Energy Futures Prices: Term Structure Models with Kalman Filter Estimation

Abstract

We present a class of multi-factor stochastic models for energy futures prices, similar to the interest rate futures models recently formulated by Heath (1998). We do not postulate directly the risk-neutral processes followed by futures prices, but define energy futures prices in terms of a spot price, not directly observable, driven by several stochastic factors. Our formulation leads to an expression for futures prices which is well suited to the application of Kalman filtering techniques together with maximum likelihood estimation methods. Based on these techniques we perform an empirical study of a one and a two-factor model for futures prices for natural gas.

I Introduction

Prices of energy commodities, like electricity and natural gas, have traditionally been regulated, with the financial risks associated with the costs of running a utility company collectively borne by the users. As the United States and Europe are moving toward a deregulated environment and as new financial instruments tailored to individual demand profiles are being developed, it is important to introduce models that account for the risks that the sellers and buyers of such energy instruments would face.

In this paper we offer a general multi-factor model designed to account for the observed stochastic behavior of energy futures prices, in the spirit of the interest rate model proposed by Heath (1998) for bond futures. Like other models of the term structure of commodity futures prices, such as Cortazar and Schwartz (1994), and Miltersen and Schwartz (1998), our model fits within the general Heath, Jarrow and Morton (1992) no-arbitrage framework. Additionally, similar to the work of Schwartz and Smith (1997), we offer a connection between the model for the futures prices and a model for an underlying spot price. The spot price is assumed to be a given function of underlying state variables following, exogenously specified, generalized Ornstein-Uhlenbeck stochastic processes under the risk-neutral measure.

The expression for futures prices, as functions of the state variables, is then derived from the definition of the futures price as the risk-neutral conditional expectation of the underlying spot at the maturity of the futures contract.

We view our contribution to the literature as twofold: on the theoretical side we extend the framework presented in Miltersen and Schwartz (1998) to account for seasonal patterns in the futures curve and also offer a Kalman filter formulation for the general model, including seasonality and an arbitrary number of random factors. On the empirical side we offer a study of the natural gas market, with estimates for the parameters for a seasonal stochastic processes with one and two factors.

The choice of generalized Ornstein-Uhlenbeck processes for the state variables underlying the spot and futures model has the desirable properties of incorporating mean-reversion, an empirical feature of commodity prices, and of being able to account for the observed term structure of volatilities and correlations of futures prices. At the same time, the general model we discuss has lognormally distributed futures prices under the risk-neutral measure, which leads to closed-form Black-Scholes type formulas for European options written on the futures. The analytical tractability makes this class of models very appealing in view of the potential applications, such as futures price curve building and option pricing. Another advantage of our approach is that it leads to a state space formulation of the futures model which is well suited for the use of a powerful model estimation technique which combines Kalman filtering and maximum likelihood methods. We implement this empirical estimation method for the case in which the model has time-homogeneous instantaneous volatilities for futures prices and the market prices of risk relating risk-neutral and real-world probability measures are assumed constant. Using historical information on futures prices on natural gas over the period September 1997 to August 1998 we are able to accurately estimate the values of the model parameters for two special cases of the general model: a one-factor and a two-factor model with seasonal adjustments for futures prices in different months of the year.

One important feature that underlies our choice of general model is the fact that we explicitly allow the possibility of a futures curve that is non-monotonic with respect to the term of the futures contract. In particular, we can account for the seasonal increases and decreases in futures prices observed in the summer vs. the winter months in the futures curves of electricity and natural gas, through the presence of a deteministic seasonality factor in the expression of futures prices. Although certain non-monotonic patterns for futures prices of financial assets lead to arbitrage opportunities, such patterns are often observed in futures prices of energy as well as other commodities. It is important to realize that the presence of such patterns in the futures curve of energy assets does not offer arbitrage opportunities, since the assets delivered are not fungible. A simple example is to consider on-peak electricity prices, i.e. electricity delivered 6 am to 10 pm, Monday through Friday, as compared to off-peak electricity prices. Since on-peak prices are significantly higher than off-peak prices, to take advantage of the price differential one would need to purchase electricity during off-peak hours and deliver it during on-peak hours. However, due to inefficiency of storage of electricity, the "arbitrage" opportunity offered by this strategy is eliminated. Consistent with no-arbitrage pricing, all tradable assets in our model, that is, futures contracts and contracts with payoffs based on futures prices, are martingales under the risk-neutral measure. Since the spot price in our model does not correspond to the price of a tradable asset it is neither observable, nor a martingale under the risk-neutral measure.

The rest of the paper is organized as follows. Section **II** describes the general multi-factor model. We begin by defining the general framework of the model and by summarizing the relations between absence of arbitrage, existence of a martingale measure and the implied restrictions on the form of the real-world and the risk-neutral world futures price processes. We then introduce the state variables which follow generalized Ornstein-Uhlenbeck type stochastic processes and in terms of which we define the futures prices. Section **III** presents, under some additional, simplifying assumptions, the state space formulation of the problem, amenable to the use of the Kalman filter and maximum likelihood parameter estimation methods. In Section **IV** an empirical study is presented for two particular futures price models: a one-factor model with seasonality and mean-reversion in the underlying spot price stochastic movements and a two-factor model with seasonality and with the underlying spot price having mean-reverting short-term stochastic movements and long-term movements following geometric Brownian motion. The empirical results are based on a one-year historical time series of futures prices for natural gas contracts. Section **V** summarizes and concludes the paper.

II Formulation of the model

II.A The general framework

We consider a trading interval $[0, T^*]$, where T^* is a fixed time horizon. Uncertainty in the economy is modeled by the filtered probability space $(\Omega, \mathbb{F}, \mathbb{P})$ with $\mathbb{P}$ the real world probability measure. Events in the economy are revealed over time according to the filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T^*]}$ generated by n $(n \geq 1)$ independent standard Brownian motion factors $W_t^1, W_t^2, \ldots W_t^n$.

The market model that we are considering contains the *energy commodity futures prices* of different maturities T $(0 < T < T^*)$ as the prices of primary traded securities. For each $T \in [0, T^*]$ we let $F(t, T)$, for $t \leq T$, denote the T-maturity futures price process. In addition to these futures prices, the market model contains the price of an additional primary security, a *money market account*. Its price process B_t is defined by: $dB_t = r_t B_t dt$, $B_0 = 1$, in terms of the short-term interest rate process r_t which is assumed to be deterministic.

We assume that the market security prices are functions of m *state vari-*

ables $\xi_t^1, \xi_t^2, \ldots, \xi_t^m$ which follow Itô processes defined by:

$$d\xi_t^i \;=\; \mu_t^i\, dt + \sum_{j=1}^{n} \sigma_t^{ij}\, dW_t^j, \qquad i = 1, \ldots, m \tag{1}$$

The coefficients μ_t^i and σ_t^{ij} are adapted stochastic processes on $(\Omega, \mathbb{F}, \mathbb{P})$ which are sufficiently well behaved for (1) to define an Itô process (see Oksendal (1995), ch.IV). We further assume that μ_t^i and σ_t^{ij} depend only on $\boldsymbol{\xi}_t = (\xi_t^1, \xi_t^2, \ldots, \xi_t^m)$ and t, so that $\boldsymbol{\xi}_t$ follows an m-dimensional diffusion process defined by the system of stochastic differential equations (1).

As a function of the state variables, a generic futures price $F(t,T) = f(\boldsymbol{\xi}_t, t, T)$ follows a diffusion process (the function f is assumed sufficiently well behaved for this to hold):

$$d_t F(t,T) = F(t,T)[\alpha(t,T)\, dt + \sum_{j=1}^{n} \sigma_j(t,T)\, dW_t^j] \tag{2}$$

where

$$\alpha(t,T) = \frac{1}{f}[\frac{\partial f}{\partial t} + \sum_{i=1}^{m} \mu_t^i \frac{\partial f}{\partial \xi_t^i} + \frac{1}{2} \sum_{i,l=1}^{m} \sum_{j=1}^{n} \sigma_t^{ij} \sigma_t^{lj} \frac{\partial^2 f}{\partial \xi_t^i \partial \xi_t^l}]$$

$$\sigma_j(t,T) = \frac{1}{f}[\sum_{i=1}^{m} \sigma_t^{ij} \frac{\partial f}{\partial \xi_t^i}]$$

We comment now on the arbitrage-free property of the futures market model and the existence of an equivalent martingale measure. The material in the rest of this subsection is based in part on results presented in Duffie ((1996), ch.6,8), Heath, Jarrow and Morton (1992), Musiela and Rutkowski ((1997), ch.10,13).

We will assume that in our market model any investment portfolio involves only a finite number of futures contracts of different maturities as well as cash investment in the money market account. Let $0 < T_1 < T_2 < \cdots < T_N \leq T^*$ be an arbitrary collection of maturities. We extend the processes $F(t, T_k)$ over the whole interval $[0, T^*]$ by setting $F(t, T_k) = 0$ for $t \in (T_k, T^*]$. A *futures trading strategy* associated to the set of maturities $T_1, \dots, T_N$ consists of an $(N+1)$-dimensional $\mathbb{F}$-adapted stochastic process $(\phi_t^1, \phi_t^2, \dots, \phi_t^N, \psi_t)$. The coordinates ϕ_t^k, $k = 1, \dots, N$, represent the number of long or short positions in the T_k-maturity futures contract at time t and satisfy $\phi_t^k = 0$ for $t \in (T_k, T^*]$. The amount of cash in the money market account at time t is ψ_t. Since futures prices are specified so that the value of futures contracts remains zero, for every $t \in [0, T^*]$ the value V_t of the portfolio $(\phi_t^1, \phi_t^2, \dots, \phi_t^N, \psi_t)$ is given by:

$$V_t = \psi_t B_t \tag{3}$$

The futures trading strategy is called *self-financing* if the stochastic process

V_t satisfies:

$$dV_t = \sum_{k=1}^{N} \phi_t^k \, dF(t, T_k) + \psi_t \, dB_t \tag{4}$$

The market model is *arbitrage-free* if there are no self-financing trading strategies which give rise to arbitrage opportunities. An *arbitrage opportunity* would be represented by a self-financing trading strategy $(\phi_t^1, \phi_t^2, \ldots, \phi_t^N, \psi_t)$ over the time interval $[0, T]$, associated to a set of maturities $0 < T_1 < T_2 < \cdots < T_N \leq T$, whose portfolio value process V_t satisfies $V_0 = 0$ and $\mathbb{P}(V_T \geq 0) = 1$, $\mathbb{P}(V_T > 0) > 0$ (see (Musiela and Rutkowski (1997), §10.1) or (Duffie (1996), ch.6) for the additional technical conditions on self-financing trading strategies involved in the definition of an arbitrage-free market for a continuous-time model).

Then, aside from technical conditions, the following statements **(A)**-**(D)** are equivalent:

(A) The market model is arbitrage-free.

(B) There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that all futures price processes are martingales under $\mathbb{Q}$.

(C) There exists an adapted n-dimensional stochastic process $\boldsymbol{\gamma}_t = (\gamma_t^1, \gamma_t^2, \ldots, \gamma_t^n)$ on $(\Omega, \mathbb{F}, \mathbb{P})$, the *market price of risk*, such that

$\mathbb{E}_{\mathbb{P}}\big[\exp\big(\frac{1}{2}\sum_{j=1}^{n}\int_0^{T^*}|\gamma_t^j|^2\,dt\big)\big] < \infty$, and such that, for any maturity $T \le T^*$, the following relation between the drift and the volatilities of the futures price process (2) is satisfied:

$$\alpha(t,T) \;=\; \sum_{j=1}^{n} \sigma_j(t,T)\gamma_t^j \tag{5}$$

(D) There exists a probability measure $\mathbb{Q}$ equivalent to $\mathbb{P}$ such that for any self-financing futures trading strategy its discounted portfolio value $\widetilde{V}_t = B_t^{-1}V_t$ is a martingale under $\mathbb{Q}$.

For any $T \le T^*$, given $\boldsymbol{\gamma}_t$ as in **(C)**, the *martingale (risk-neutral) measure* $\mathbb{Q}$ equivalent to $\mathbb{P}$ on $(\Omega, \mathcal{F}_T)$ satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \;=\; \exp\Big(-\sum_{j=1}^{n}\int_0^T \gamma_t^j dW_t^j - \frac{1}{2}\sum_{j=1}^{n}\int_0^T |\gamma_t^j|^2 dt\Big) \qquad \mathbb{P}\text{-a.s.} \tag{6}$$

and the processes

$$\widetilde{W}_t^j = W_t^j + \int_0^t \gamma_s^j ds, \quad t \in [0,T], \qquad j = 1,\ldots,n \tag{7}$$

are standard Brownian motions under $\mathbb{Q}$. Thus, under the martingale measure $\mathbb{Q}$ defined by (6), the futures price process $F(t,T)$ is defined by

$$d_tF(t,T) \;=\; F(t,T)\sum_{j=1}^{n}\sigma_j(t,T)d\widetilde{W}_t^j \tag{8}$$

To ensure the martingale property of the futures price process $F(t,T)$ under the measure $\mathbb{Q}$ the volatilities $\sigma_j(t,T)$ have to satisfy the condition $\mathbb{E}_{\mathbb{Q}}\big[\exp\big(\frac{1}{2}\sum_{j=1}^{n}\int_0^T \sigma_j(t,T)^2\,dt\big)\big] < \infty$.

As in the Heath, Jarrow and Morton (1992) interest rate model, once a set of volatility surfaces $\sigma_j(t,T)$ has been specified, absence of arbitrage in the futures market restricts the drifts $\alpha(t,T)$ of the real world futures price processes (2) to those of the form (5), where the only degrees of freedom are in the specification of the n-dimensional process $\boldsymbol{\gamma}_t$. However, given a general adapted process $\boldsymbol{\gamma}_t$, futures prices can exhibit a rich set of possible drifts under the real world measure $\mathbb{P}$.

In the following sections, assuming that the above statements **(A)**-**(D)** are true, we are going to develop an explicit model for energy futures prices. We will start by postulating the stochastic processes followed by the state variables ξ_t^i under the risk-neutral measure $\mathbb{Q}$, which give deterministic volatilities $\sigma_j(t,T)$ for the futures price processes (8). This leads to lognormally distributed futures prices under the risk-neutral measure $\mathbb{Q}$. No further assumptions, besides the regularity conditions necessary for **(A)**-**(D)** to be satisfied, will be imposed on the market price of risk process $\boldsymbol{\gamma}_t$ at this stage. In Section **III**, however, we shall make the additional, restrictive assumption that the process $\boldsymbol{\gamma}_t$ defining the change of measure from the physical measure $\mathbb{P}$ to the equivalent martingale measure $\mathbb{Q}$ is constant.

II.B State variables and spot price processes

In the framework introduced in the previous subsection we now postulate the arbitrage-free property of our market model and thus the existence of a probability measure $\mathbb{Q}$ equivalent to the real world measure $\mathbb{P}$ under which all energy futures prices of different maturities follow martingale processes.

Let S_t denote the *spot price* of the energy commodity at time t. This is *not* assumed to be the observed quoted spot price, in case the corresponding commodity has one, but rather an unobserved variable which serves the formal role of the underlying asset for derivatives like futures or forward contracts.

We assume that S_t can be decomposed as the product of several components, one of which might be a seasonality factor. More precisely, introducing

$$X_t = \ln S_t$$

we assume that X_t can be expressed as a sum of the m state variables

$\xi_t^1, \xi_t^2, \ldots, \xi_t^m$, that is:

$$X_t = \sum_{i=1}^{m} \xi_t^i \tag{9}$$

The state variables ξ_t^i are assumed to each follow a stochastic process defined under the martingale measure $\mathbb{Q}$ by the stochastic differential equation:

$$d\xi_t^i = (\widetilde{\alpha}_t^i - k_t^i \xi_t^i)\, dt + \sum_{j=1}^{n} \sigma_t^{ij}\, d\widetilde{W}_t^j \tag{10}$$

with k_t^i, σ_t^{ij} and $\widetilde{\alpha}_t^i$ deterministic functions of time. For $\sigma_t^{ij} = \sigma^{ij}$ and $k_t^i = k^i$ non-zero constants the above stochastic differential equation for ξ_t^i defines an Ornstein-Uhlenbeck mean-reverting process, with mean-reverting rate k^i and mean-reverting level $\frac{\widetilde{\alpha}_t^i}{k^i}$, while for $k^i = 0$ it reduces to a Brownian motion process with drift $\widetilde{\alpha}_t^i$. If the state variable ξ_t^i represents the seasonality component we might consider it as a deterministic function of time by taking $k^i = \sigma^{i1} = \cdots = \sigma^{in} = 0$ and specifying an appropriate time functional dependence for $\widetilde{\alpha}_t^i$.

The stochastic differential equation of the state variable ξ_t^i under the real world measure $\mathbb{P}$ is, using (7) and (10),

$$d\xi_t^i = (\alpha_t^i - k_t^i \xi_t^i) dt + \sum_{j=1}^{n} \sigma_t^{ij}\, dW_t^j\,, \tag{11}$$

where

$$\alpha_t^i = \widetilde{\alpha}_t^i + \sum_{j=1}^{n} \sigma_t^{ij} \gamma_t^j . \tag{12}$$

Let $0 \le t \le T \le T^*$. Solving the stochastic differential equation (10) we have:

$$\xi_T^i = \beta_T^i \big[(\beta_t^i)^{-1} \xi_t^i + \int_t^T (\beta_s^i)^{-1} \widetilde{\alpha}_s^i ds + \sum_{j=1}^{n} \int_t^T (\beta_s^i)^{-1} \sigma_s^{ij} \, d\widetilde{W}_s^j \big] \tag{13}$$

where β_t^i is the solution of the ordinary differential equation:

$$\frac{d\beta_t^i}{dt} = -k_t^i \, \beta_t^i, \qquad \beta_0^i = 1$$

Equation (13) shows that the distribution of the state variables $\xi_T^1, \ldots, \xi_T^m$ conditional on $\mathcal{F}_t$ is multivariate normal with the ith component mean equal to

$$\mathbb{E}_{\mathbb{Q}}[\xi_T^i \mid \mathcal{F}_t] = \beta_T^i \big[(\beta_t^i)^{-1} \xi_t^i + \int_t^T (\beta_s^i)^{-1} \widetilde{\alpha}_s^i ds \big] \tag{14}$$

and with covariance matrix

$$\mathrm{Cov}_{\mathbb{Q}}[\xi_T^i, \xi_T^l \mid \mathcal{F}_t] = \sum_{j=1}^{n} \beta_T^i \beta_T^l \int_t^T (\beta_s^i)^{-1} (\beta_s^l)^{-1} \, \sigma_s^{ij} \sigma_s^{lj} \, ds \tag{15}$$

It follows that the distribution of $X_T = \ln S_T$ conditional on $\mathcal{F}_t$ is normal with mean and variance:

$$\mathbb{E}_{\mathbb{Q}}[X_T \mid \mathcal{F}_t] = \sum_{i=1}^{m} \beta_T^i \big[(\beta_t^i)^{-1} \xi_t^i + \int_t^T (\beta_s^i)^{-1} \widetilde{\alpha}_s^i ds \big] \tag{16}$$

(17) $$\mathrm{Var}_{\mathbb{Q}}[X_T \mid \mathcal{F}_t] \;=\; \sum_{i,l=1}^{m}\sum_{j=1}^{n} \beta_T^i \beta_T^l \int_t^T (\beta_s^i)^{-1}(\beta_s^l)^{-1}\, \sigma_s^{ij}\sigma_s^{lj}\, ds$$

II.C Futures price process

As in Subsection **II.A**, $F(t,T)$ denotes the futures price at time t of an energy futures contract with maturity date T. The futures price $F(t,T)$ is a $\mathbb{Q}$-martingale defined as the expectation under the risk-neutral measure $\mathbb{Q}$, conditional on $\mathcal{F}_t$, of the spot price S_T at the maturity date T:

(18) $$F(t,T) = \mathbb{E}_{\mathbb{Q}}[S_T \mid \mathcal{F}_t]$$

Since $X_T = \ln S_T$ is normally distributed under $\mathbb{Q}$, one has:

(19) $$F(t,T) \;=\; \mathbb{E}_{\mathbb{Q}}[\mathrm{e}^{X_T} \mid \mathcal{F}_t] \;=\; \exp\big(\mathbb{E}_{\mathbb{Q}}[X_T \mid \mathcal{F}_t] + \frac{1}{2}\mathrm{Var}_{\mathbb{Q}}[X_T \mid \mathcal{F}_t]\big)$$

This leads to the following expression for the futures price $F(t,T)$ as a function of the state variables

(20) $$F(t,T) \;=\; \exp\big(\sum_{i=1}^{m} \beta_T^i (\beta_t^i)^{-1}\xi_t^i + A(T,t)\big)$$

where

$$A(T,t) = \sum_{i=1}^{m} \beta_T^i \int_t^T (\beta_s^i)^{-1}\widetilde{\alpha}_s^i ds \;+\; \frac{1}{2}\sum_{i,l=1}^{m}\sum_{j=1}^{n} \beta_T^i\beta_T^l \int_t^T (\beta_s^i)^{-1}(\beta_s^l)^{-1}\,\sigma_s^{ij}\sigma_s^{lj}\, ds$$

The futures price $F(t,T)$, has a lognormal distribution under the martingale measure $\mathbb{Q}$. That is, $\Phi(t,T) = \ln F(t,T)$, conditional on $\mathcal{F}_0$, is normally distributed under $\mathbb{Q}$ with mean:

$$\begin{aligned} \mu_\Phi(t,T) &= \mathbb{E}_{\mathbb{Q}}[\ln F(t,T)|\mathcal{F}_0] \\ &= \sum_{i=1}^{m}\left[\beta_T^i \xi_0^i + \beta_T^i \int_0^T (\beta_s^i)^{-1}\widetilde{\alpha}_s^i ds\right] \\ &\quad + \frac{1}{2}\sum_{i,l=1}^{m}\sum_{j=1}^{n}\beta_T^i\beta_T^l \int_t^T (\beta_s^i)^{-1}(\beta_s^l)^{-1}\,\sigma_s^{ij}\sigma_s^{lj}\,ds \end{aligned} \tag{21}$$

and variance:

$$\begin{aligned} \sigma_\Phi^2(t,T) &= \mathrm{Var}_{\mathbb{Q}}[\ln F(t,T)|\mathcal{F}_0] \\ &= \sum_{i,l=1}^{m}\sum_{j=1}^{n}\beta_T^i\beta_T^l \int_0^t (\beta_s^i)^{-1}(\beta_s^l)^{-1}\,\sigma_s^{ij}\sigma_s^{lj}\,ds \end{aligned} \tag{22}$$

Applying Itô's Lemma to (20) and using (10), it follows that under the martingale measure $\mathbb{Q}$ the stochastic differential equation for the futures price process $F(t,T)$ has the form (8)

$$d_tF(t,T) \;=\; F(t,T)\sum_{j=1}^{n}\sigma_j(t,T)d\widetilde{W}_t^j \tag{23}$$

with the volatility function $\sigma_j(t,T)$ given by the expression:

$$\sigma_j(t,T) = \sum_{i=1}^{m}\beta_T^i(\beta_t^i)^{-1}\,\sigma_t^{ij} \tag{24}$$

Under the initial assumptions regarding the stochastic processes followed by the state variables (k_t^i and σ_t^{ij} in the stochastic differential equation (10) are deterministic functions of time), the resulting model for futures exhibits *deterministic volatilities* of futures prices with both time and maturity dependence. If we require that the volatilities $\sigma_j(t,T)$ be *time-homogeneous* then it follows that the quantities k_t^i and σ_t^{ij} must be constant and we set $k_t^i = k_i$ and $\sigma_t^{ij} = \sigma^{ij}$. In this case (24) becomes

$$\sigma_j(t,T) = \sum_{i=1}^{m} \sigma^{ij} \mathrm{e}^{-k_i(T-t)} \tag{25}$$

The total squared instantaneous volatility of the process $F(t,T)$ is in this case a deterministic function of $T-t$:

$$\sigma_{F(t,T)}^2 = \sum_{j=1}^{n} \sigma_j(t,T)^2 = \sum_{j=1}^{n} \Big[\sum_{i=1}^{m} \sigma^{ij} \mathrm{e}^{-k_i(T-t)} \Big]^2. \tag{26}$$

Its functional form suggests that even under the time-homogeneity assumption on the volatility coefficients of the random factors in the futures price process, the model is capable of calibration to a wide range of observed volatility term structures. [1]

[1] An equation similar to (23)-(25) was postulated in Heath (1998) for the process followed by interest rate futures prices under the risk-neutral measure. Here, however, we did not take equation (23) as our starting point, but, as in Schwartz and Smith (1997), we started by postulating the stochastic multi-factor risk-neutral process (9)-(10) followed by a spot price underlying the commodity futures price. Our approach allows a direct state

III Model estimation

III.A Additional model assumptions

In order to obtain estimates of the model parameters, we fit the model futures prices to the observed time series of prices via the Kalman filter and maximum likelihood method under additional simplifying assumptions.

(i) The market price of risk process $\boldsymbol{\gamma}_t = (\gamma_t^1, \ldots, \gamma_t^n)$ is constant.

(ii) The number of state variables is $m = n+1$, with $\xi_t^1, \xi_t^2, \ldots, \xi_t^n$ stochastic components and with $q(t) = \xi_t^{(n+1)}$ the seasonality component, assumed deterministic and periodic with period set to be equal to one year.

(iii) The quantities from equation (10), k_t^i, σ_t^{ij} and $\widetilde{\alpha}_t^i$, are constant and we denote them with $k_i = k_t^i, \sigma^{ij} = \sigma_t^{ij}$ and $\widetilde{\alpha}_i = \widetilde{\alpha}_t^i$. Then, considering (i), also α_t^i from equation (11) is constant, and we set $\alpha_i = \alpha_t^i$.

Hence, the class of models we are considering has time-homogeneous instantaneous volatilities for futures prices as given by expression (26).

space representation given by equation (20) for futures prices, which does not depend on the initial forward curve $F(0,T)$. We shall use the state space formulation in Section **III** to estimate the process parameters using Kalman filtering and maximum likelihood.

For computational purposes, we introduce the Brownian motion factors $Z^1, \ldots, Z^n$, where:

$$\sigma_i dZ_t^i = \sum_{j=1}^{n} \sigma^{ij} dW_t^j, \quad i = 1, \ldots, n \tag{27}$$

with correlations $dZ_t^i \cdot dZ_t^l = \rho_{il} dt,\ i, l = 1, \ldots, n$. Thus:

$$\begin{aligned} \sigma_i^2 &= \sum_{j=1}^{n} (\sigma^{ij})^2 \\ \rho_{il}\sigma_i\sigma_l &= \sum_{j=1}^{n} \sigma^{ij}\sigma^{lj} \end{aligned}$$

With these notations the stochastic differential equation for the state variables ξ_t^i is:

$$d\xi_t^i = (\alpha_i - k_i\xi_t^i)dt + \sigma_i dZ_t^i\,, \quad i = 1, \ldots, n \tag{28}$$

For any $T \geq t$, equation (28) has the solution:

$$\xi_T^i \;=\; \mathrm{e}^{-k_i(T-t)}\,\xi_t^i \;+\; \frac{\alpha_i}{k_i}\left[1 - \mathrm{e}^{-k_i(T-t)}\right] + \int_t^T \mathrm{e}^{-k_i(T-s)}\,\sigma_i\, dZ_s^i \tag{29}$$

With the new notations and assumptions, equation (20) for the futures price $F(t,T)$ as a function of the state variables $\xi_t^1, \ldots, \xi_t^n$ and the seasonality component $Q(t) = \exp q(t)$ becomes:

$$F(t,T) \;=\; Q(T)\exp\left(\sum_{i=1}^{n} \mathrm{e}^{-k_i(T-t)}\xi_t^i + A(T-t)\right) \tag{30}$$

where

$$A(T-t) = \sum_{i=1}^{n} \frac{\widetilde{\alpha}_i}{k_i} \left[1 - \mathrm{e}^{-k_i(T-t)}\right] + \frac{1}{2} \sum_{i,l=1}^{n} \frac{\rho_{il}\sigma_i\sigma_l}{k_i + k_l} \left[1 - \mathrm{e}^{-(k_i+k_l)(T-t)}\right]$$

From (28) and (30) one can derive the expression for the stochastic processes followed by futures prices:

$$d_t F(t,T) = F(t,T)\left[\sum_{i=1}^{n} (\alpha_i - \widetilde{\alpha}_i)\mathrm{e}^{-k_i(T-t)} dt + \sum_{i=1}^{n} \sigma_i \mathrm{e}^{-k_i(T-t)} dZ_t^i\right] \tag{31}$$

In view of (12) and (27), the above stochastic differential equation satisfies the restriction (5) on the drift of the futures price process.

Let Δt denote a time period length. Then (31) leads to the following expressions for the mean and variance of the log returns of futures prices over the period Δt:

$$\begin{aligned} \mathbb{E}_{\mathbb{P}}[\ln(\tfrac{F(t,T)}{F(t-\Delta t,T)})] &= \sum_{i=1}^{n} \frac{\alpha_i - \widetilde{\alpha}_i}{k_i} \mathrm{e}^{-k_i(T-t)}[1 - \mathrm{e}^{-k_i \Delta t}] \\ &\quad - \frac{1}{2} \sum_{i,l=1}^{n} \frac{\rho_{il}\sigma_i\sigma_l}{k_i + k_l} \mathrm{e}^{-(k_i+k_l)(T-t)}[1 - \mathrm{e}^{-(k_i+k_l)\Delta t}] \\ \mathrm{Var}_{\mathbb{P}}[\ln(\tfrac{F(t,T)}{F(t-\Delta t,T)})] &= \sum_{i,l=1}^{n} \frac{\rho_{il}\sigma_i\sigma_l}{k_i + k_l} \mathrm{e}^{-(k_i+k_l)(T-t)}[1 - \mathrm{e}^{-(k_i+k_l)\Delta t}] \end{aligned} \tag{32}$$

Moreover, the covariance of log returns over the period Δt for two different futures contracts is given by:

(33)

$$\mathrm{Cov}_{\mathbb{P}}[\ln(\frac{F(t,T_1)}{F(t-\Delta t,T_1)}), \ln(\frac{F(t,T_2)}{F(t-\Delta t,T_2)})] = \sum_{i,l=1}^{n} \frac{\rho_{il}\sigma_i\sigma_l}{k_i + k_l} \mathrm{e}^{-k_i(T_1-t)-k_l(T_2-t)}[1-\mathrm{e}^{-(k_i+k_l)\Delta t}]$$

III.B State space form representation, Kalman filter and maximum likelihood method

The futures price model described by equations (29) and (30) can be cast in *state space form* which can then be used in conjunction with the Kalman filter and the maximum likelihood estimation method for the empirical implementation of the model. A general description of the state space formulation and of the Kalman filter can be found in Hamilton (1994) or Harvey (1994). We detail below the state space formulation for our models.

Let $\mathbf{y}_t$, $t = t_0, t_1, t_2, \ldots, t_{\text{final}}$, be the observed multivariate time series with N elements:

(34)
$$\mathbf{y}_t = \begin{pmatrix} y_{1t} & = & \Phi(t, t+\tau_1) \\ & \vdots & \\ y_{Nt} & = & \Phi(t, t+\tau_N) \end{pmatrix}$$

where $\Phi(t, t+\tau) = \ln F(t, t+\tau)$ is the logarithm of the futures price at time t with expiry date at $T = t + \tau$. We assume that the observations $\mathbf{y}_t$ are given at equally spaced time steps and we let $\Delta t = t_i - t_{i-1}$ denote the length of the time step.

From equation (30) it follows that the observables $\mathbf{y}_t$ are related to the state vector

$$\boldsymbol{\xi}_t = \begin{pmatrix} \xi_t^1 \\ \vdots \\ \xi_t^n \end{pmatrix} \tag{35}$$

through the *observation equation*:

$$\mathbf{y}_t = \mathbb{Z}\boldsymbol{\xi}_t + \mathbf{d}_t + \boldsymbol{\varepsilon}_t \tag{36}$$

where

$$\mathbb{Z}_{pi} = \mathrm{e}^{-k_i \tau_p}, \quad i = 1, \ldots, n, \quad p = 1, \ldots, N$$

$$(\mathbf{d}_t)_p = A(\tau_p) + q(t + \tau_p), \quad p = 1, \ldots, N$$

To account for possible errors in the data we introduced in equation (36) a vector $\boldsymbol{\varepsilon}_t$ of serially uncorrelated, identically distributed disturbances. Each $\boldsymbol{\varepsilon}_t$ has a multivariate normal distribution, $\boldsymbol{\varepsilon}_t \sim \mathcal{N}(0, \mathbb{R})$, with mean zero and with covariance matrix taken for computational simplicity to be diagonal $\mathbb{R} = \begin{pmatrix} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \omega_N^2 \end{pmatrix}$.

According to equation (29), the state vector $\boldsymbol{\xi}_t$ follows a first order Markov process defined by the *state equation*:

$$\boldsymbol{\xi}_t = \mathbb{T}\boldsymbol{\xi}_{t-\Delta t} + \mathbf{c} + \boldsymbol{\eta}_t \tag{37}$$

where

$$\mathbb{T}_{il} = (\mathrm{e}^{-k_i \Delta t})\delta_{il}, \quad i, l = 1, \ldots, n$$

$$c_i = \frac{\alpha_i}{k_i}\left(1 - \mathrm{e}^{-k_i \Delta t}\right), \quad i = 1, \ldots, n$$

and $\boldsymbol{\eta}_t$ is a vector of serially uncorrelated, identically distributed disturbances, with each $\boldsymbol{\eta}_t$ drawn from the same multivariate normal distribution, $\boldsymbol{\eta}_t \sim \mathcal{N}(0, \mathbb{V})$, with covariance matrix $\mathbb{V}_{il} = \frac{\rho_{il}\sigma_i\sigma_l}{k_i+k_l}\left[1 - \mathrm{e}^{-(k_i+k_l)\Delta t}\right]$. The disturbances $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\eta}_t$ are assumed to be uncorrelated with each other in all time periods. We also assume that the initial state vector $\boldsymbol{\xi}_0$, although unknown, is non-stochastic and fixed, and its components will be included among the model parameters to be estimated.

The observation and state equation matrices, $\mathbb{Z}, \mathbf{d}_t, \mathbb{R}, \mathbb{T}, \mathbf{c}, \mathbb{V}$, depend on the unknown parameters of the model. One of the main goals in the empirical implementation of the model is to find estimates for these parameters. This can be accomplished by maximizing the likelihood function with respect to the unknown parameters through an optimization procedure.

For notational simplicity, we collect the unknown model parameters in a vector $\boldsymbol{\theta}$ and we denote by $\mathcal{Y}_t = \{\mathbf{y}_t, \mathbf{y}_{t-\Delta t}, \ldots, \mathbf{y}_{t_1}, \mathbf{y}_{t_0}\}$ the information set at time t. As usual for time series models, the observations $\mathbf{y}_{t_0}, \mathbf{y}_{t_1}, \ldots, \mathbf{y}_{t_{\text{final}}}$ are not independent. Their joint probability density function (pdf), the *likelihood function*, is given by

$$L(\mathbf{y}; \boldsymbol{\theta}) = \prod_t g(\mathbf{y}_t \mid \mathcal{Y}_{t-\Delta t}) \tag{38}$$

where $g(\mathbf{y}_t \mid \mathcal{Y}_{t-\Delta t})$ denotes the pdf of $\mathbf{y}_t$ conditional on the information set $\mathcal{Y}_{t-\Delta t}$ available in the previous time period. Under the stated assumptions regarding $\boldsymbol{\xi}_0$ and the disturbances $\boldsymbol{\varepsilon}_t$ and $\boldsymbol{\eta}_t$, the distribution of $\mathbf{y}_t$ conditional on $\mathcal{Y}_{t-\Delta t}$ is normal with mean $\widehat{\mathbf{y}}_{t|t-\Delta t} = \mathbb{E}_{\mathbb{P}}[\mathbf{y}_t \mid \mathcal{Y}_{t-\Delta t}]$ and covariance matrix $\mathbb{C}_t$. With $\boldsymbol{\nu}_t = \mathbf{y}_t - \widehat{\mathbf{y}}_{t|t-\Delta t}$ denoting the vector of prediction errors, the logarithm of the likelihood function is given by:

$$\begin{aligned} \log L(\mathbf{y}; \boldsymbol{\theta}) = &- \frac{N(t_{\text{final}} - t_1)}{2\Delta t} \log 2\pi - \frac{1}{2} \sum_t \log |\det \mathbb{C}_t| \\ &- \frac{1}{2} \sum_t \boldsymbol{\nu}_t \mathbb{C}_t^{-1} \boldsymbol{\nu}_t \end{aligned} \tag{39}$$

For a given set of parameters $\boldsymbol{\theta}$, the likelihood function can be evaluated through the Kalman filtering procedure. This is a recursive procedure for computing the optimal estimate $\widehat{\boldsymbol{\xi}}_t$ of the state vector $\boldsymbol{\xi}_t$ based on the information available at time t, that is, based on the observed time series $\mathbf{y}_t$ up to and including time t. This estimate $\widehat{\boldsymbol{\xi}}_t$ is the conditional mean $\widehat{\boldsymbol{\xi}}_t = \mathbb{E}_{\mathbb{P}}[\boldsymbol{\xi}_t \mid \mathcal{Y}_t]$. Besides $\widehat{\boldsymbol{\xi}}_t$, the Kalman filter yields in each time period t the prediction error $\boldsymbol{\nu}_t$ and the covariance matrix $\mathbb{C}_t$ which enter into the expression of the likelihood function. The best-fit parameters $\boldsymbol{\theta}$ are those which maximize the logarithm of the likelihood function described in equation (39).

IV Results

We give estimates for two specific models from the class of models formulated in Subsection **III.A**, with one and two stochastic factors.

IV.A One-factor model

The model is described by a deterministic seasonality factor and one random factor. The state vector contains only one stochastic component ξ_t which is assumed to follow an Ornstein-Uhlenbeck mean-reverting process, with mean-reversion rate k, volatility σ, mean reverting level α for the real-world process and mean-reverting level $\widetilde{\alpha}$ for the risk-neutral process. The observation and state equations are:

$$(\mathbf{y}_t)_p = \mathrm{e}^{-k\tau_p}\xi_t + (\mathbf{d}_t)_p + (\boldsymbol{\varepsilon}_t)_p, \qquad p = 1, \dots, N \tag{40}$$

$$\xi_t = \mathrm{e}^{-k\Delta t}\xi_{t-\Delta t} + \frac{\alpha}{k}\left(1 - \mathrm{e}^{-k\Delta t}\right) + \eta_t \tag{41}$$

where

$$\boldsymbol{\varepsilon}_t \sim \text{i.i.d.}\,\mathcal{N}(0, \mathbb{R}) \qquad \mathbb{R} = \begin{pmatrix} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \omega_N^2 \end{pmatrix}$$

$$\eta_t \sim \text{i.i.d.}\,\mathcal{N}(0, \mathbb{V}) \qquad \mathbb{V} = \frac{\sigma^2}{2k}\left(1 - \mathrm{e}^{-2k\Delta t}\right)$$

$$(\mathbf{d}_t)_p = q(t + \tau_p) + \frac{\widetilde{\alpha}}{k}\left(1 - \mathrm{e}^{-k\tau_p}\right) + \frac{\sigma^2}{4k}\left(1 - \mathrm{e}^{-2k\tau_p}\right)$$

We take the seasonality component $q(t)$ to be a periodic step function with $Q(t) = \mathrm{e}^{q(t)} = s_m$, if time t belongs to the month m (the period is one year). The convention means that we assign a fixed seasonality factor s_m for each month m of the year. Moreover, these monthly values are normalized so that $\prod_{m=1}^{12} s_m = 1$. The vector of unknown parameters to be estimated is $\boldsymbol{\theta} = (k, \sigma, \alpha, \widetilde{\alpha}, \xi_0, (\omega_1^2, \dots, \omega_N^2), (s_1, \dots, s_{12}))$.

IV.B Two-factor model

The model is described by a deterministic seasonality factor and two random factors. The state vector contains two stochastic components $\boldsymbol{\xi}_t = (\xi_t^1, \xi_t^2)$.

Component ξ_t^1 follows an Ornstein-Uhlenbeck mean-reverting process with mean-reversion rate k, volatility σ_1, mean-reverting level $\alpha_1 = 0$ for the real-world process and mean-reverting level $\widetilde{\alpha}_1$ for the risk-neutral process. Component ξ_t^2 follows a Brownian motion process with volatility σ_2, drift rate α_2 for the real-world process, and drift rate $\widetilde{\alpha}_2$ for the risk-neutral process. The initial state vector is taken to be $\boldsymbol{\xi}_0 = (\xi_0^1 = 0, \xi_0^2)$. The observation and state equations are:

$$(\mathbf{y}_t)_p = \mathrm{e}^{-k\tau_p}\xi_t^1 + \xi_t^2 + (\mathbf{d}_t)_p + (\boldsymbol{\varepsilon}_t)_p, p = 1, \ldots, N \tag{42}$$

$$\begin{pmatrix} \xi_t^1 \\ \xi_t^2 \end{pmatrix} = \begin{pmatrix} \mathrm{e}^{-k\Delta t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \xi_{t-\Delta t}^1 \\ \xi_{t-\Delta t}^2 \end{pmatrix} + \begin{pmatrix} 0 \\ \alpha_2 \Delta t \end{pmatrix} + \begin{pmatrix} \eta_t^1 \\ \eta_t^2 \end{pmatrix} \tag{43}$$

where

$$\boldsymbol{\varepsilon}_t \sim \text{i.i.d.}\, \mathcal{N}(0, \mathbb{R}) \qquad \mathbb{R} = \begin{pmatrix} \omega_1^2 & & 0 \\ & \ddots & \\ 0 & & \omega_N^2 \end{pmatrix}$$

$$\eta_t \sim \text{i.i.d.}\, \mathcal{N}(0, \mathbb{V}) \qquad \mathbb{V} = \begin{pmatrix} \frac{(\sigma_1)^2}{2k}\left(1 - \mathrm{e}^{-2k\Delta t}\right) & \frac{\rho_{12}\sigma_1\sigma_2}{k}\left(1 - \mathrm{e}^{-k\Delta t}\right) \\ \frac{\rho_{12}\sigma_1\sigma_2}{k}\left(1 - \mathrm{e}^{-k\Delta t}\right) & (\sigma_2)^2\Delta t \end{pmatrix}$$

$$\begin{aligned}(\mathbf{d}_t)_p = q(t+\tau_p) + \frac{\widetilde{\alpha}_1}{k}\left(1 - \mathrm{e}^{-k\tau_p}\right) + \widetilde{\alpha}_2\tau_p + \frac{(\sigma_1)^2}{4k}\left(1 - \mathrm{e}^{-2k\tau_p}\right) \\ + \frac{\rho_{12}\sigma_1\sigma_2}{k}\left(1 - \mathrm{e}^{-k\tau_p}\right) + \frac{1}{2}(\sigma_2)^2\tau_p\end{aligned}$$

The assumptions on the seasonality factor are the same as in the case of the one-factor model. The vector of unknown parameters to be estimated is $\boldsymbol{\theta} = (k, \sigma_1, \widetilde{\alpha}_1, \sigma_2, \alpha_2, \widetilde{\alpha}_2, \rho_{12}, \xi_0^2, (\omega_1^2, \ldots, \omega_N^2), (s_1, \ldots, s_{12}))$.

IV.C Empirical results for natural gas futures

For the empirical implementation of the models we used a dataset including the historical time series of natural gas futures prices, quoted daily from 09/02/97 to 08/31/98, for 15 contracts with times to expiration 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15 months (Henry Hub natural gas futures, Bloomberg data). The main steps we performed towards the estimation of the model parameters $\boldsymbol{\theta}$ were the following. We chose an initial guess for $\boldsymbol{\theta}$. Then, the following iterative procedure was performed until an appropriate convergence criterion has been satisfied: for a specified value of $\boldsymbol{\theta}$, we ran the Kalman filter recursions over the whole time interval of the available time series data; after each such complete run, the log-likelihood function was evaluated and an optimization procedure, designed for this particular problem, was performed in order to get new estimates for the parameters $\boldsymbol{\theta}$. At each iteration the value of the log-likelihood function was compared against the values in the previous two iterations and, if the differences between the current value and the old values were positive and smaller than a specified amount (we chose 0.001) the iterative procedure was stopped.

Tables **1** and **2** list the results for the parameter estimates and the corresponding standard deviations of the estimation errors for the one-factor and the two-factor models. A summary of the results follows.

Mean reversion. For both the one-factor as well as the two-factor model there is evidence of mean-reversion in the observed time series of futures prices. The two-factor model yields a mean-reversion rate k for the mean-reverting component with $\frac{1}{k} \approxeq 2.9$ months. In the one-factor model the value obtained for the mean-reverting rate k is, as expected, between 0 (the mean-reverting rate for a Brownian motion factor) and the value of the mean-reverting rate from the two-factor model, with $\frac{1}{k} \approxeq 12$ months.

Instantaneous volatility. The two-factor model yields an annualized volatility for the short-term, mean-reverting component of approximately 131%, much higher than the volatility of approximately 13% of the Brownian motion, long-term component. The mean-reverting component describes short-term price deviations from an equilibrium price level, while the fluctuations of the equilibrium price level are modeled by the Brownian motion component. For the one-factor model the estimated annualized volatility of the mean-reverting component has an intermediate value of approximately 36%.

Seasonality factor. Both models yield similar estimates for the monthly seasonality indices $s_1, s_2, \ldots, s_{12}$, higher for the winter months ($\sim 1.0-1.1$) and lower during the summer months ($\sim 0.95-0.96$). The pattern of the seasonality indices is triangular with the peak during the month of January.

Risk premia. The empirical estimation of the risk premia is based on the state space formulation described in Subsection **III.B**. The observation equation (36) describes prices of traded instruments, and thus involves parameters which enter into the risk-neutral processes. On the other hand, state equation (37), describes the observed time evolution of prices of traded instruments, which follows from the real-world processes. Hence, the approach of using the state space formulation together with the Kalman filter and maximum likelihood method for model estimation from the observed time series of futures prices, provides parameter estimates for both the risk-neutral process and the real-world process. Thus we are able to obtain estimates of the market risk-premia associated with the stochastic factors involved. For the one-factor model the risk-premium for the underlying mean-reverting stochastic factor is, according to the results from Table **1**, equal to

$\alpha - \widetilde{\alpha} \approxeq 12\%$ per year. From the results in Table **2**, for the two-factor model, the risk-premium associated to the short-term, strongly fluctuating mean-reverting factor is $\alpha_1 - \widetilde{\alpha}_1 \approxeq 225\%$ per year, and the risk-premium for the Brownian motion factor is $\alpha_2 - \widetilde{\alpha}_2 \approxeq 24\%$ per year. However, as the tables show, the standard deviations of the estimation errors for the risk-premia are high.

Term structure of volatilities and correlations. Table **4** lists the model implied annualized volatilities of daily log returns of futures prices, derived from expression (32) for the log returns variance using the estimated set of model parameters. It also lists the empirical volatilities of futures prices computed as the annualized standard deviations of their observed daily log returns. Figure (2) shows the one-factor and two-factor model volatilities compared to the empirical volatility estimates. We notice that the fit of the model volatilities to the empirical ones is rather poor for the short-term contracts, but improves as we move towards contracts with longer time to expiry. Regarding correlations between price movements of different futures contracts, in the one-factor model different points on the futures curve are perfectly correlated. However, as Table **6** shows, the empirical correlations between the log return series for different futures contracts can be far from the value of 1. This can not be accounted for in a one-factor model. On the other hand, in the two-factor model one can derive the model implied correlations between the daily log returns of futures series with different times to expiry. Using expressions (32) and (33) for the variances and covariances of log returns, we computed the model implied correlations for the estimated set of model parameters. The results are listed in Table **5** and selected results are plotted in Figures **3** and **4** against the empirical correlation estimates.

Prediction errors. Table **3** shows the mean and the standard deviation of the daily prediction errors in the futures price for each contract. These daily prediction errors represent the difference between the observed futures prices and the model predicted futures prices based upon running the Kalman filter up to the previous day with the model parameters set at their estimated values. As illustrated by Figure **1**, the two-factor model yields lower standard deviations for the prediction errors compared to the one-factor model. We also notice that for both models the short-term contracts have the highest standard deviations of prediction errors and the contracts with intermediate-to-long time to expiry have the lowest.

Comparing the values of the likelihood functions (see Tables **1** and **2**) and the standard deviations of predictions errors (Figure **1**) the two-factor model appears to be a better fit to the observed prices.

Figures **5** and **6** show the observed and the model implied futures price curve on 8/31/98, the last day of the time series used for the model estimation, for contracts with times to expiry ranging from 1 month to 36 months. The model implied curve follows from expression (30), with the parameters set at their estimated value and with the state vector value obtained by running the Kalman filter up to that day.

V Conclusions

We have described a general multi-factor model of futures prices which can be used to model the behavior of futures contracts for energy commodities. The futures prices are defined in terms of a spot price which, in general, does not represent the price of a tradable asset. Although the spot price does not correspond to a tradable asset and is not always observable, its introduction provides the link between futures prices and the stochastic factors driving the tradable assets in our market model. We provided a state space formulation of the model in which futures prices are expressed in terms of unobserved state variables. Based on the state space formulation we implemented Kalman filter techniques and maximum likelihood estimation to determine the model parameters.

For the empirical model estimation we considered two cases of the general model: (i) a one-factor model with one stochastic state variable driven by a mean-reverting process and with another deterministic state variable describing the seasonality dependence of the futures price; (ii) a two-factor model with two stochastic state variables, one driven by a mean-reverting process and another by a Brownian motion, and with another deterministic state variable for seasonality. The data used was a one year time series of natural gas futures. From both models there is evidence of mean-reversion in the data. The results regarding the one-day ahead prediction errors in the futures prices show that the two-factor model performs better than the model with only one stochastic factor. In addition, the superiority of the two-factor model is evident in the fact that it is able to capture the observed correlation structure of futures prices. The class of models we introduced also allowed us to capture the seasonality pattern present in the data.

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Table 1: Estimated parameters and standard deviations of the estimation errors for the *one-factor model*

Model Parameters	**Value**	**Standard Error**
k (years^{-1})	0.99953	0.00488
σ (annualized)	0.35775	0.01911
ξ_0	0.79335	0.02349
α (years^{-1})	0.90452	0.35929
$\widetilde{\alpha}$ (years^{-1})	0.78939	0.00111
Standard deviations of observed log prices noise ω_i		
(contract time to expiry)		
1 month	0.16183	0.00076
2 months	0.14306	0.00052
3 months	0.11586	0.00028
4 months	0.08419	0.00011
5 months	0.05153	0.00002
6 months	0.02614	0.00000
7 months	0.01222	0.00000
8 months	0.00666	0.00000
9 months	0.00554	0.00000
10 months	0.00661	0.00000
11 months	0.00893	0.00000
12 months	0.01242	0.00000
13 months	0.01441	0.00000
14 months	0.01351	0.00000
15 months	0.01356	0.00000
Seasonality monthly indices		
s_1	1.10516	0.00071
s_2	1.05877	0.00068
s_3	1.01174	0.00065
s_4	0.96740	0.00062
s_5	0.95378	0.00061
s_6	0.95223	0.00061
s_7	0.95299	0.00061
s_8	0.95588	0.00062
s_9	0.96032	0.00062
s_{10}	0.97662	0.00064
s_{11}	1.03240	0.00067
s_{12}	1.09000	0.00071
Log Likelihood Function Value: 8511.04		

Table 2: Estimated parameters and standard deviations of the estimation errors for the *two-factor model*

Model Parameters	**Value**	**Standard Error**
Mean-reverting component		
k (years^{-1})	4.14604	0.04671
σ_1 (annualized)	1.31467	0.03441
ξ_0^1	0	(kept constant)
α_1 (years^{-1})	0	(kept constant)
$\widetilde{\alpha}_1$ (years^{-1})	-2.25214	0.34862
Brownian motion component		
σ_2 (annualized)	0.13127	0.00597
ξ_0^2	1.22649	0.08863
α_2 (years^{-1})	0.20438	0.10515
$\widetilde{\alpha}_2$ (years^{-1})	-0.03077	0.00135
ρ_{12}	0.62885	0.05525
Standard deviations of observed log prices noise ω_i		
(contract time to expiry)		
1 month	0.06670	0.00006
2 months	0.02654	0.00000
3 months	0.01989	0.00000
4 months	0.02141	0.00000
5 months	0.01950	0.00000
6 months	0.01788	0.00000
7 months	0.01545	0.00000
8 months	0.01255	0.00000
9 months	0.01074	0.00000
10 months	0.00890	0.00000
11 months	0.00766	0.00000
12 months	0.00832	0.00000
13 months	0.00938	0.00000
14 months	0.01030	0.00000
15 months	0.01206	0.00000
Seasonality monthly indices		
s_1	1.11454	0.00075
s_2	1.06652	0.00072
s_3	1.01407	0.00068
s_4	0.96247	0.00065
s_5	0.94533	0.00063
s_6	0.94517	0.00063
s_7	0.94904	0.00063
s_8	0.95475	0.00064
s_9	0.96031	0.00064
s_{10}	0.97538	0.00065
s_{11}	1.03460	0.00070
s_{12}	1.09864	0.00074
Log Likelihood Function Value: 10053.94		

Table 3: Results on the daily prediction errors on futures prices

Contract time to expiry (months)	One-factor model			Two-factor model		
	Mean Error	Std.Dev. Error	Mean Abs Error	Mean Error	Std.Dev. Error	Mean Abs Error
1	-0.007	0.419	0.339	-0.072	0.178	0.140
2	0.030	0.380	0.284	-0.033	0.098	0.076
3	0.046	0.312	0.224	-0.001	0.085	0.063
4	0.040	0.224	0.156	0.010	0.078	0.056
5	0.025	0.133	0.091	0.007	0.063	0.047
6	0.011	0.070	0.050	0.000	0.053	0.039
7	0.002	0.044	0.032	-0.005	0.046	0.035
8	-0.001	0.036	0.025	-0.006	0.039	0.030
9	0.000	0.034	0.023	-0.003	0.035	0.026
10	-0.001	0.032	0.022	-0.004	0.032	0.022
11	-0.001	0.032	0.022	-0.004	0.029	0.020
12	-0.002	0.035	0.025	-0.004	0.030	0.021
13	-0.002	0.037	0.028	-0.004	0.031	0.023
14	0.004	0.034	0.027	0.002	0.031	0.023
15	0.007	0.034	0.026	0.005	0.033	0.024

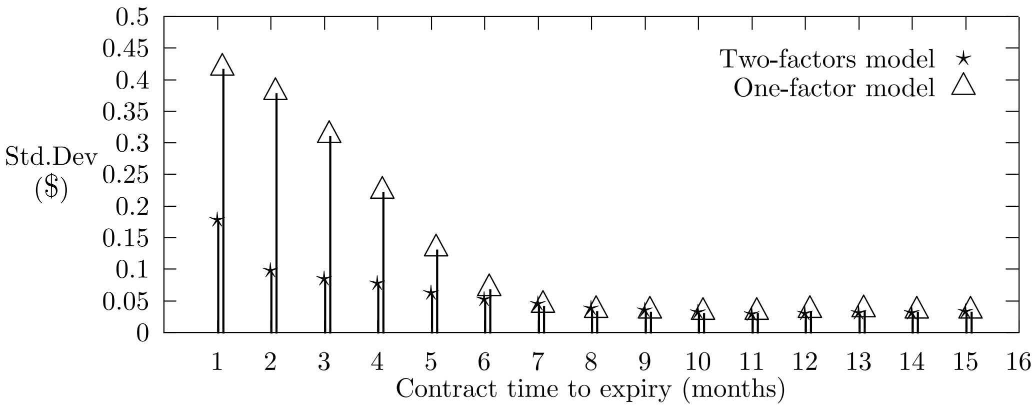

Figure 1: One- and Two-factor model standard deviations of daily prediction errors of futures prices

Table 4: Empirical and model implied volatilities of daily log returns of futures prices

Contract time to expiry	Empirical Volatility (%)	One-factor model Volatility (%)	One-factor model Estimation std. error(%)	Two-factor model Volatility (%)	Two-factor model Estimation std. error(%)
1 month	49.1	34.3	29.1	118.4	48.5
2 months	45.3	31.5	26.8	86.5	36.7
3 months	37.0	29.0	24.6	64.0	28.5
4 months	31.7	26.7	22.7	48.3	22.6
5 months	25.8	24.5	20.9	37.2	18.1
6 months	22.1	22.6	19.2	29.6	14.5
7 months	19.8	20.8	17.7	24.3	11.5
8 months	18.0	19.1	16.3	20.7	9.0
9 months	16.6	17.6	15.0	18.3	7.0
10 months	15.3	16.2	13.8	16.6	5.4
11 months	14.2	14.9	12.7	15.5	4.1
12 months	13.4	13.7	11.7	14.8	3.2
13 months	12.8	12.6	10.8	14.3	2.6
14 months	12.2	11.6	9.9	13.9	2.1
15 months	12.0	10.7	9.1	13.7	1.8

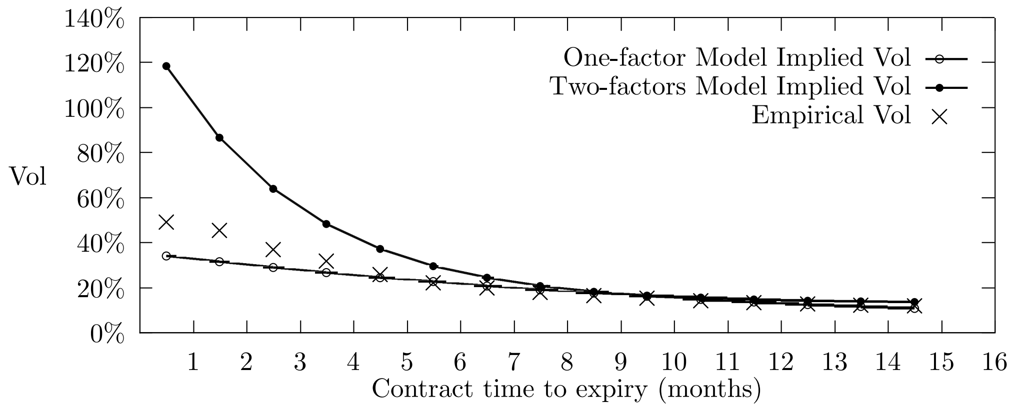

Figure 2: Empirical and model implied annualized volatilities of daily log returns of futures prices

Table 5: Correlations between daily log returns of futures prices for different contracts, implied by the two-factor model (expressed in %)

Contract time to expiry	1mo	2mo	3mo	4mo	5mo	6mo	7mo	8mo	9mo	10mo	11mo	12mo	13mo	14mo	15mo
1mo	100.0	99.9	99.7	99.2	98.1	96.4	94.0	90.9	87.5	83.9	80.7	78.0	75.9	74.1	72.8
2mo	99.9	100.0	99.9	99.5	98.7	97.2	95.0	92.2	88.9	85.6	82.6	80.0	77.8	76.2	74.9
3mo	99.7	99.9	100.0	99.8	99.3	98.1	96.2	93.7	90.8	87.7	84.9	82.4	80.4	78.8	77.6
4mo	99.2	99.5	99.8	100.0	99.7	99.0	97.5	95.4	92.9	90.1	87.6	85.3	83.4	82.0	80.9
5mo	98.1	98.7	99.3	99.7	100.0	99.7	98.7	97.1	95.1	92.7	90.5	88.5	86.8	85.5	84.5
6mo	96.4	97.2	98.1	99.0	99.7	100.0	99.6	98.6	97.1	95.3	93.4	91.7	90.3	89.1	88.2
7mo	94.0	95.0	96.2	97.5	98.7	99.6	100.0	99.6	98.7	97.4	96.0	94.6	93.5	92.5	91.8
8mo	90.9	92.2	93.7	95.4	97.1	98.6	99.6	100.0	99.7	98.9	97.9	96.9	96.0	95.3	94.7
9mo	87.4	88.9	90.8	92.9	95.1	97.1	98.7	99.7	100.0	99.7	99.2	98.5	97.9	97.3	96.9
10mo	83.9	85.6	87.7	90.1	92.7	95.3	97.4	98.9	99.7	100.0	99.8	99.4	99.0	98.6	98.3
11mo	80.7	82.6	84.9	87.6	90.5	93.4	96.0	97.9	99.2	99.8	100.0	99.9	99.6	99.4	99.2
12mo	78.0	80.0	82.4	85.3	88.5	91.7	94.6	96.9	98.5	99.4	99.9	100.0	99.9	99.8	99.6
13mo	75.8	77.8	80.4	83.4	86.8	90.3	93.5	96.0	97.9	99.0	99.6	99.9	100.0	99.9	99.9
14mo	74.1	76.2	78.8	82.0	85.5	89.1	92.5	95.3	97.3	98.6	99.4	99.8	99.9	100.0	99.9
15mo	72.8	74.9	77.6	80.9	84.5	88.2	91.8	94.7	96.9	98.3	99.2	99.6	99.9	99.9	100.0

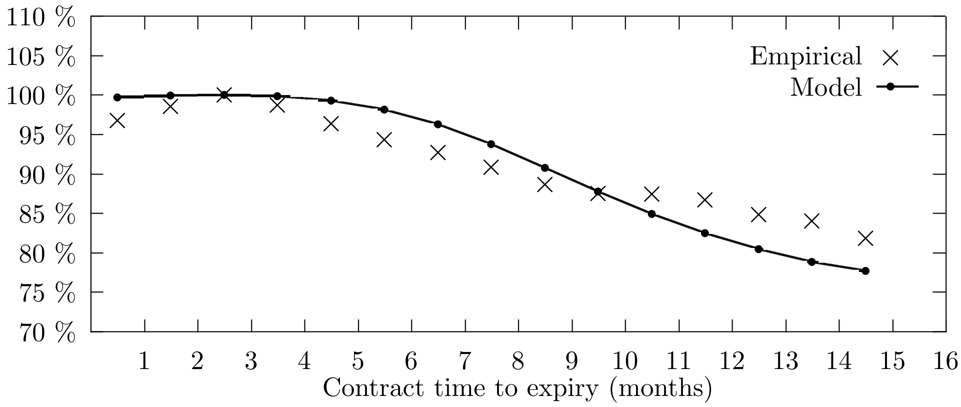

Figure 3: Empirical and *two-factor model* implied correlations of daily log returns between the 3 months contract and other contracts

Table 6: Empirical correlations between daily log returns of futures prices for different contracts (expressed in %)

Contract time to expiry	1mo	2mo	3mo	4mo	5mo	6mo	7mo	8mo	9mo	10mo	11mo	12mo	13mo	14mo	15mo
1mo	100.0	99.2	96.8	94.1	90.6	88.7	87.3	85.3	83.2	82.1	82.3	81.6	79.1	78.6	76.0
2mo	99.2	100.0	98.6	96.2	92.8	90.9	89.5	87.4	85.6	84.5	84.6	83.9	81.6	80.8	78.2
3mo	96.8	98.6	100.0	98.7	96.3	94.4	92.7	90.9	88.7	87.5	87.5	86.7	84.9	84.0	81.8
4mo	94.1	96.2	98.7	100.0	98.7	97.0	95.6	93.8	91.9	90.7	90.4	89.5	87.5	86.6	84.8
5mo	90.6	92.8	96.3	98.7	100.0	99.0	97.8	96.3	94.7	93.4	93.0	92.0	90.1	89.1	87.5
6mo	88.7	90.9	94.4	97.0	99.0	100.0	99.3	98.1	96.6	95.5	94.9	93.8	92.0	91.2	89.8
7mo	87.3	89.5	92.7	95.6	97.8	99.3	100.0	99.4	98.3	97.4	96.7	95.5	93.8	92.6	91.3
8mo	85.3	87.4	90.9	93.8	96.3	98.1	99.4	100.0	99.2	98.4	97.7	96.4	94.8	93.4	92.1
9mo	83.2	85.6	88.7	91.9	94.7	96.6	98.3	99.2	100.0	99.4	98.5	97.1	95.5	94.1	92.7
10mo	82.1	84.5	87.5	90.7	93.4	95.5	97.4	98.4	99.4	100.0	99.4	98.2	96.8	95.5	94.1
11mo	82.3	84.6	87.5	90.4	93.0	94.9	96.7	97.7	98.5	99.4	100.0	99.4	98.3	97.1	95.7
12mo	81.6	83.9	86.7	89.5	92.0	93.8	95.5	96.4	97.1	98.2	99.4	100.0	99.1	98.0	96.5
13mo	79.1	81.6	84.9	87.5	90.1	92.0	93.8	94.8	95.5	96.8	98.3	99.1	100.0	99.3	98.3
14mo	78.6	80.8	84.0	86.6	89.1	91.2	92.6	93.4	94.1	95.5	97.1	98.0	99.3	100.0	99.3
15mo	76.0	78.2	81.8	84.8	87.5	89.8	91.3	92.1	92.7	94.1	95.7	96.5	98.3	99.3	100.0

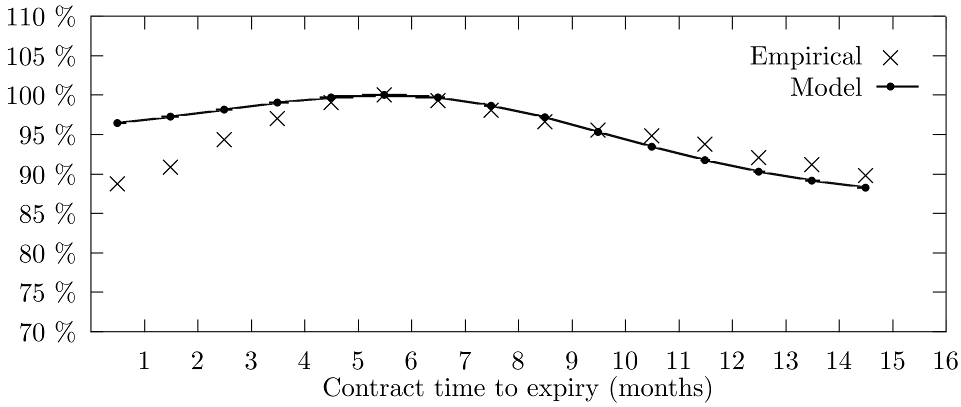

Figure 4: Empirical and *two-factor model* implied correlations of daily log returns between the 6 months contract and other contracts

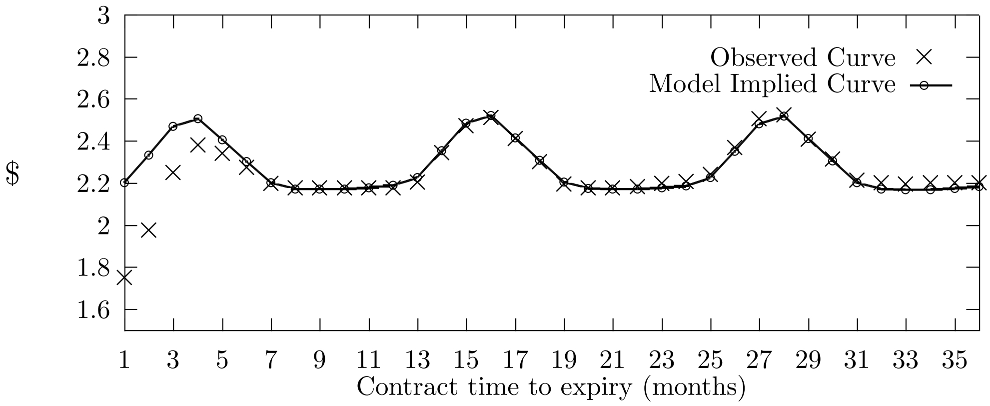

Figure 5: Observed futures curve and the *one-factor model* implied curve on 8/31/98

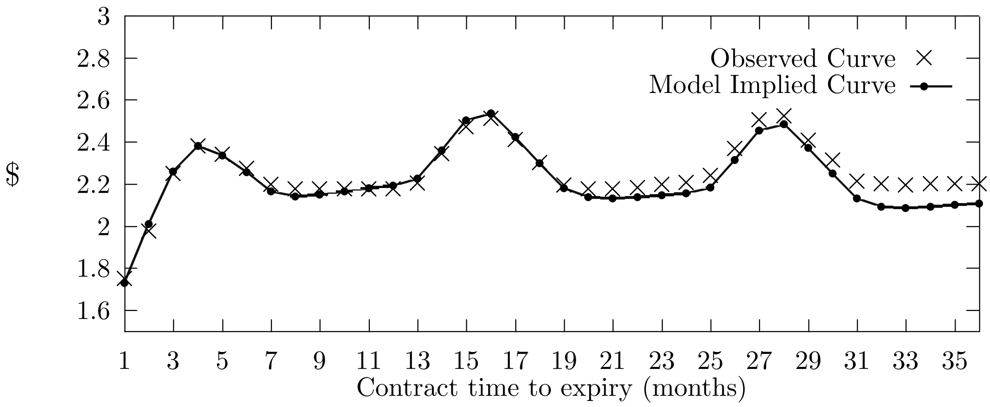

Figure 6: Observed futures curve and the *two-factor model* implied curve on 8/31/98

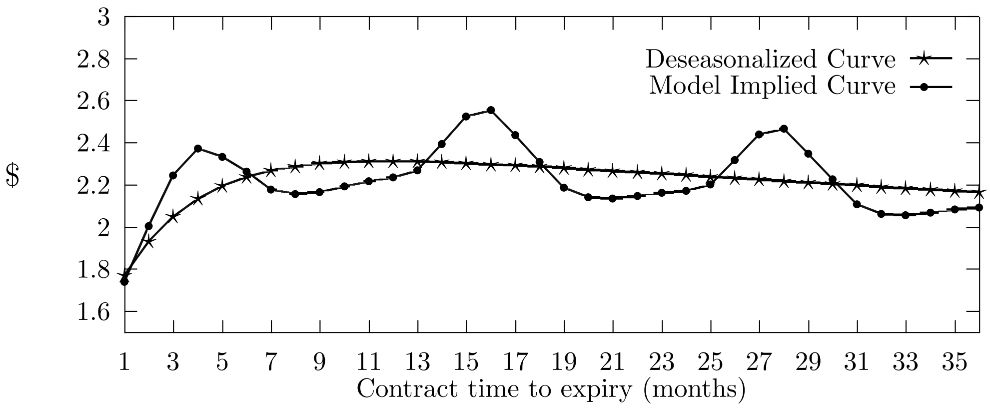

Figure 7: Futures curve and deseasonalized curve on 8/31/98 from the *two-factor model*