Advanced Mathematics of Finance Honours Project: *The Pricing of Cliquets*

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Abstract

A Cliquet is a type of structured product which is equivalent to a collection of forward starting at-the-money options where the strike is reset periodically to the current level of the underlying. There are a number of problems associated with the pricing of Cliquets, mainly associated with estimating the forward volatility curves of the underlying.
We examine two approaches to pricing Cliquets: An analytic solution is derived in the first approach under the Black-Scholes framework. In the second approach we deal with the pricing using stochastic volatility model. In particular, a risk-neutral representation is derived for the stochastic volatility model. The effectiveness of this derived stochastic volatility model is demonstrated in a Monte Carlo study.

Contents

Chapter 1

Introduction

1.1 The Cliquet Option

Cliquets, Cliquet Options or Ratchets all refer to a type of retail structured product with underlying usually some equity or equity index. A Cliquet can be defined as a series of forward starting at-the-money options, where the strike is periodically reset to the current level of the underlying. Payoffs can be taken at the reset dates or accumulated to pay at the maturity of the Cliquet. Cliquets enable investors to lock in gains at pre-specified dates in the future when the option's strike is effectively reset.

The Cliquet is suitable for investors with a medium term investment horizon. It is less risky than ordinary medium term options, as there is less specific risk i.e. the reset facility gives the buyer a 'second' and 'third' chance. This increases the chance of payout, but must be balanced with the higher premium cost. As a series of 'pre-purchased' options, the Cliquet is attractive to passive investors as it requires no intermediate management. They have traditionally been attractive to retail and private investors when embedded in deposits and bonds [e.g. Accumulators and Predators] as they provide a low risk (capital guaranteed) exposure to equity and Bond markets. Investors may use Cliquets to take advantage of assumptions about future volatility.

1.2 Objectives of the Research

Cliquets have been traded for over a decade although their pricing has always been debated. The problem arises in the path-dependent nature of the

Cliquet payoff - for example, in a Cliquet that resets annually, the strike price after 3 years is defined to be the spot price of the underlying after 2 years. If the volatility associated with the underlying changes after 2 years, this would give a different payoff to the Cliquet had there been no change in the volatility.
This report aims to deal with some of the problems associated with pricing Cliquets. Two approaches are considered:

1. Pricing Cliquets in the Black-Scholes framework:
   We derive an analytic solution to the Cliquet option by applying the Equivalent Martingale Pricing approach.

2. Pricing Cliquets using a 'Risk Neutral' Stochastic Volatility model
   We consider a stochastic volatility term in the SDE for the stock price and derive a risk-neutral representation of the model by constructing a riskless portfolio.

The next two chapters of this report discuss the 2 approaches above in greater detail. Chapter 4 gives a demonstration of the effectiveness of the derived Stochastic Volatility model in comparison to the Black-Scholes model used in the first approach. The remainder of the report, deals with issues of calibratrion and methods used to price the Cliquets in the Stochastic Volatility model. The Matlab functions used for pricing are listed in the Appendix.

Chapter 2

An Analytic Solution to the Cliquet Price

2.1 Forward Start Options

Let S_t denote the value of the underlying instrument at time t. *Forward start options* are options whose strike is unknown at t_0 (say) - the time of pricing, and will be determined at some later date $t_i > t_0$. The strike is determined according to some pre-agreed formula (e.g. at-the-money, 10% in-the-money,etc.). Thus, an at-the-money forward start option with strike determined at time t_i and maturity $t_j > t_i$ has the following payoff at time t_j:

$$payoff = max\left[S_{t_j} - S_{t_i}, 0\right] \quad (call) \tag{2.1}$$

$$payoff = max\left[S_{t_i} - S_{t_j}, 0\right] \quad (put) \tag{2.2}$$

2.2 Derivation of the Pricing Formula

Let S_t denote the value of the underlying instrument at time t. Consider a Cliquet of calls - with n resets and maturity T. Let the payoffs be at reset times denoted by $t_1, t_2, \ldots, t_n$ such that $0 = t_0 < t_1 < \cdots < t_{n-1} < t_n = T$.

Then the above Cliquet has the following n payoffs:

$$\begin{aligned} v_1 &= max\,[S_{t_1} - S_{t_0}, 0] \;\; at\; t_1 \\ v_2 &= max\,[S_{t_2} - S_{t_1}, 0] \;\; at\; t_2 \\ &\vdots \\ v_n &= max\left[S_{t_n} - S_{t_{n-1}}, 0\right] \;\; at\; t_n = T \end{aligned} \quad (2.3)$$

We may replicate this multiple payoff (2.3) by constructing a portfolio consisting of a standard at-the-money call option and a series of forward-starting at-the-money call options i.e. :

1. A standard at-the-money call option with maturity t_1
2. A forward start at-the-money call option with maturity t_i and strike reset time t_{i-1} for each $i = 2 \ldots n$

By No-Arbitrage, the value of the Cliquet at time t_0 must equal the value of the above replicating porfolio. Note that in an identical fashion, we may derive a replicating porfolio for a Cliquet of puts. The pricing of the Cliquet is thus reduced to pricing the forward start options.

Pricing the Forward Start Option

Consider the traditional Black-Scholes framework for pricing derivatives:

1. No transaction costs
2. No arbitrage opportunities
3. Complete Markets
4. Constant volatilty
5. No default risk
6. Constant risk-free rate of interest $= r$

Let S_t denote the value of the underlying instrument at time t. We have that

$$under\ \mathbf{P}: \qquad dS_t = \mu S_t dt + \sigma S_t dW_t$$

Choose the numeraire asset to be the money market account with value $A(t) = e^{rt}$. Using the *Equivalent Martingale Pricing* approach we have that:

$$under\ \mathbf{Q}: \qquad dS_t = rS_t dt + \sigma S_t d\tilde{W}_t$$

where **Q** represents the Risk-Neutral Measure.

Consider the first forward start option used to replicate the Cliquet payoff in (2.3) i.e. with maturity at t_2 and strike determination at time t_1. Let $c(t)$ denote the value of this forward start call at time t. Then

$$c(t_2) = max\,(S_{t_2} - S_{t_1}, 0)$$

By the *Equivalent Martingale Pricing* approach we have that the relative derivative price:

$$\left\{ \frac{c(t)}{A(t)} \right\}_{t\varepsilon[0,T]}$$

is a **Q**-martingale.
Using this property, we first examine the value of the relative derivate price at time t_1:

$$\begin{aligned} \frac{c(t_1)}{A(t_1)} &= \mathbf{E}^{\mathbf{Q}}\left[\frac{c(t_2)}{A(t_2)}|\mathcal{F}_{t_1}\right] && (2.4) \\ &= \mathbf{E}^{\mathbf{Q}}\left[\frac{max\,(S_{t_2} - S_{t_1}, 0)}{A(t_2)}|S_{t_1}, A_{t_1}\right] \quad S_{t_1}, A_{t_1} \subset \mathcal{F}_{t_1} \\ \Rightarrow\ c(t_1) &= e^{-r(t_2-t_1)}\mathbf{E}^{\mathbf{Q}}\left[max\,(S_{t_2} - S_{t_1}, 0)\,|S_{t_1}, A_{t_1}\right] && (2.5) \end{aligned}$$

The RHS of equation (2.5) is simply the Black-Scholes price at time t_1 of an at-the-money call option with time to maturity $t_2 - t_1$. We may thus evaluate $c(t_1)$ as follows:

$$c(t_1) = S_{t_1}\left[N(d_1) - e^{-r(t_2-t_1)}N(d_2)\right] \qquad (2.6)$$

$$where\ \ d_1 = \frac{(r + \frac{1}{2}\sigma^2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} = \frac{(r + \frac{1}{2}\sigma^2)\sqrt{t_2 - t_1}}{\sigma} \qquad (2.7)$$

$$and\ \ d_2 = \frac{(r - \frac{1}{2}\sigma^2)(t_2 - t_1)}{\sigma\sqrt{t_2 - t_1}} = \frac{(r - \frac{1}{2}\sigma^2)\sqrt{t_2 - t_1}}{\sigma} \qquad (2.8)$$

Equation (2.6) thus gives us the value for the forward start call at time t_1. To obtain its value at time t_0 we use the martingale property once more:

$$\begin{aligned} \frac{c(t_0)}{A(t_0)} &= \mathbf{E}^{\mathbf{Q}}\left[\frac{c(t_1)}{A(t_1)}|\mathcal{F}_{t_0}\right] && (2.9) \\ &= e^{-rt_1}\mathbf{E}^{\mathbf{Q}}\left[c(t_1)|S_{t_0}, A_{t_0}\right] \quad S_{t_0}, A_{t_0} \subset \mathcal{F}_{t_0} \end{aligned}$$

substituting (2.6) we obtain

$$\begin{aligned}
c(t_0) &= e^{-rt_1}\mathbf{E}^{\mathbf{Q}}\left[S_{t_1}\left[N(d_1) - e^{-r(t_2-t_1)}N(d_2)\right]|S_{t_0}, A_{t_0}\right] \\
&= e^{-rt_1}\left[N(d_1) - e^{-r(t_2-t_1)}N(d_2)\right]\mathbf{E}^{\mathbf{Q}}\left[S_{t_1}|S_{t_0}, A_{t_0}\right] \\
&= e^{-rt_1}\left[N(d_1) - e^{-r(t_2-t_1)}N(d_2)\right]S_{t_0}e^{t_1} \quad \text{,since } S_{t_1} \text{ is lognormally distributed}
\end{aligned}$$

$$\Rightarrow c(t_0) = S_0\left[N(d_1) - e^{-r(t_2-t_1)}N(d_2)\right]$$

We now have the price of the forward start call option at $t = 0$. Generalising the above result for maturity t_i and strike determination time t_{i-1}, we get:

$$c(0) = S_0\left[N(d_{1,i}) - e^{-r(t_i-t_{i-1})}N(d_{2,i})\right] \tag{2.10}$$

$$where \;\; d_{1,i} = \frac{(r+\frac{1}{2}\sigma^2)\sqrt{t_i - t_{i-1}}}{\sigma} \tag{2.11}$$

$$and \;\; d_{2,i} = \frac{(r-\frac{1}{2}\sigma^2)\sqrt{t_i - t_{i-1}}}{\sigma} \tag{2.12}$$

Note that:

$$c(0) = c_{BS}(S_0, K = S_0, \tau = t_i - t_{i-1})$$

where c_{BS} is Black-Scholes price of a standard call option with strike K and maturity τ.

We may now obtain the price of the Cliquet by calculating the individual prices of each of the forward start options. Hence, the value of the Cliquet of calls defined at the start of Section(2.2) can be defined as:

$$price = S_0 \sum_{i=1}^{n}\left[N(d_{1,i}) - e^{-r(t_i-t_{i-1})}N(d_{2,i})\right]$$

where d_1 and d_2 are as defined above.

Using put-call parity, we derive the value of a Cliquet of puts as:

$$price = S_0 \sum_{i=1}^{n}\left[e^{-r(t_i-t_{i-1})}N(-d_{2,i}) - N(-d_{1,i})\right]$$

The above pricng formulae may be applied to intervals $(t_i - t_{i-1})$ not necessarily of the same length. In the market, it is most common for the reset period to be fixed i.e.

$$(t_i - t_{i-1}) = \Delta t = \frac{T}{n} \qquad \forall \; i = 1 \ldots n$$

This further simplifies the solution, since the forward start option prices are actually functions of $(t_i - t_{i-1}) = \Delta t$ rather than the times t_i and t_{i-1} explicitly.

So, for a Cliquet of calls with maturity T and n resets, the price is given by:

$$price = nS_0 \left[N(d_1) - e^{-r\Delta t} N(d_2)\right] \tag{2.13}$$

and similarly for a Cliquet of puts:

$$price = nS_0 \left[e^{-r\Delta t} N(-d_2) - N(-d_1)\right] \tag{2.14}$$

$$where \;\; d_1 = \frac{(r + \frac{1}{2}\sigma^2)\sqrt{\Delta t}}{\sigma} \tag{2.15}$$

$$and \;\; d_2 = \frac{(r - \frac{1}{2}\sigma^2)\sqrt{\Delta t}}{\sigma} \tag{2.16}$$

Chapter 3

A 'Risk Neutral' Stochastic Volatility Model

It it clearly observed that market volatility is stochastic. The Black-Scholes model does not allow for this, and results in mis-pricing derivatives, especially those such as Cliquets which have forward volatility exposure. Due to the path dependent nature of the Cliquet, we must consider a model for the term structure of volatility. Hence, a stochastic volatility model will enable us to give a better estimation of the price of the Cliquet.

The major problem associated with this type of model is that volatilty itself is not a traded asset. This forces us into the realm of incomplete markets. It is possible, however, to overcome this, and an approximation is made to lead us to Risk Neutral Evaluation.

3.1 Construction of the Model

Let the risk-free rate of interest r be constant. Let S_t denote the value of the underlying instrument at time t with volatility σ_t, where σ_t follows a stochastic process.
Consider the following SDE's for S_t and σ_t under $\mathbf{P}$:

$$dS_t = \mu S_t dt + \sigma_t S_t dW_{t,1} \tag{3.1}$$

$$d\sigma_t = \gamma\sigma_t dt + \rho\beta\sigma_t dW_{t,1} + \sqrt{1-\rho^2}\beta\sigma_t dW_{t,2} \tag{3.2}$$

where ρ represents the correlation between the underlying and its volatilty, and β the volatility of volatility.

Let $f = f(t, S, \sigma)$ be the value of a derivative at time t with underlying

S_t.

To derive the processes for S_t and σ_t under $\mathbf{Q}$ in the Risk-Neutral world, we follow a similar argument to the orignal derivation of the Black-Scholes PDE. We formulate a risk-neutral portfolio (Π) consisting of:

1. A long position in the derivative with value $f = f(t, S, \sigma)$
2. A short position in Δ units of the underlying
3. A short position in Φ units of a short-term call option, maturity Δt, on the underlying with value $C = C(t, S, \sigma)$

The value of the portfolio at time t is given by:

$$\Pi = f - \Delta S - \Phi C \tag{3.3}$$

Consider the change in the portfolio over a infinitesimal time period dt:

$$d\Pi = df - \Delta dS - \Phi dC \tag{3.4}$$

We apply Ito's Lemma to obtain equations for df and dC:

$$df = \frac{\partial f}{\partial t}dt + \frac{\partial f}{\partial S}dS + \frac{\partial f}{\partial \sigma}d\sigma + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} + \rho\beta\sigma^2 S \frac{\partial^2 f}{\partial S \partial \sigma} + \frac{1}{2}\beta^2\sigma^2 \frac{\partial^2 f}{\partial \sigma^2}\right) dt \tag{3.5}$$

$$dC = \frac{\partial C}{\partial t}dt + \frac{\partial C}{\partial S}dS + \frac{\partial C}{\partial \sigma}d\sigma + \left(\frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} + \rho\beta\sigma^2 S \frac{\partial^2 C}{\partial S \partial \sigma} + \frac{1}{2}\beta^2\sigma^2 \frac{\partial^2 C}{\partial \sigma^2}\right) dt \tag{3.6}$$

To simplify notation, we define the differential operator ℓ as:

$$\ell \equiv \frac{1}{2}\sigma^2 S^2 \frac{\partial^2}{\partial S^2} + \rho\beta\sigma^2 S \frac{\partial^2}{\partial S \partial \sigma} + \frac{1}{2}\beta^2\sigma^2 \frac{\partial^2}{\partial \sigma^2} \tag{3.7}$$

Re-writing (3.5) and (3.6), with derivatives written as subscripts we obtain:

$$df = f_t dt + f_S dS + f_\sigma d\sigma + \ell f dt \tag{3.8}$$

$$dC = C_t dt + C_S dS + C_\sigma d\sigma + \ell C dt \tag{3.9}$$

Substituting (3.5) and (3.6) into (3.4), we obtain:

$$d\Pi = f_t dt + f_S dS + f_\sigma d\sigma + \ell f dt - \Delta dS - \Phi\left(C_t dt + C_S dS + C_\sigma d\sigma + \ell C dt\right) \tag{3.10}$$

Where ℓ is the differential operator defined in (3.7). In order to obtain a risk-neutral setting, we make an **approximation**:
We assume the value of the short-term call option to be the *Black-Scholes* price. A riskless portfolio is now obtained by choosing appropriate values for Δ and Φ in (3.10) i.e. we set:

$$\Phi = \frac{f_\sigma}{C_\sigma} \tag{3.11}$$

$$\Delta = f_S - \Phi C_S = f_S - \frac{C_S}{C_\sigma} f_\sigma \tag{3.12}$$

(3.10) becomes:

$$d\Pi = (f_t + \ell f - \Phi(C_t + \ell C))\, dt \tag{3.13}$$

By No-Arbitrage, the return on a riskless portfolio must equal the risk-free rate. Thus, we must have:

$$d\Pi = r\Pi dt \tag{3.14}$$

$$\Rightarrow \quad (f_t + \ell f - \Phi\,(C_t + \ell C))\, dt = r\,(f - \Delta S - \Phi C)\, dt \tag{3.15}$$

Substituting for Δ

$$\begin{aligned} \Rightarrow \quad f_t + \ell f &= \Phi(C_t + \ell C) + r\,(f - [f_S - \Phi C_S]S - \Phi C) \\ &= \Phi\,[C_t + \ell C - rC + rSC_S] + rf - rf_S \end{aligned} \tag{3.16}$$

Consider $C_t + \ell C$:

$$C_t + \ell C = C_t + \frac{1}{2}\sigma^2 S^2 C_{SS} + \rho\beta\sigma^2 S C_{S\sigma} + \frac{1}{2}\beta^2\sigma^2 C_{\sigma\sigma} \tag{3.17}$$

Recall our approximation, C is the Black-Scholes price, and must hence satisfy the Black-Scholes PDE:

$$C_t + rSC_S + \frac{1}{2}\sigma^2 S^2 C_S S - rC = 0 \tag{3.18}$$

$$\Rightarrow \quad C_t + \frac{1}{2}\sigma^2 S^2 C_S S = rC + rSC_S \tag{3.19}$$

Substituting (3.19) into (3.17) we obtain:

$$C_t + \ell C = rC + rSC_S + \rho\beta\sigma^2 S C_{S\sigma} + \frac{1}{2}\beta^2\sigma^2 C_{\sigma\sigma} \tag{3.20}$$

Now substituting (3.20) into (3.16) we obtain:

$$f_t + \ell f = \Phi\left[\rho\beta\sigma^2 S C_{S\sigma} + \frac{1}{2}\beta^2\sigma^2 C_{\sigma\sigma}\right] + rf - rf_S \tag{3.21}$$

Recall $\Phi = \frac{f_\sigma}{C_\sigma}$, substituting this in and re-arranging:

$$f_t + \ell f - \rho\beta\sigma^2 S \frac{C_{S\sigma}}{C_\sigma} f_\sigma - \frac{1}{2}\beta^2\sigma^2 \frac{C_{\sigma\sigma}}{C_\sigma} f_\sigma + r f_S - r f = 0 \tag{3.22}$$

Now, by our approximation:

$$\begin{aligned} C &= S\left[N(d_1) - X e^{-r\Delta t} N(d_2)\right] &(3.23)\\ where\ \ d_1 &= \frac{\log\left(\frac{S}{X}\right) + (r + \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} &(3.24)\\ and\ \ d_2 &= \frac{\log\left(\frac{S}{X}\right) + (r - \frac{1}{2}\sigma^2)\Delta t}{\sigma\sqrt{\Delta t}} &(3.25) \end{aligned}$$

The following Greeks may then be substituted into (3.22):

$$\begin{aligned} C_S &= N(d_1) &(3.26)\\ C_\sigma &= S\sqrt{\Delta t}\, N'(d_1) &(3.27)\\ C_{S\sigma} &= N'(d_1)\left[-\frac{[\log\left(\frac{S}{X}\right) + r\Delta t]}{\sigma^2\sqrt{\Delta t}} + \frac{1}{2}\sigma\sqrt{\Delta t}\right] &(3.28)\\ C_{\sigma\sigma} &= S\sqrt{\Delta t} N''(d_1)\left[-\frac{[\log\left(\frac{S}{X}\right) + r\Delta t]}{\sigma^2\sqrt{\Delta t}} + \frac{1}{2}\sigma\sqrt{\Delta t}\right] &(3.29) \end{aligned}$$

$$\text{where} \qquad N'(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2}$$

In order to simplify the resulting equation, we choose:

$$X = S e^{r\Delta t}$$

This gives us:

$$\begin{aligned} d_1 &= \frac{1}{2}\sigma\sqrt{\Delta t} \quad\ and \quad\ d_2 = -\frac{1}{2}\sigma\sqrt{\Delta t} &(3.30)\\ C_{S\sigma} &= N'(d_1)\left[\frac{1}{2}\sqrt{\Delta t}\right] &(3.31)\\ C_{\sigma\sigma} &= S\sqrt{\Delta t} N''(d_1)\left[\frac{1}{2}\sqrt{\Delta t}\right] &(3.32) \end{aligned}$$

Now

$$N''(d_1) = N'(d_1).(-d_1)$$

$$\Rightarrow \quad C_{\sigma\sigma} = S\sqrt{\Delta t}N'(d_1)(-\frac{1}{2}\sigma\sqrt{\Delta t})\left[\frac{1}{2}\sqrt{\Delta t}\right]$$
$$= -\frac{1}{4}S\sigma N'(d_1)(\Delta t)^{\frac{3}{2}} \quad (3.33)$$

We substitute the above Greeks into (3.22) and neglect $C_{\sigma\sigma} \approx 0$ since it contains a higher order term of Δt. The resulting equation is:

$$f_t + \ell f - \frac{1}{2}\rho\beta\sigma^2 S f_\sigma + r f_S - r f = 0 \quad (3.34)$$

Writing in full:

$$f_t + r f_S - \frac{1}{2}\rho\beta\sigma^2 S f_\sigma + \frac{1}{2}\sigma^2 S^2 f_{SS} + \rho\beta\sigma^2 S f_{S\sigma} + \frac{1}{2}\beta^2\sigma^2 f_{\sigma\sigma} - r f = 0 \quad (3.35)$$

Then $f(t, S, \sigma)$ satisfies (3.35) with some boundary condition

$$f(T, S, \sigma) = g(S)$$

By the Feynman-Kac Theorem, $f(t, S, \sigma)$ has the representation:

$$f(t, S, \sigma) = \mathbf{E}^{\mathbf{Q}}\left[e^{-r(T-t)}g(S_T)|S_t, \sigma_t\right] \quad (3.36)$$

where $\mathbf{Q}$ denotes a measure under which S_t and σ_t follow the following processes:

$$dS_t = rS_t dt + \sigma_t S_t d\tilde{W}_{t,1} \quad (3.37)$$
$$d\sigma_t = -\frac{1}{2}\rho\beta\sigma^2 dt + \rho\beta\sigma_t d\tilde{W}_{t,1} + \sqrt{1-\rho^2}\beta\sigma_t d\tilde{W}_{t,2} \quad (3.38)$$

We have thus (under our assumptions) derived a representation of the stochastic volatility model in the Risk-Neutral world. Not surprisingly, the drift term for the underlying instrument/security is the risk-free rate r. It is also interesting to observe that the model is independent of the drift term of the stochastic volatiltiy σ_t under $\mathbf{P}$ - the real world measure.

Chapter 4

A Monte Carlo Study of the Stochastic Volatility Model

Since the Risk-Neutral Stochastic Volatility [SV] Model derived in the last chapter has no analytic solution, we employ the technique of Monte Carlo simulation to generate sample paths of the underlying price. These paths will be used to price path-dependent derivatives such as Cliquets in particular, although, the model can be used to value *any* type of derivative on a single underlying instrument paying no dividends.

We fist demonstrate some of the features of the model [1].

4.1 Generating Stock Prices

A Monte Carlo approach is implemented to generate sample paths for S_t and σ_t over $[0, T]$. We first divide the interval into n evenly spaced time periods each of length $\Delta t = \frac{T}{n}$. We apply the Euler-Maruyama method to (3.38), and generate values for σ_t by using the recursive equation:

$$\sigma_{i+1} = \sigma_i - \frac{1}{2}\rho\beta\sigma_i^2\Delta t + \rho\beta\sigma_i\sqrt{\Delta t}\,\epsilon_{1,i} + \sqrt{1-\rho^2}\beta\sigma_i\sqrt{\Delta t}\,\epsilon_{2,i} \qquad (4.1)$$
$$\forall i = 0\ldots(n-1)$$

We generate stock price paths using the recursive equation:

$$S_{i+1} = S_i exp\left((r - \frac{1}{2}\sigma_i^2)\Delta t + \sigma_i\sqrt{\Delta t}\,\epsilon_{1,i}\right) \qquad (4.2)$$

[1] In the analyis following, the underlying will also be referred to as the stock price

where σ_i = value of σ_t after i periods
where S_i = value of S_t after i periods
and $\epsilon_{1,i}, \epsilon_{2,i} \sim \mathcal{N}(0, 1)$

4.2 Implied Distributions

The probability distribution associated with possible stock prices in a model can be referred as the implied distribution of the model. In the commonly used Black-Scholes model, the implied distribution is the Lognormal distribution. We note however that this distribution is not observed with actual prices in the market. This is due to the so called **'fat tail'** effect of the market i.e. the market expects tail events to occur with a far greater probability than is implied by lognormally distributed prices. This type of distribution in the market is termed leptokurtotic (excessively peaked and fat-tailed). As a result of this, options which payoff in these circumstances need to be priced higher than the price implied by the Black-Scholes model.

We consider the above problem associated with the lognormal distribution and therefore compare its distribution with that obtained from the Stochastic Volatility model developed. Various parameters are considered to reveal the implied distributions that are possible using our Stochasic Volatility model.

The histograms in figure (4.2) are generated from terminal values of stock price paths generated using a Monte Carlo simulation. The model parameters used in the simulations were:

Parameter	Value
S_0	100
r	12%
σ_0	30%
T	1 year

By examining the histograms produced in figure (4.2), it is apparent that our SV model is capable of producing various types of leptokurtotic distributions by varying the parameters ρ and β. Note how the implied distrtribution shifts to the left when ρ is increased.Increasing β [volatility of volatility] has the effect of increasing the peak of the distribution.

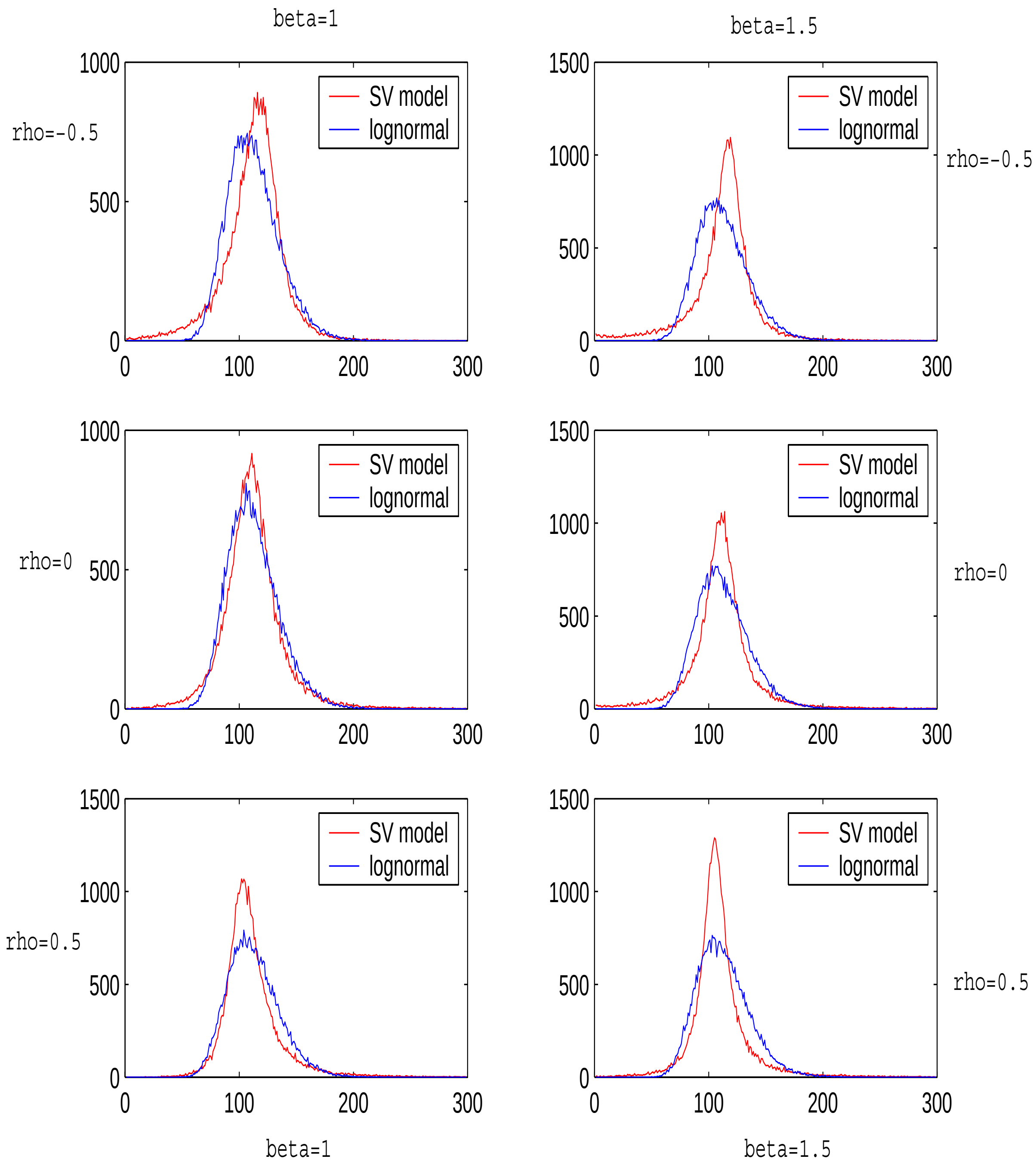

Figure 4.1: Histograms produced using the SV model

4.3 Implied Volatilty

The market prices of vanilla options [puts/calls] most often differ from the corresponding option price obtained from the Black-Scholes equation. The implied volatility of an option can be defined as that volatility σ [constant] which satisfies:

$$C_{BS}(S_0, r, \sigma, T, K) = C_{Market}$$

where C_{Market} is the observed market price of the option and C_{BS} is the corresponding Black-Scholes price of the option.

4.3.1 The Implied Volatilty Smile

By calculating implied volatilities of options in the market with different strikes, we obtain what is known as the implied volatility "smile". Cliquets are a collection of **foward-starting** options. This means that Cliquets have exposure to changes in forward implied volatilty. Our SV model has the ability to generate a option prices according to some type of volatility smile which evolves over time - this has obvious benefits in pricing Cliquets as well as any derivatives which have exposure to future implied volatility changes. To demonstrate this, a series of stock prices paths were first generated up to time T by our SV model using Monte Carlo simulation. A series of vanilla call options with different strikes were then priced using the terminal values $[S_T]$ for the stock. Finally, the resulting vanilla call prices were inverted (using a numerical procedure) to obtain implied volatilities at each level of the strike. The results for different values of ρ and β are shown in figure (4.3.1).The following parameters were used:

Parameter	Value
S_0	100
r	12%
σ_0	30%
T	0.5 yrs

Figure (4.2) shows the variation in the implied volatility curve for different correlation ρ. We observe that, in general, negative correlations correspond to negative skewness; and positive correlation for positive skewness.
Figure (4.3) illustrates the effect of changing β. Increasing β has the effect of increasing the curvature of the smile and thus shifting its center.

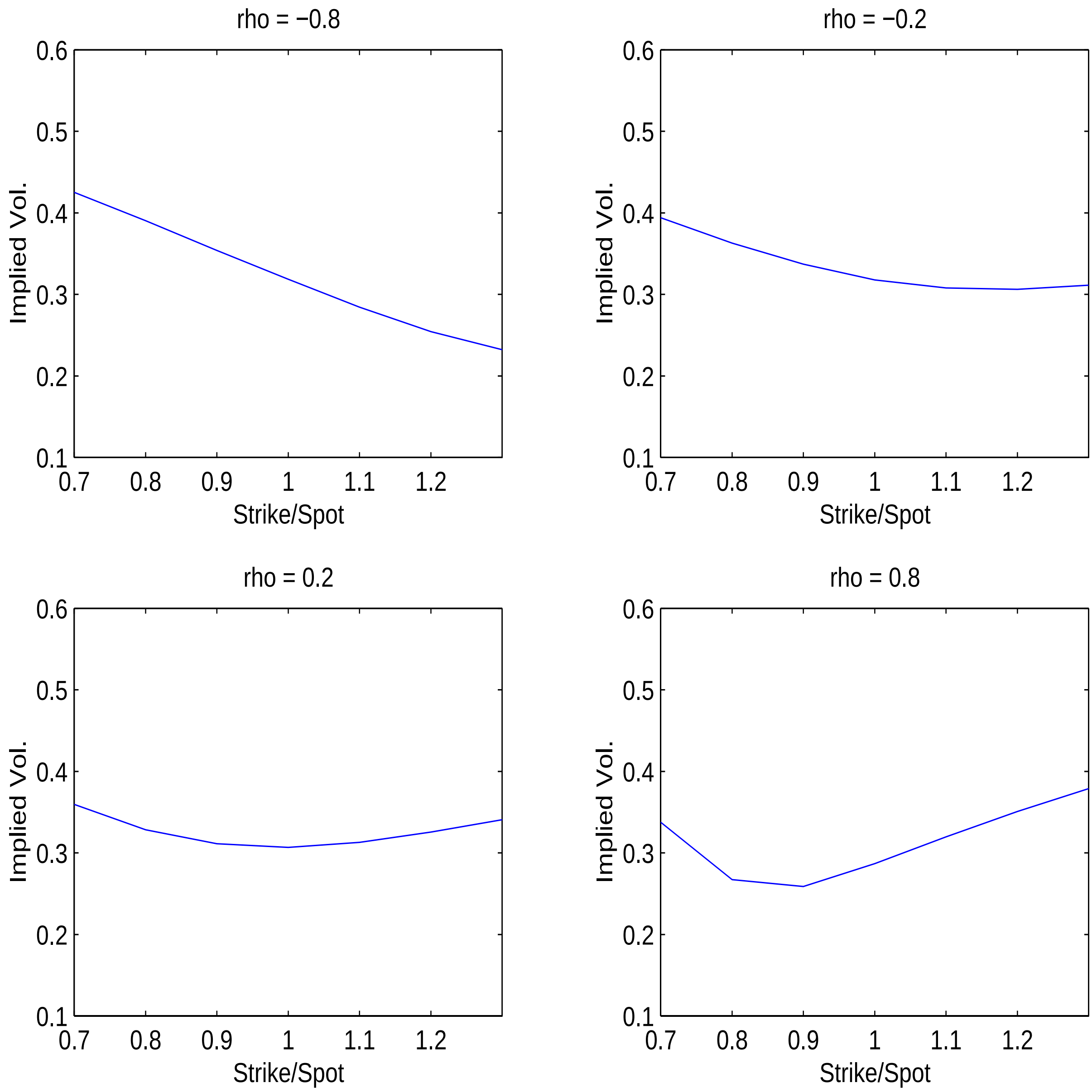

Figure 4.2: Implied Volatility Smiles using SV model for different values of ρ with $\beta = 1$

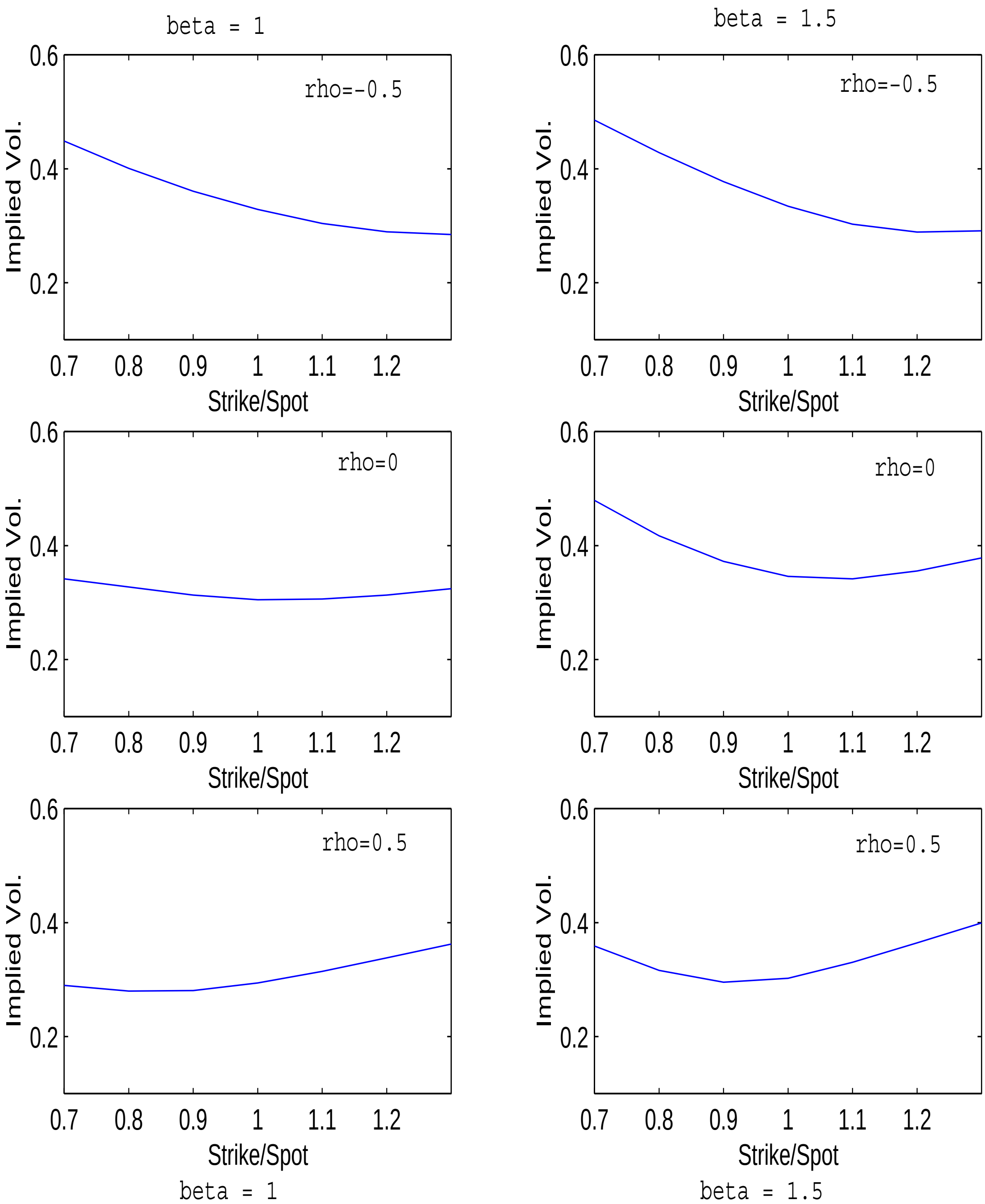

Figure 4.3: Implied Volatility Smiles using SV model for different values of ρ and β

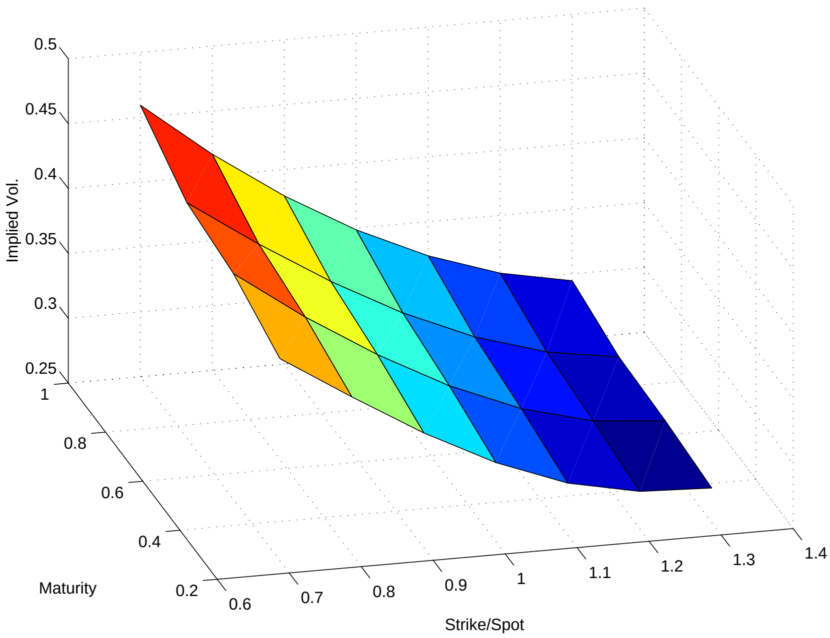

Figure 4.4: Implied Volatility Surface generated up to $T = 1$ year using SV model with $\rho = -0.5$ and $\beta = 1$

4.3.2 The Implied Volatilty Surface

Each implied volatility smile corresponds to a set of options with a specific maturity. This curve evolves as the maturity of the options changes. We obtain an implied volatility surface by combining several volatility smiles for options of increasing maturity. The existence of an implied volatility surface in the market is evidence of the term structure of volatility. Our SV model

This phenomenon must be considered when choosing parameters for the model.

We illustrate the term structure of volatility generated by our SV model by constructing implied volatility surfaces with different parameter combinations in figures (4.4),(4.5),(4.6) and (4.7).

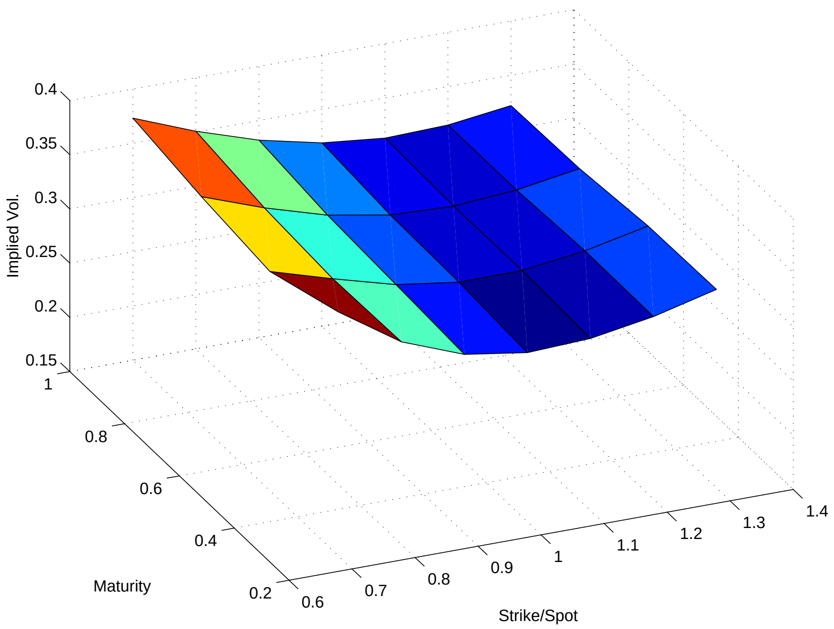

Figure 4.5: Implied Volatility Surface generated up to $T = 1$ year using SV model with $\rho = 0$ and $\beta = 1$

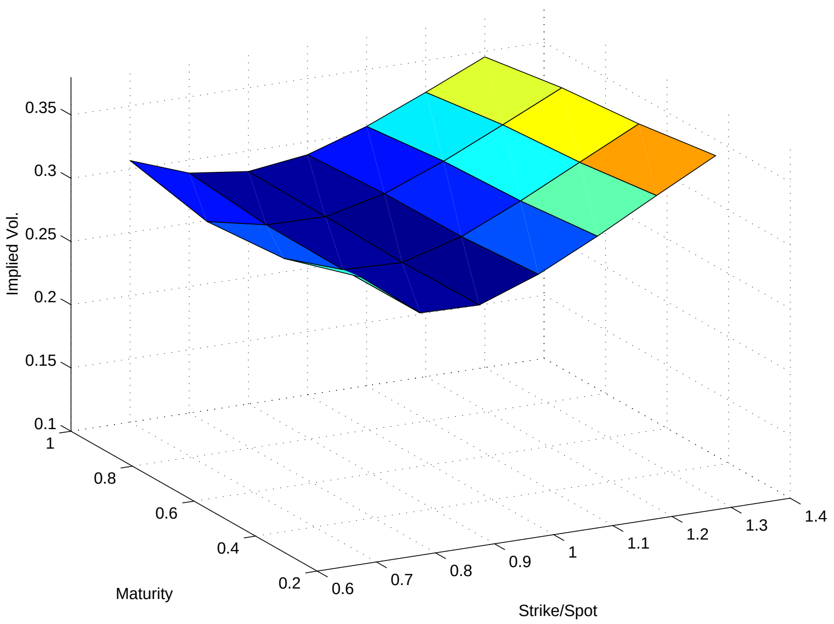

Figure 4.6: Implied Volatility Surface generated up to $T = 1$ year using SV model with $\rho = 0.5$ and $\beta = 1$

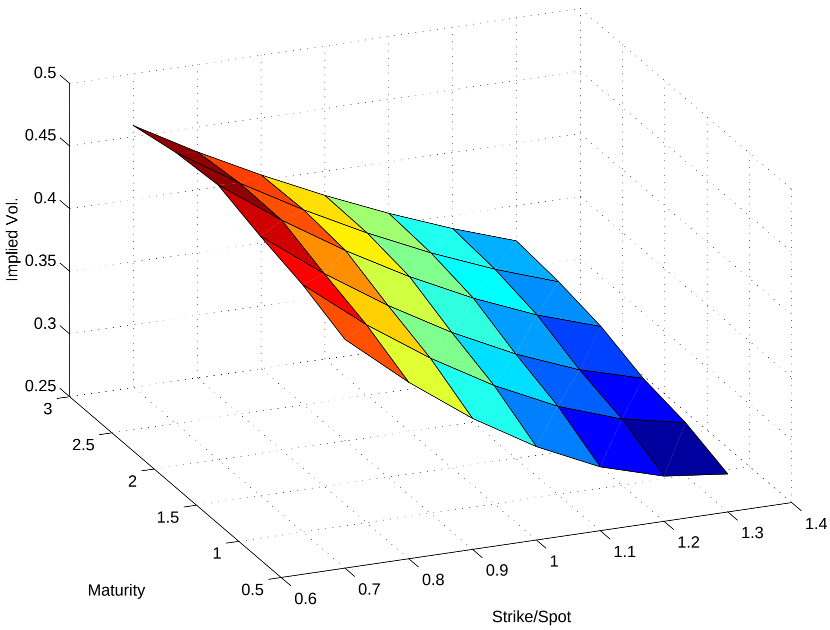

Figure 4.7: Implied Volatility Surface generated up to $T = 3$ years using SV model with $\rho = -0.5$ and $\beta = 1$

Chapter 5

Calibration & Pricing

5.1 Calibration

As demonstrated in the last chapter, the SV model derived is capable of producing a variety of implied volatility smiles/skews. In our aim to price Cliquets more accurately, we require a model for the evolution of the forward implied volatility that reflects the current market situation. We must therefore seek to calibrate our model to match(approximately) the current implied volatility surface of the market. In this way, our model will become consistent with the current market prices of vanilla puts and calls. This formulation has useful implications for hedging when the model is used to calculate Greeks, as many exotic derivatives such as Cliquets are hedged using vanilla puts and calls.

5.2 Pricing

The Cliquet pricing is performed using the Monte Carlo approach discussed in Section (4.1). The Matlab function `SVmodel.m` generates an m x n matrix containing m rows of stock-price paths each consisting of n stock prices over the period $[0, T]$. The exprected payoffs of the Cliquet are then calculated from this matrix and discounted at the risk-free rate r. The function `cliquetSV.m` [listed in the appendix] generates Cliquet prices using the above approach.

Chapter 6

Conclusion

The Black-Scholes framework is widely used in finance - it is easy to use and understand. There is, however, sufficient evidence in the existence of an implied volatility surface in the market, to demonstrate the deviation of market prices from their Black-Scholes equivalents. Adding to this, the uncertainty associated with changing volatility in the market, and we find that in reality, stock prices are not lognormally distributed.

We have developed a model that takes the above into consideration, and as result, is capable of generating a term structure of volatility. By choosing the model parameters carefully, we may produce an implied volatility surface that resembles that of the market. This is crucial in the interest of pricing Cliquets accurately since they are directly exposed to the stochastic nature of the implied volatility surface.

The Stochastic Volatility model derived in this report thus gives us a far better representation of market observed prices for the underlying and will thereby give a better estimation of the Cliquet premium.

Appendix A

Matlab Code

The following Matlab functions as listed:

1. `cliquetBS.m` :
   Computes the Price of a Cliquet in the Black-Scholes framework
2. `SVModel.m` :
   Returns an m x n matrix of stock price paths using the SV Model
3. `cliquetSV.m` :
   Computes the Price of a Cliquet using the SV Model

```
function [call_price,put_price] = cliquetBS(s0,r,sig,T,n)

% Generates the price of Cliquet with maturity T and n resets over [o,T]
% where s0 = initial value of underlying
% r = risk-free rate (NACC)
% sig = volatility of the underlying
% n = # of resets in [0,T]

reset_period=T/n;
call_price=n*fwdstartcall(s0,r,sig,T,T-reset_period);
put_price=n*fwdstartput(s0,r,sig,T,T-reset_period);

function ret = fwdstartcall(s0,r,sig,T,t1)

d1=(r+sig^2/2)*sqrt(T-t1)/sig;
d2=d1-sig*sqrt(T-t1);

ret = s0*(cumnorm(d1)-exp(-r*(T-t1))*cumnorm(d2));
```

```
function ret = fwdstartput(s0,r,sig,T,t1)

d1=(r+sig^2/2)*sqrt(T-t1)/sig;
d2=d1-sig*sqrt(T-t1);

ret = s0*(exp(-r*(T-t1))*cumnorm(-d2)-cumnorm(-d1));

% ====================================
% == Compute P[Z<=x], for Z~N(0,1). ==
% ====================================
%
function ret = cumnorm(x)
ret = 0.5+erf(x/sqrt(2))/2;
```

```
function [S] = SVModel(s0,r,v0,T,m,n,rho,beta)

% Generates an m x n matrix S of stock price paths
% where s0 = initial value of underlying
% where v0 = initial value of volatility
% r = risk-free rate (NACC)
% m = number of sample paths to generate
% n = number of steps taken [0,T]
% T = maturity
% rho = the correlation between underlying and volatility
% beta  = the volatility of volatility

dt=T/n;

rho2=sqrt(1-rho^2);
S(1:m,1)=s0;
V(1:m,1)=v0;
sqdt=sqrt(dt);
Z1=randn(m,n);
Z2=randn(m,n);

mu=-0.5*rho*beta*dt;

for k=1:n
   S(:,k+1)=S(:,k).*exp( (r-0.5*(V(:,k).^2))*dt + sqdt*V(:,k).*Z1(:,k));
   V(:,k+1)=V(:,k) + mu*(V(:,k).^2) + beta*sqdt*V(:,k).*(rho*Z1(:,k) + ...
          ... rho2*Z2(:,k));
end
S(isnan(S))=0;
```

```
function [call_price,put_price]=cliquetSV(s0,r,v0,T,beta,rho,m,step,freq)

% Computes the prices the price of Cliquet using the SV Model

% where s0 = initial value of underlying
% where v0 = initial value of volatility
% r = risk-free rate (NACC)
% m = number of sample paths to generate
% n = number of steps taken [0,T]
% T = maturity
% rho = the correlation between underlying and volatility
% beta  = the volatility of volatility

% freq = # of resets in [0,T]
% step = # of time steps per RESET period

n=freq*step;  % the number of steps for entire duration of cliquet
% generate mxn matrix of stock price paths
S=SVModel(s0,r,v0,T,m,n,rho,beta);

for i=1:freq
   %calculate payoff for cliquet of calls
   pay1=S(:,step*i +1)-S(:,step*(i-1) +1);
   pay2=-pay1;  %calculate payoff for cliquet of puts

   pay1(pay1<0)=0;
   pay2(pay2<0)=0;
   disc=i*T/freq;
   calls(i)=exp(-r*disc)*mean(pay1);
   puts(i)=exp(-r*disc)*mean(pay2);
end

call_price=sum(calls);
put_price=sum(puts);
```

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