Partial Autocorrelation Function (PACF)

In practice, we will not know the order p of the AR process, but the partial autocorrelation function (PACF) will help to determine the order p.

1. For any possible order k of AR model, we consider solving the $k \times k$ system of Yule-Walker equations

$$\rho(j) = \phi_{1k}\rho(j-1) + \phi_{2k}\rho(j-2) + \cdots + \phi_{kk}\rho(j-k), \qquad j = 1, 2, \ldots, k.$$

2. In matrix form $P_k \Phi_k = \rho_k$, with solution $\Phi_k = P_k^{-1}\rho_k$, where

$$P_k = \begin{bmatrix} 1 & \rho(1) & \rho(2) & \ldots & \rho(k-1) \\ \rho(1) & 1 & \rho(1) & \ldots & \rho(k-2) \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \rho(k-1) & \rho(k-2) & \rho(k-3) & \ldots & 1 \end{bmatrix}, \Phi_k = \begin{bmatrix} \phi_{1k} \\ \phi_{2k} \\ \vdots \\ \phi_{kk} \end{bmatrix}, \rho_k = \begin{bmatrix} \rho(1) \\ \rho(2) \\ \vdots \\ \rho(k) \end{bmatrix}.$$

The last coefficient in the solution, ϕ_{kk}, is called the partial autocorrelation at lag k.

3. Important feature of PACF for AR(p) model

If process $\{Y_t\}$ is a AR(p) then PACF ϕ_{kk}=0 for $(k > p)$.

4. Sampling properties of sample PACF $\hat{\phi_{kk}}$ for AR(p) model.

In an AR(P) process, when $k > p$, $\hat{\phi_{kk}}$ has an approximately normal distribution with E($\hat{\phi_{kk}}$) ≈ 0, var ($\hat{\phi_{kk}}$) $\approx 1/T$. So $\pm\frac{2}{\sqrt{T}}$ provides limits to judge the statistical significance of $\hat{\phi_{kk}}$.

Model Building Strategy.

model specification → model fitting → model diagnostics

1, Check Stationary: check the behavior of ACF $\rho(k)$.

2. General features of ACF $\rho(k)$:

   – MA(q) model: ACF cuts off after lag q.

   – AR(p) model: ACF has form of exponential decay or damped sinusoid or a mixture of both.

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– ARMA(p,q) model: ACF can have patterns similar to those in corresponding AR model.

3. General features of PACF $\hat{\phi_{kk}}$:

   – MA(q) model: PACF can be exponential decay, damped sinusoid or a mixture of both.

   – AR(p) model: PACF cuts off after lag p.

   – ARMA(p.q) model: PACF can be exponential decay, damped sinusoid or a mixture of both.

4. Criteria of a good fitting model:

   – The residual behave like a white noise process.

   – The chosen model has smaller variance of the residuals.

   – The model should require the smallest number of parameters as possible that will adequately represent the data (principle of parsimony). Model selection criteria such as AIC and BIC can be used.

Exercises

Note that in ARMA or AR model, $\phi(B)Y_t = \delta + \theta(B)w_t$, $Y_t = \delta + \theta(B)w_t$, if process $\{Y_t\}$ has infinite MA representation. i.e., $Y_t = \mu + \psi(B)w_t$, $\mu = \frac{\delta}{1-\phi_1-\ldots-\phi_p}$.

1. From a series of length $T = 250$, we computed the sample autocorrelations of $r(1) = .626$, $r(2) = .260$,$r(3) = .066$, sample variance $c(0) = 10.745$ and $\bar{Y} = 9.429$. We assume AR(2) with constant term is appropriate. Obtain preliminary (method of moment) estimates of the parameters $\hat{\phi}_1$, $\hat{\phi}_2$ of the AR(2).

2. Suppose a time series of T=200 observation gave the sample ACF: $r(1) = .47$, $r(2) = -.07$,$r(3) = .09$,$r(4) = .05$, $r(5) = -.03$,$r(6) = .10$, $r(7) = -.03$ with sample variance $c(0) = 6.0$. Specify a low order ARMA.

3. Suppose a time series of T=250 observation gave the sample ACF: $r(1) = .70$, $r(2) = .47$,$r(3) = .34$,$r(4) = .22$, $r(5) = .16$,$r(6) = .10$, $r(7) = .06$ and PACF: $\hat{\phi_{11}} = .7$,$\hat{\phi_{22}} = -.039$, $\hat{\phi_{33}} = .05$, $\hat{\phi_{44}} = -.054$, $\hat{\phi_{55}} = .043$, $\hat{\phi_{66}} = -.041$,$\hat{\phi_{77}} = .007$, with sample variance $c(0) = 4.0$. Specify a low order ARMA.