new solutions, *Risk*, **14(3)**, 87-90

[34] Riboulet, G. and T. Roncalli [2002], A fast algorithm to simule elliptical copulas with sector correlations, Groupe de Recherche Opérationnelle, Crédit Lyonnais, *Internal Document*

[35] Rockafellar, R.T. and S. Uryasev [2002], Conditional value-at-risk for general loss distributions, *Journal of Banking & Finance*, **26(7)**, 1443-1471

[36] Roncalli, T. [2003], Le Risque de crédit, ENSAI, *Lectures Notes*

[37] Schönbucher, P.J. and D. Schubert [2001], Copula-dependent default risk in intensity models, Department of Statistics, Bonn University, *Working Paper*

[38] Schweizer, B. and E. Wolff [1981], On nonparametric measures of dependence for random variables, *Annals of Statistics*, **9**, 879-885

[39] Sklar, A. [1959], Fonctions de repartition à $n$ dimensions et leurs marges, *Publications de l'Institut de Statistique de l'Université de Paris*, **8**, 229-231

[40] Szegö, G. [2002], Measures of risk, *Journal of Banking and Finance*, **26(7)**, 1253-1272

[41] Tasche, D. [2000], Conditional expectation as quantile derivative, Zentrum Mathematik, TU München, *Working Paper*

[42] Tiit, E-M. and H-L. Helemäe [1997], Boundary distributions with fixed marginals, in V. Beneš and J. Štěpán (eds.), *Distributions with Given Marginals and Moment Problems*, Kluwer Academic Publishers, Dordrecht

[43] Wang, S.S. [1999], Aggregation of correlated risk portfolios: models & algorithms, CAS Committee on Theory of Risk, *preprint*