Static and Dynamic Approach to the Cox-Ingersoll-Ross (CIR) Model and Empirical Evaluation of the Market Price of Risk

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Agenda

- **Explanation of the term structure**
- **Illustration of the Cox-Ingersoll-Ross (CIR) model**
- **Presentation of our research**
- **Conclusion**

To Borrow and To Lend

The future is always indeterminate …

…so in borrowing activity there are two problems:

1) The uncertainty

e.g. the loan could be paid after the expiry, or could not be paid, or could be paid only in part

2) The temporal procrastination

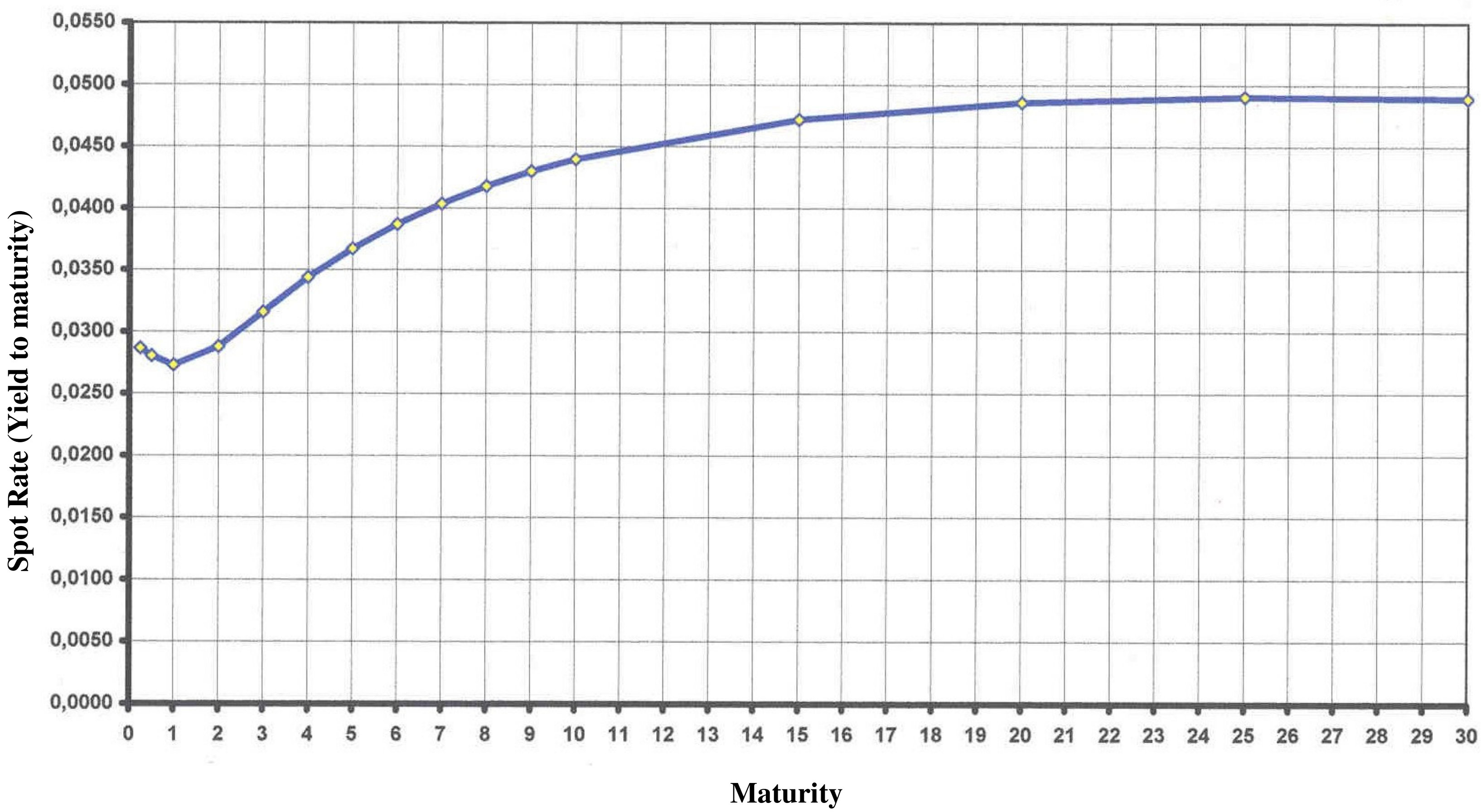
During the life of a bond, the lender can not take advantage of other investment opportunities

The value of the interest rates should reflect these issues

The Term Structure

The curve that shows yield to maturity with respect to maturity is called

Term Structure



The market expectations

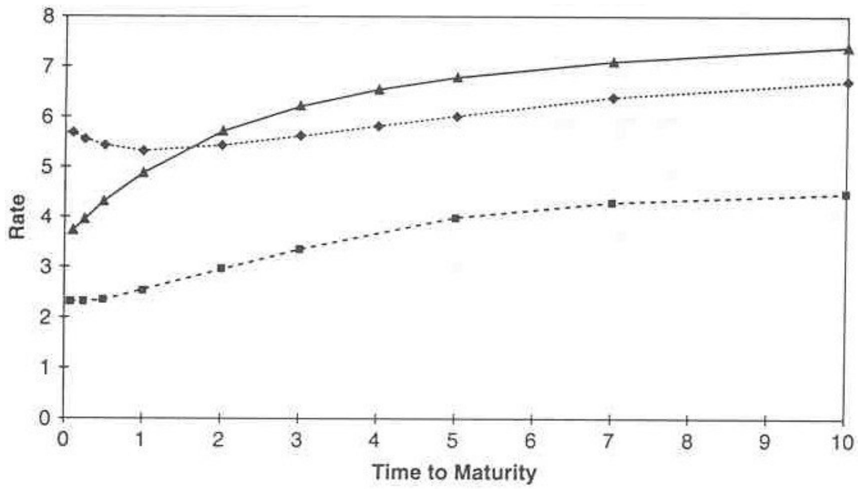
The term structure can display a wide variety of shapes because it also depends on…

the expectations of the market

For example:

if the market strongly believes in a decrement of the rates, no one would find it convenient to sell a long maturity bond at the current price …

… so the long term interest rates could become lower than the short term interest rates.



We will see how the analysis of the market by means of **a non-linear model** (the CIR model) can help to evaluate the risk that investors see in the market

The Cox-Ingersoll-Ross Model

The CIR model describes the dynamics of the short rate by a stochastic differential equation

The short rate is the yield to maturity of a bond with instantaneous maturity

$$dr_t = k(\mu - r_t)\,dt + \sigma\sqrt{r_t}\,dz$$

- **Speed of adjustment** (k)
- **Long term average rate (MEAN REVERTING)** (μ)
- **Implied volatility** ($\sigma\sqrt{r_t}$)
- **Standard Brownian motion** (dz)

$$E(r_t) = \mu + (r_0 - \mu)e^{-kt} \quad \Rightarrow \quad \lim_{t\to\infty} E(r_t) = \mu$$

Properties of the CIR Model

The CIR model assumes that all bond prices depend on the movement of r_t and that all bond prices move in tandem depending on one factor of risk (perfect correlation across maturities)

At first, this seems non-intuitive

how can we assume that there is a single factor of risk?

Litterman and Scheinkman [‘91]: **the term structure tends to make parallel shift**

The parallel movements explain over 80%

of the yield curve movements

Dybvig [‘89]: **one-factor models offer an appropriate first-order approximation**

From an empirical point of view a one factor model (as the CIR model) can be considered acceptable!

The Local Expectation Hypothesis

In some sense… the LEH manages the time procrastination for the CIR model

The expected return

$$E\left[\frac{dB}{B}\right] = r + \left(\lambda r \frac{B_r}{B}\right)$$

The expected risk

The market price of risk is assumed to be (-λ) and not λ because B_r is in general negative.

- B : Price of a generic zero-coupon bond
- B_r : Derivative of the price with respect to the short rate
- $r\left(B_r/B\right)$: Bond's elasticity with respect to the short rate
- $(-\lambda)$: Market price of risk

To estimate -λ means to know the expectations of the market for the future: we can determine the form of the intrinsic term structure of the market

Implementation of the CIR model

We implement the CIR model with two different methods:

a) **Static implementation**

b) **Dynamic implementation**

Our dataset is composed by:

- Euribor Rates for maturities under 1 year (3, 6 months);
- Swap Rates for maturities from 1 to 30 years (from 1 to 10, 15, 20, 25, 30)

The Static Implementation

For each day we apply the non linear least squares method <u>by cross section</u>

We obtain the parameters Φ_1, Φ_2, Φ_3 and the short rate r_t

$$R(t,T) = \frac{-\ln\left(P(t,T)\right)}{T-t} \Rightarrow P(t,T) = F(t,T)e^{-G(t,T)r} \Rightarrow \begin{cases} F(t,T) = \left[\dfrac{\Phi_1 e^{\Phi_2(T-t)}}{\Phi_2\left(e^{\Phi_1(T-t)}-1\right)+\Phi_1}\right]^{\Phi_3} \\ G(t,T) = \left[\dfrac{\left(e^{\Phi_1(T-t)}-1\right)}{\Phi_2\left(e^{\Phi_1(T-t)}-1\right)+\Phi_1}\right] \end{cases}$$

From the parameters Φ_1, Φ_2, Φ_3 we can extract the parameters of the model

$$\begin{cases} \Phi_1 = \sqrt{(k+\lambda)^2 + 2\sigma^2} \\ \Phi_2 = \dfrac{(k+\lambda)+\Phi_1}{2} \\ \Phi_3 = \dfrac{2k\mu}{\sigma^2} \end{cases}$$

It is not possible to separate the speed of adjustment k from the market price of risk -λ.

Static Implementation Description

For each day we apply the non linear least squares method (by cross section)
we obtain the parameters Φ_1, Φ_2, Φ_3 and the short rate r_t

In particular, we obtain 4 time series:

One for the short rate r_t
One for the parameter Φ_1
One for the parameter Φ_2
One for the parameter Φ_3

From these parameters we could obtain the parameters of the model.

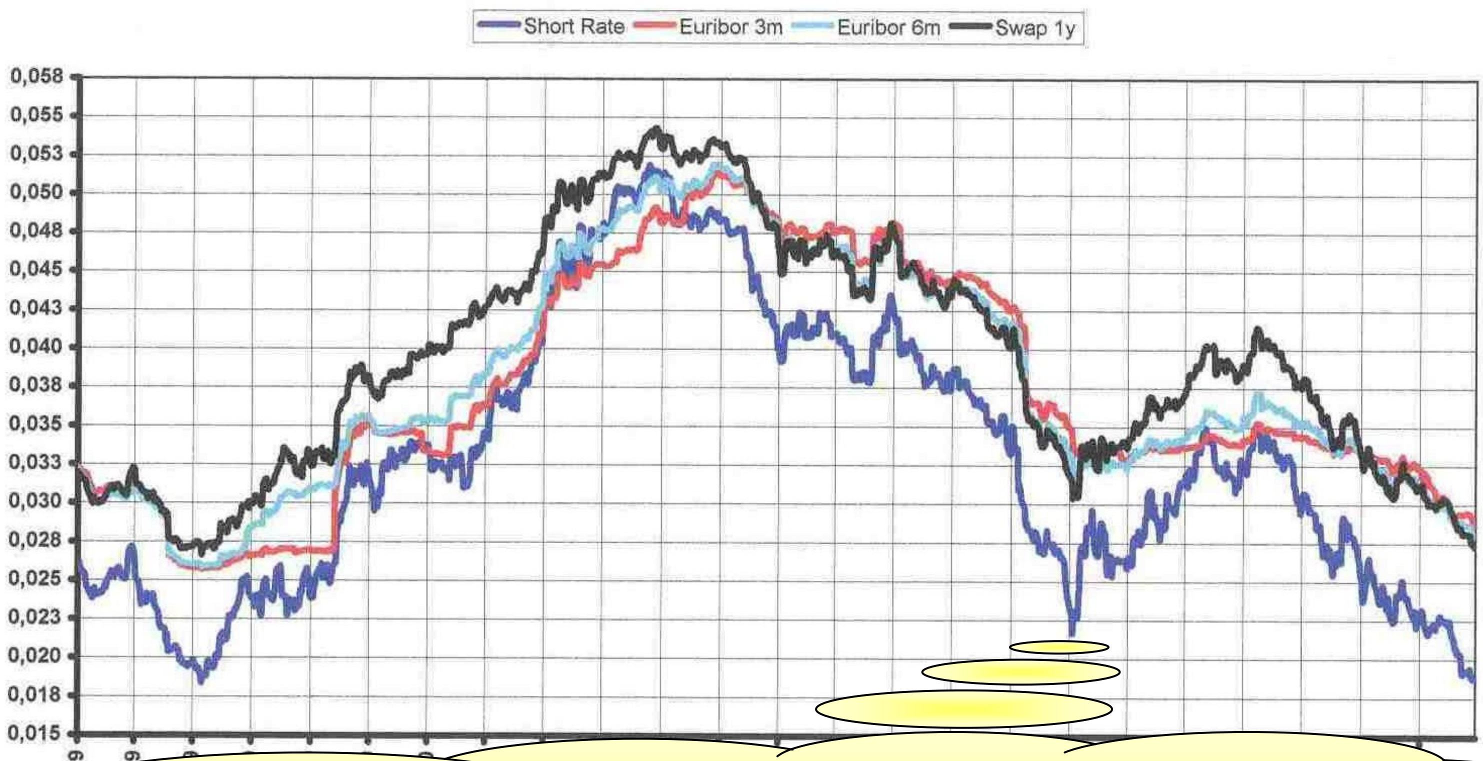
We denote this method as **STATIC** to stress that for each day we use data of that **single day (cross section)**

The static implementation of the CIR model describes the market situation of a single day

Results of Static Implementation

Short rate vs Euribor and 1-year Swap rates

The behaviour of the short rate appears very close but (as we expected) lower than the Euribor rates and the 1-year Swap rate.



The short rate behaviour agrees with the nature of "instantaneous maturity" of the short rate

The Dynamic Implementation

The CIR model defines the stochastic evolution of the short rate so…

The ”natural” method to implement the CIR model should be:

- **Choose a proxy for the short rate and consider its time evolution in the last n-days**
- **Compute the model parameters with likelihood methods (e.g. EMM or GMM)**

We denote this method as **DYNAMIC** to stress that for each day we only use data of **the short rate for the last n days**

The dynamic implementation of the CIR model describes the average market situation of the last n days

The dynamic implementation has three problems:

1) What is the best representative of the short rate?

2) In the case of a shock, the model can not immediately show the change in the market

3) If we only use a benchmark of the short rate, we can not introduce information in the model about all quoted maturities.

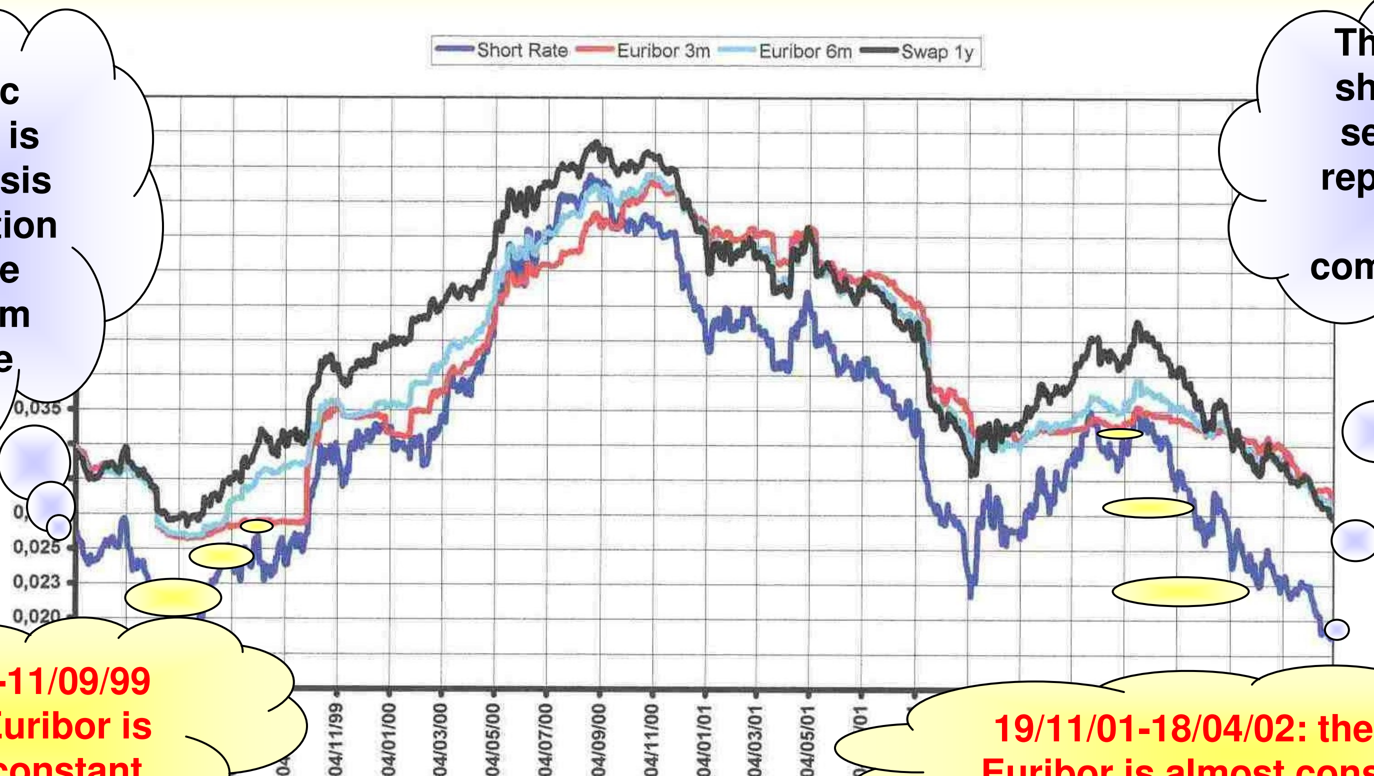
The proxy of the Short Rate

The natural candidates to the rule of proxy for the short rate are:

- The 3 months Euribor rate
- The short rate obtained by the static implementation of the CIR model

We believe that the better choice is the short rate obtained by the static implementation

The 3 months Euribor rate shows anomalous behaviour:



The static short rate is the synthesis of information inside the whole term structure

The static short rate seems to represent a good compromise

03/06/99-11/09/99 the 3m Euribor is almost constant

19/11/01-18/04/02: the 3m Euribor is almost constant

The Dynamic Implementation

With the static short rate as proxy of the "real" short rate, we compute the parameters of the CIR model by means of martingale estimations techniques

$$dr_t = k\left(\mu - r_t\right)dt + \sigma\sqrt{r_t}\,dz$$

$$\Downarrow$$

$$dr_t = \left(a + br_t\right)dt + \sigma\sqrt{r_t}\,dz$$

$$r_{t+1} = r_t + \Delta\left(a + br_t\right) + \sigma\sqrt{r_t}\,dW + \ldots$$

$$\ldots + \frac{\sigma^2}{4}\left(dW^2 - \Delta\right) + b\sigma\sqrt{r_t}\,dZ + \frac{1}{2}\Delta^2 b\left(a + br_t\right) + \frac{\sigma}{2\sqrt{r_t}}\left(a + br_t - \frac{\sigma^2}{4}\right)\left(dW\Delta - dZ\right)$$

$$\begin{cases} dW = U_1\sqrt{\Delta} \\ dZ = \Delta^{3/2}\left(U_1 + \dfrac{U_2}{2}\right) \end{cases}$$

The Martingale Estimation

Martingale Estimations Formulas:

$$b = \frac{1}{\Delta} \ln \left[\frac{n \sum_t \left(\frac{r_t}{r_{t-1}} \right) - \sum_t r_t \sum_t \left(\frac{1}{r_{t-1}} \right)}{n^2 - \sum_t r_{t-1} \sum_t \left(\frac{1}{r_{t-1}} \right)} \right]$$

$$a = \frac{b}{1 - e^{b\Delta}} \frac{n e^{b\Delta} - \sum_t \left(\frac{r_t}{r_{t-1}} \right)}{\sum_t \left(\frac{1}{r_{t-1}} \right)}$$

$$\sigma^2 = \frac{\sum_t \left\{ r_t - \frac{1}{b} \left[\left(a + b r_{t-1} \right) e^{b\Delta} - a \right] \right\}^2 \frac{1}{r_{t-1}}}{\sum_t \left\{ \frac{\left(a + 2 b r_{t-1} \right) e^{2b\Delta} - 2 \left(a + b r_{t-1} \right) e^{b\Delta} + a}{2 r_{t-1} b^2} \right\}}$$

Before applying the estimators, we analysed their convergence by means of simulations of CIR paths, in order to decide the reliability of the techniques and to decide the number of data necessary for the implementation

The problem is very delicate:
it's necessary to use more data as possible to obtain reliable statistics, but it is also necessary to use fewer data as possible to describe only the actual market situation

The best choice is a number of 600 daily data and a discretization step of $\Delta = 1/250$ (250 trading days in a year)

The Dynamic Procedure

The dynamic procedure:
1) We use as dataset the time series of the static short rate 2) We start at 601st day and we estimate the CIR parameters with martingale estimations formulas, using the last 600 data. 3) Then, we continue with the 602nd day and we estimate the parameters using only the last 600 data. 4) And so on, until the last day of our archive.

At the end of this procedure we obtain 3 new time series, respectively for the parameters k, μ, and σ

It is important to remember that each triplet (for each day) describes the average market of the last 600 days and not the whole term structure of a single day (as in the static implementation)

Comparison between static and dynamic procedure

Dynamic implementation vs Static Implementation From 23/04/02 to 31/12/02 - $\Delta = 1/250$						
Parameter	**Method**	**Mean**	**St. Dev.**	**Rel. Var.**	**Min**	**Max**
Speed of Adjustment k	**Dynamic k**	0.7449	0.2812	37.8%	0.1975	1.6179
	Static ($k+\lambda$)	0.4078	0.1248	30.6%	0.1791	0.5937
Long term average rate μ	**Dynamic**	3.5882	1.0179	28.4%	0.9159	6.6734
	Static(hp λ=0)	5.7112	0.3441	6.0%	5.1494	6.4191
Implied Volatility $\sigma\sqrt{r}$	**Dynamic**	0.7798	0.0569	7.3%	0.6462	0.9016
	Static	1.1552	0.2067	17.9%	0.8632	1.5650

The mean value of the parameters obtained from the dynamic implementation and from the static implementation have the same magnitude

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The dynamic implied volatility is lower than the static implied volatility

The Annual Volatility

The dynamic implied volatility is very close to the explicit volatility of the short rate

Dynamic implementation vs Static Implementation From 23/04/02 to 31/12/02 Δ = 1/250				
Parameter	**Method**	**Mean**	**St. Dev.**	**Rel. Var.**
Implied (annual) volatility	**Static**	1.1552	0.2066	17.9%
Implied (annual) volatility	**Dynamic**	0.7798	0.0568	7.3%
Annual Volatility	**Short Rate Proxy**	0.6577	0.5872	89.3%

$$r_{t+1} = r_t + \Delta(a + br_t) + \sigma\sqrt{r_t}\sqrt{\Delta}dU$$

We can compute the implied volatility dividing the daily fluctuations of the proxy by $\sqrt{\Delta}$

We can rewrite the discretization up to the first order in Δ

The dynamic method can explain very well the volatility of the market

The Market Price of Risk

The speed of adjustment obtained by the static implementation

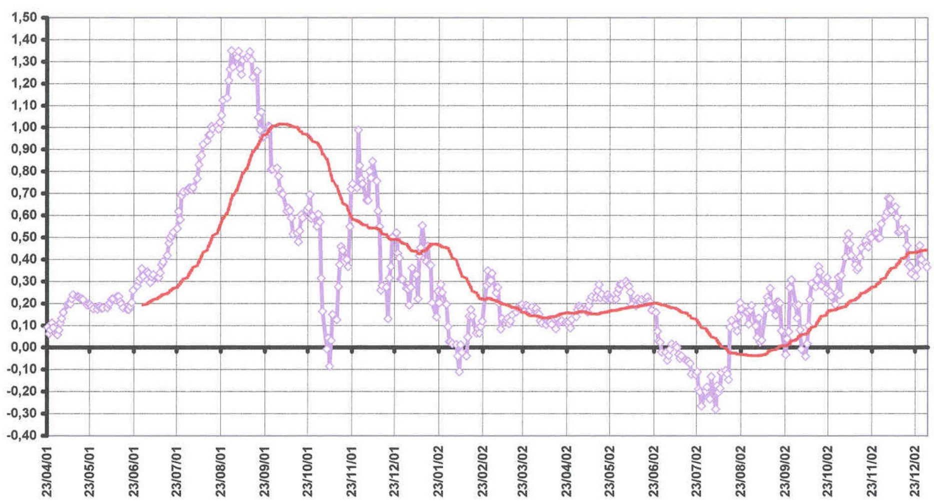
The speed of adjustment obtained by the dynamic implementation

$$K_{static} = K_{dynamic} + \lambda$$

From (-λ) it is possible to achieve the expected return

$$E\left[\frac{dB}{B}\right] = r + \lambda r \frac{B_r}{B}$$

By comparing the static and the dynamic speed of adjustment, it is possible to achieve an empirical estimation of the market price of risk (-λ)

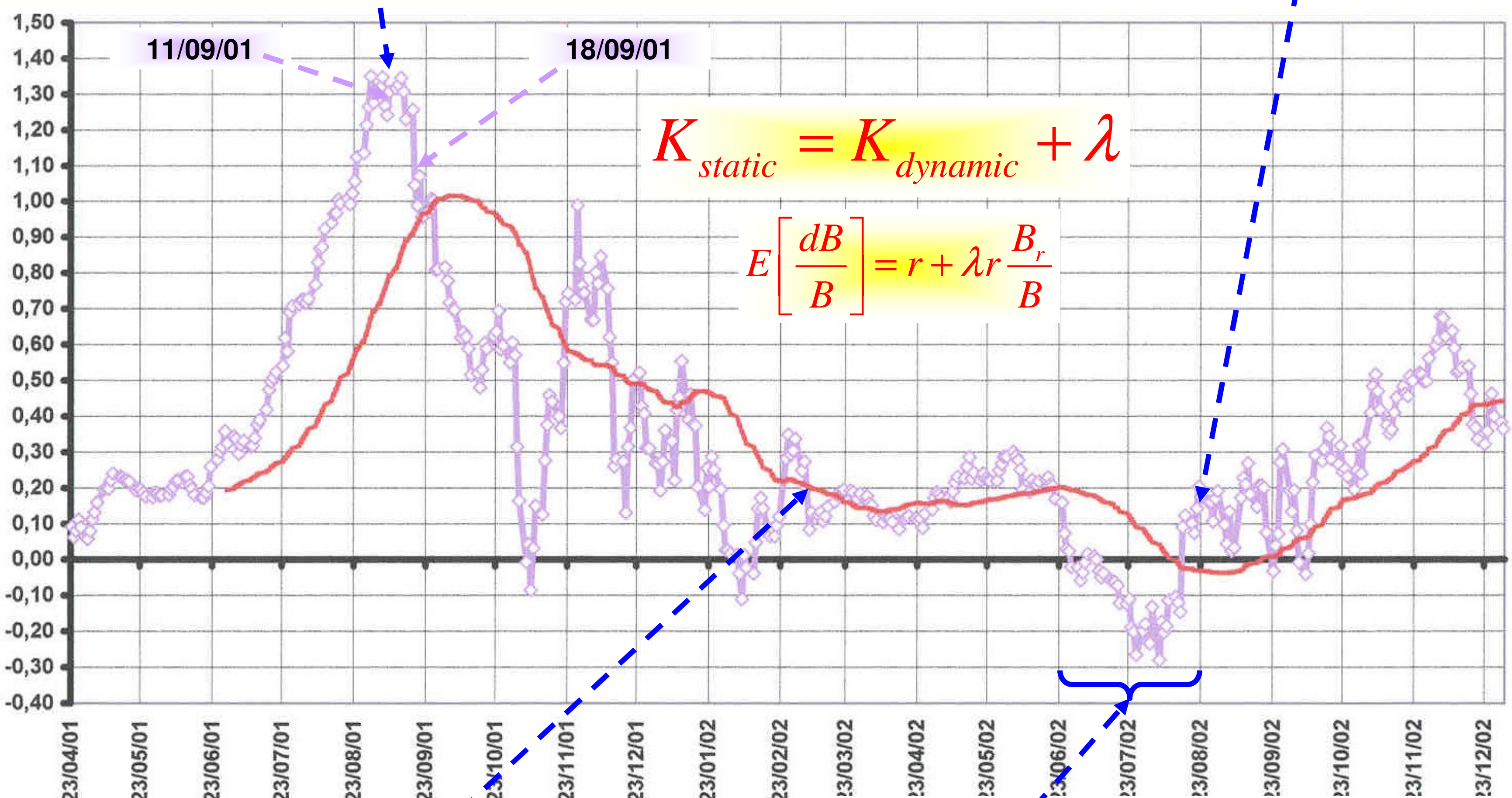


$$\begin{cases} E[-\lambda] = 0.3370 \\ \text{St.Dev}[-\lambda] = 0.3246 \end{cases}$$

The Market Price of Risk

After 11/09/01, (-λ) goes down and shows a nervous behaviour for the next 2 or 3 m (Twin Towers tragedy; expectation of a great reduction of the Reference Rate)

After the middle of 08/02 (-λ) comes back positive but nervous: signal of indecision in the market



After January 2002, (-λ) is small and almost stable, according to the idea of a continuing crisis

Between 06/02 and 08/02, (-λ) remains under 0: there is the expectation of a new possible reduction of the rates (actually happened).

The Intrinsic Term Structure

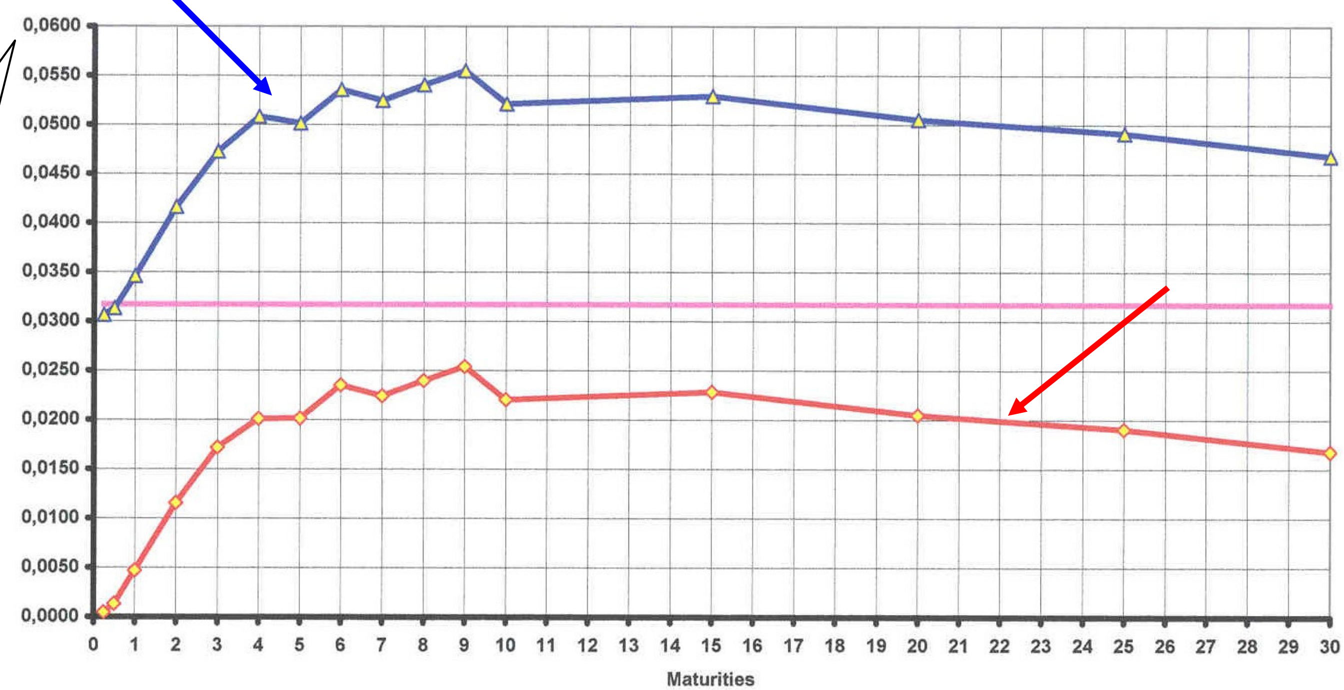
This is the expected return

$$E\left[\frac{dB}{B}\right] = r + \left(\lambda r \frac{B_r}{B}\right)$$

This is the expected risk

THE INTRINSIC TERM STRUCTURE

By classical discretization of the derivatives B_r it is possible to calculate the expected risk and return. The intrinsic term structure is the average on the time of the expected return.



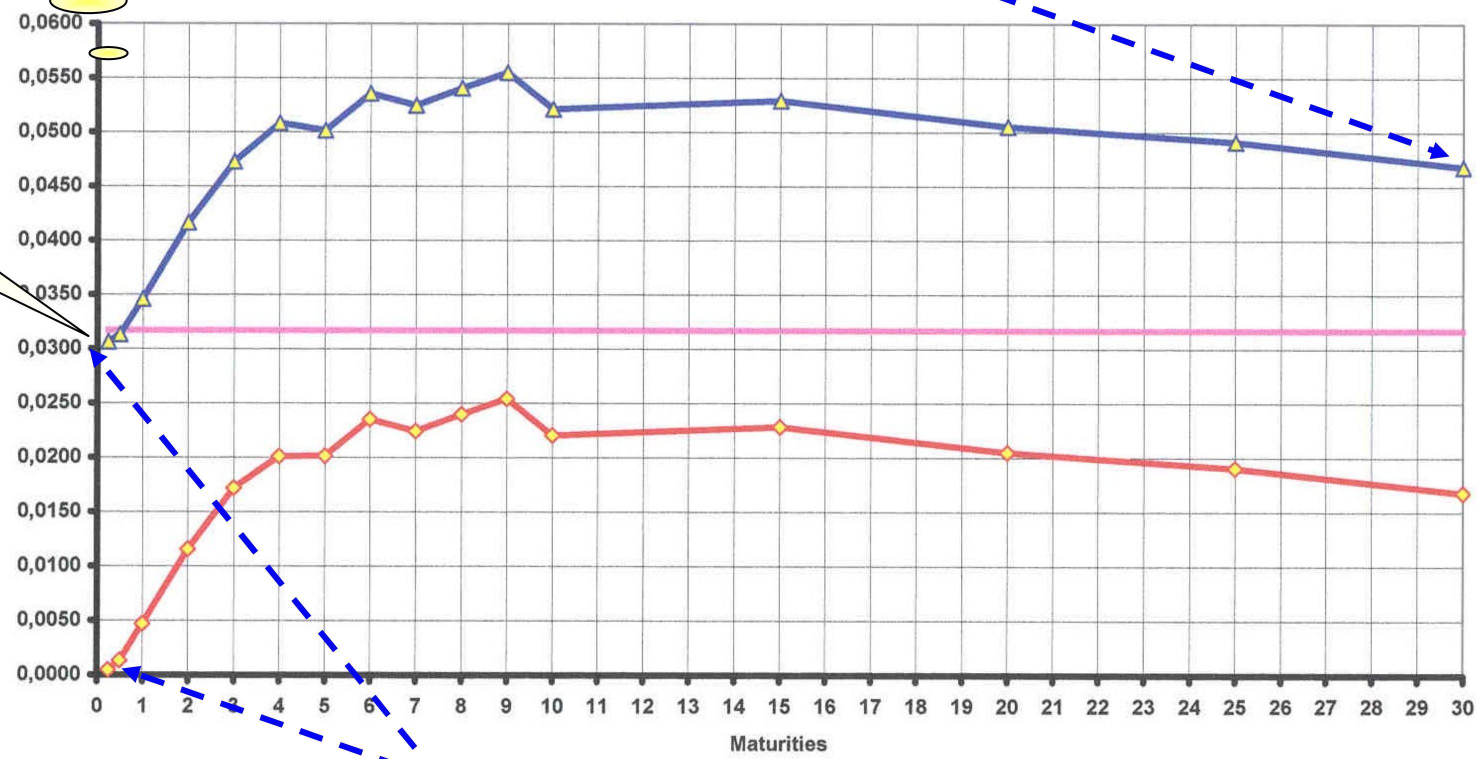
Obviously the intrinsic term structure won't be exactly equal to the future term structures: the market is not forecastable and the expectations of the market will change with the time

Observations on the Intrinsic Term Structure

The expected returns provide a reasonable term structure

This result is an indirect confirmation of the coherence of the study

For very long maturities the market seems to have a more stable idea of the cost of risk. In some sense, the long maturities are so far that they become less sensible to the fluctuations of the market.



For very short maturities the expected risk has to be small, so the expected returns can not be seriously influenced by the fluctuations of the market

Observations About the Intrinsic Term Structure

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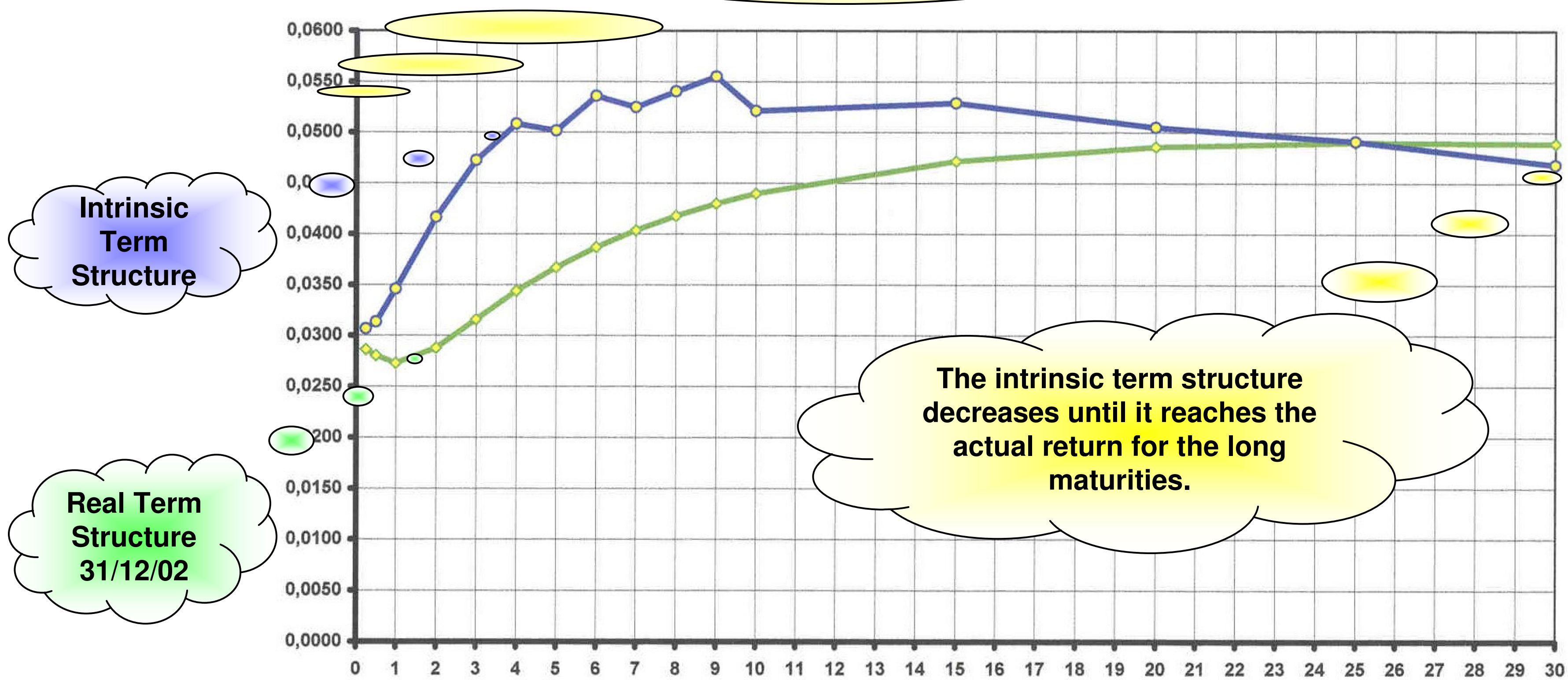

This is another confirmation of the stability of the long maturities with respect to the fluctuations of the market.

Conclusions

By means of a non-linear model…

…it is possible to try to "explain" the expectations of the market by an empirical evaluation of the market price of risk

The next steps:

To analyse the reliability of the model to forecast the term structure at short and medium time horizon

To evaluate the stability of the intrinsic term structure focusing on the anomalous fluctuations of the expected returns

To quantify the probability associated to particular situations of the market described by particular shapes of the term structure