Using Simulation for Option Pricing

John Charnes
U. Kansas School of Business
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Introduction

- Increased complexity of numerical computation in financial theory and practice has put demands on computational speed and efficiency
- Monte Carlo is useful for valuation of securities, estimation of their sensitivities, risk analysis, and stress testing of portfolios

Session Objectives

- Demonstrate how simulation is used for pricing derivatives
- Describe how variance reduction techniques increase precision of estimates without increasing number of runs
- Provide examples of various financial options for use by educators and others interested in applying simulation to derivative pricing with spreadsheet models

Agenda

- Overview
- Variance reduction and efficiency improvement
- Low-discrepancy sequences
- Conclusion

Overview

- Monte Carlo used for
  - Stochastic volatility applications
  - Valuation of mortgage-backed securities
  - Valuation of path-dependent options
  - Portfolio optimization
  - Interest-rate derivative claims
- Increased use by practitioners has sparked methodological developments in variance reduction and low-discrepancy sequences

Vocabulary

- *Risk-neutral pricing*
  - With no-arbitrage assumption, price of derivative security can be expressed as the expected value of its payouts discounted at risk-free rate of interest
- *Equivalent martingale measure*
  - Expectation is taken with respect to a transformation of the original probability measure

Risk-Neutral Pricing

- If option is European,

$$C_E = \mathrm{E}[e^{-rT}(S_T - K)^+]$$

  - Can be found by Black-Scholes Formula
- For American option,

$$C_A = \max_{\tau} \mathrm{E}[e^{-r\tau}(S_\tau - K)^+]$$

over all stopping times $\tau \le T$

  - Cannot be found by Black-Scholes

Simulating European Options

- Purpose
  - Even though simulation is not necessary to determine fair price of European options, it is used with European options *to test algorithms and variance reduction techniques*

How To Price European Options with Simulation

- Simulate future stock price using risk-free rate of growth and assumed level of volatility
- Evaluate discounted (at risk-free rate) cash flow for each simulated price
- Average discounted cash flows over iterations of simulation

Lognormal Model for Future Price

$$S_{t+\Delta t}^{(i)} = S_t \exp\left[\left(r - \frac{\sigma^2}{2}\right)\Delta t + \sigma\sqrt{\Delta t}Z^{(i)}\right]$$

for replication $i = 1,\ldots,n$; where

S_t is stock price at time t

r is the risk - free rate of interest

σ is the volatility

Δt is the time step

$Z^{(i)} \sim N(0,1)$

Example

- EuroCall.xls
- How many iterations (*n*) must be run to achieve a specified precision?

$$\bar{x} \pm 2\sqrt{\frac{\sigma^2}{n}}$$

- Precision is increased with larger *n* or smaller σ^2

Variance Reduction Techniques

- Antithetic Variates
- Control Variates
- Moment Matching
- Latin Hypercube Sampling
- Importance Sampling
- Conditional Monte Carlo
- Quasi-Monte Carlo

Antithetic Variates

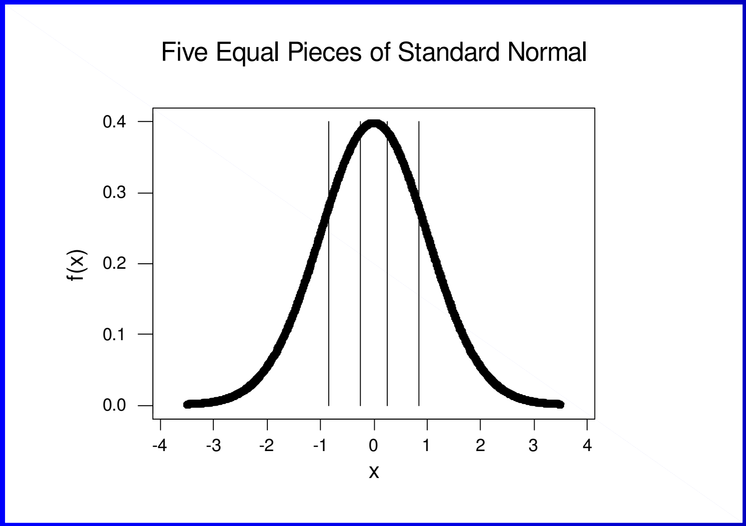
- Take average of two separate estimators that are designed to have negative correlation
- EuroCallAV.xls

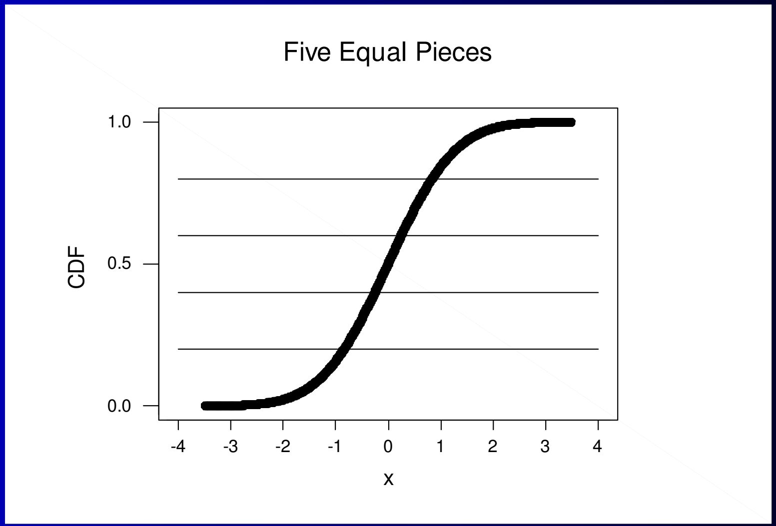
Control Variates

- Replace the evaluation of an unknown expectation with the evaluation of the difference between the unknown quantity and a related quantity whose expectation is known
- AsianCallCV.xls
  - Known expectation is price of an Asian option that pays off on geometric average
  - Unknown expectation is price of an Asian option that pays of on arithmetic average

Latin Hypercube Sampling

- Select $Z^{(i)}$'s randomly from each of k intervals having area under curve = $1/k$
- EuroCallLHS.xls





Moment-Matching

- EuroCallMM.xls
  - Generate sample terminal prices, then transform so that sample moments equal population moments
- Boyle et al. (1997) show that whenever a population moment is known, it's better to use it as a control variate than for Moment Matching

Low-discrepancy Sequences

- Discrepancy measures the extent to which points are evenly dispersed throughout a region---the more evenly dispersed the lower the discrepancy
- Low-discrepancy sequences are also known as quasi-random sequences even though they are not at all random

Quasi-random Sequence

- For any integer n, and any prime number r $\geq$ 2:
  - Expand n in terms of m places in base r

$$n = \sum_{j=0}^{m} a_j(n) r^j$$

  - Quasi-random number is "reflection about the decimal point"

$$\phi_r = \sum_{j=0}^{m} a_j(n) r^{-j-1}$$

Example (base 3):

n	Base 3	$\phi_3(n)$
1	01	1/3
2	02	2/3
3	10	1/9
4	11	4/9
5	12	7/9

Example (base 3):

n	Base 3	$\phi_3(n)$
6	20	2/9
7	21	5/9
8	22	8/9
9	100	1/27
10	101	4/27

– Added points "know" how to fill the gaps evenly

s-dimensional QR Sequence

- Let r be smallest prime number $\geq s$
  - Represent n in base r

$$n = \sum_{j=0}^{m} a_j^1(n) r^j$$

  - Find remaining elements recursively

$$a_j^n(n) = \sum_{i \le j}^{m} \binom{i}{j} a_i^{k-1}(n) r \bmod r$$

Quasi-Monte Carlo

- Gives spectacular reductions in MSE
- EuroCallQMC.xls

American Put Option

- A stock has price today = S_0
- A put option is available for purchase that gives owner the right to sell stock for strike price, K, at any time t, $0 \leq t \leq T$
- What is the fair value, P, of the put option?

American Put Option

- Early exercise feature makes valuation difficult

$$P = \max_{\tau} \mathrm{E}[e^{-r\tau}(K - S_{\tau})^{+}]$$

over all stopping times $\tau \leq T$

- In practice, find value of *Bermudan* put option, which can be exercised only at a finite number of opportunities, *k*, before expiration

Valuing Bermudan Put Options

- Analytical solution given by Geske and Johnson, JF 1984, for small *k*
- Simulation approach given by Broadie and Glasserman, JEDC 1997, for small *k,* and
- See also Fu, et al. (1999)
- Forward Monte Carlo method (Charnes and Shenoy 2000)
- Using OptQuest, package for stochastic optimization using tabu search (Glover 1997)

Free-Boundary Problem

- For each exercise opportunity, must estimate price below which put option should be exercised and above which put option should be held
- BermuPutOptAV.xls
- Uses tabu search to identify an optimal policy, then a final set of iterations to estimate value under optimal policy

Conclusion

- Interest in Monte Carlo methods for option pricing increasing because of its flexibility in handling complex derivatives
- As workstations get faster, will be able to value increasingly complex financial instruments in smaller amounts of computer time