Forward Smile and Derivative Pricing

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Outline

- What is the forward implied vol smile?
- How do we get them (fast)?
- Some new methods of computing exotics
- How different models impact pricing
- ... Oh, and someone will probably mentioned the H word...

Motivations

- Skew and Smiel observed in reality and actively traded;
- But Forward skew and smile are still open to debate;
- Model risk introduced by forward skew assumptions;
- Exotic option pricing highly model dependent (eg. Cliquets, Barriers, Variance products);
- Hedging will be off if skew evolves differently;

Introduction

- Forward-Starting options (leg of a "cliquet") starting in t

$$\text{Payoff}_T^{\text{FwdCall}} := \text{Max}\left[\frac{S_T}{S_t} - K, 0\right]$$

- Given a risk-neutral measure, time-0 price is

$$C_{t,T}(K) := \mathbb{E}_{\mathbb{Q}}\left[e^{-rT}\left(\tfrac{S_T}{S_t} - K\right)_+ \middle| \mathcal{F}_0\right],$$

- Easy in a Black-Scholes world

$$C_{t,T}^{\text{BS}}(K;\sigma_{\text{BS}}) = e^{-rt}\, C(K, 1., T-t, \sigma_{\text{BS}}).$$

where $C(K, S, \tau, \sigma)$ denotes the BS formula

Forward-starting options (ctd.)

- If vol is time-dependent but deterministic, define *forward vol* in the obivous way

$$\bar{\sigma}_{t,T} := \sqrt{\frac{\int_t^T \sigma_s^2 ds}{T-t}} \equiv \sqrt{\frac{\bar{\sigma}_{0,T}^2 \cdot T - \bar{\sigma}_{0,t}^2 \cdot t}{T-t}}$$

  substitute into BS like before to get forward start option price

- i.e. "Forward Implied Vol" = "Forward Vol"

- Everyone know this is wrong
  - Skew/Smile
  - Vol is stochastic
  - Price jumps

Forward Implied Vol

- (borrowing Rebonato's definition of (spot) implied vol) *Forward Implied Vol* is the wrong number to plug into the wrong (B-S forward-start) formula to get the right price:

$$C_{t,T}^{\text{Market}}(K) = C_{t,T}^{\text{BS}}(K; \sigma_{\text{FwdImp}})$$

- Like spot implied vol, this is also a function of strike and (constant forward) maturity....

$$\sigma_{\text{FwdImp}} = \sigma_{\text{FwdImp}}(t, T - t, K)$$

...and forward-starting time!

Past attempts at the "Vanilla Smile problem"

- Vol-by-strike, Vol-by-money (BS loyalists)
- Jump-Diffusion (Merton...)
- CEV
- Local Volatility (Dupire, Derman & Kani...)
- Stochastic Volatility (Hull & White, Heston...)
- Levy Process (Geman & Madan...)
- Universal Volatility model (Lipton...)
- Stochastic Time-change (Carr & Wu...)
- ...

”Forward Smile problem”

- Many of the smile models can calibrate to the observed implied vol surface today with small/tolerable magnitude of error (not surprising considering the number of parameters in some)
- Yet they imply very different forward implied vol surface
- As we push the forward-starting time, t, forward from 0, for example, the dynamics of constant forward maturity smile is different across models:
  Levy/Jump models: Exactly the same smile
  Local vol models: They smile less tomorrow
  Stochastic vol models: They smile more tomorrow

Levy process models

- **Intuition**

Return increments are independent and identically distributed;
Vol smile retains the same shape as time moves forward;
Hence the popular "Skew Projection" approach;

- **Example: Merton's Jump Diffusion model**

$$dS/S = (r - \delta - \lambda\,\mu_J)dt + \sigma\,dW + J\,dq$$

Log-normal jumps with Poison arrival:

$$\log[1 + J_t] \sim N\Big(\log[1 + \mu_{\mathrm{J}}] - \tfrac{1}{2}\sigma_{\mathrm{J}}^2, \sigma_{\mathrm{J}}^2\Big)$$

$\mathbb{Q}(dq_t = 1) = \lambda dt$, $\mathbb{Q}(dq_t = 0) = 1 - \lambda dt$.

Local Vol

- **local volatility function**

$$dS/S = (r - \delta)dt + \sigma(t, S_t)\, dW$$

Well known that forward smile flattens out;
Dangerous when pricing forward-starts and cliquets too cheap (Gatheral 03);

- **Intuition**

Skew/Smile generated by non-constant local vol functions;
As time progresses, the function flattens because long-dated implied vol curve flattens (CLT or simple fact);
Forward-start option in the model depends on $\{\sigma(u, S), u \in [t, T]\}$
For t large, it is almost constant in S.

Stochastic Vol

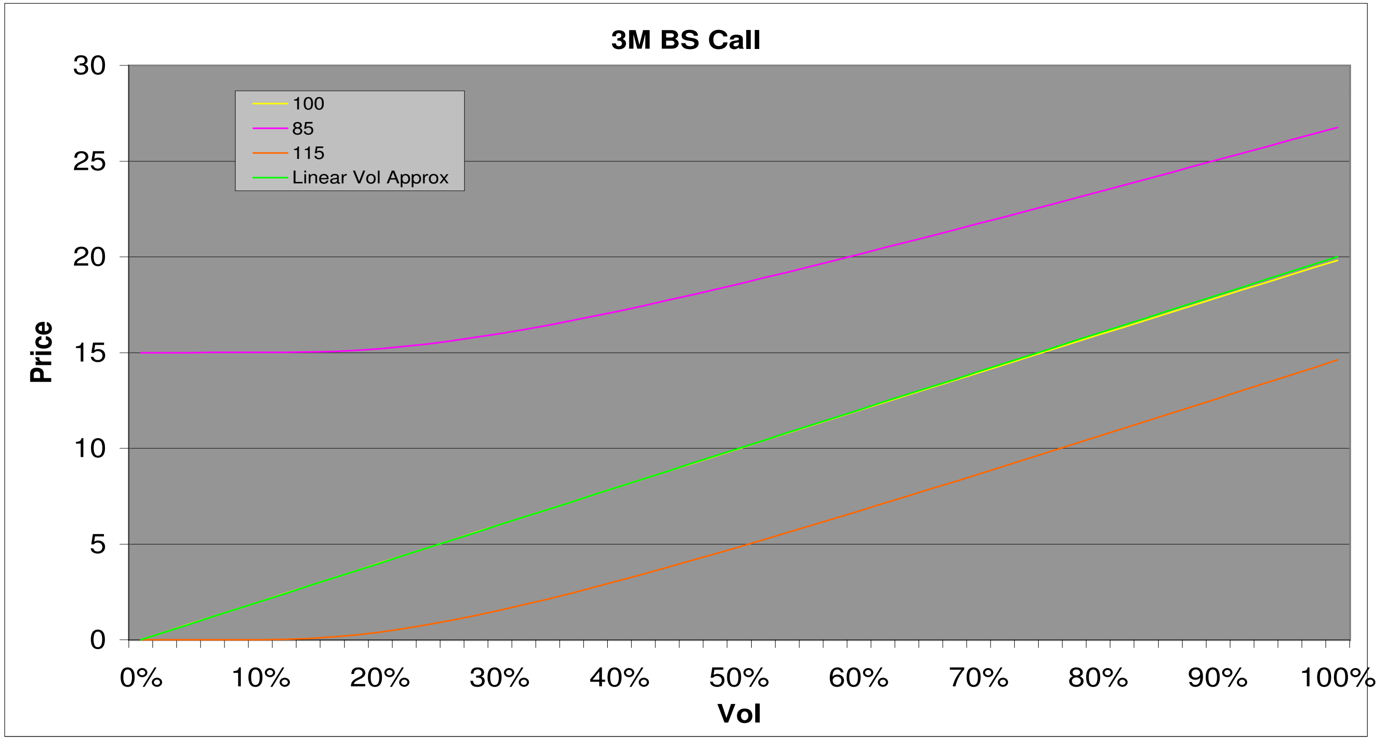
$$dS/S = (r - \delta)dt + \sigma_t \, dW$$

σ_t is another stochastic process, driven by something other than W.

- **Intuition**

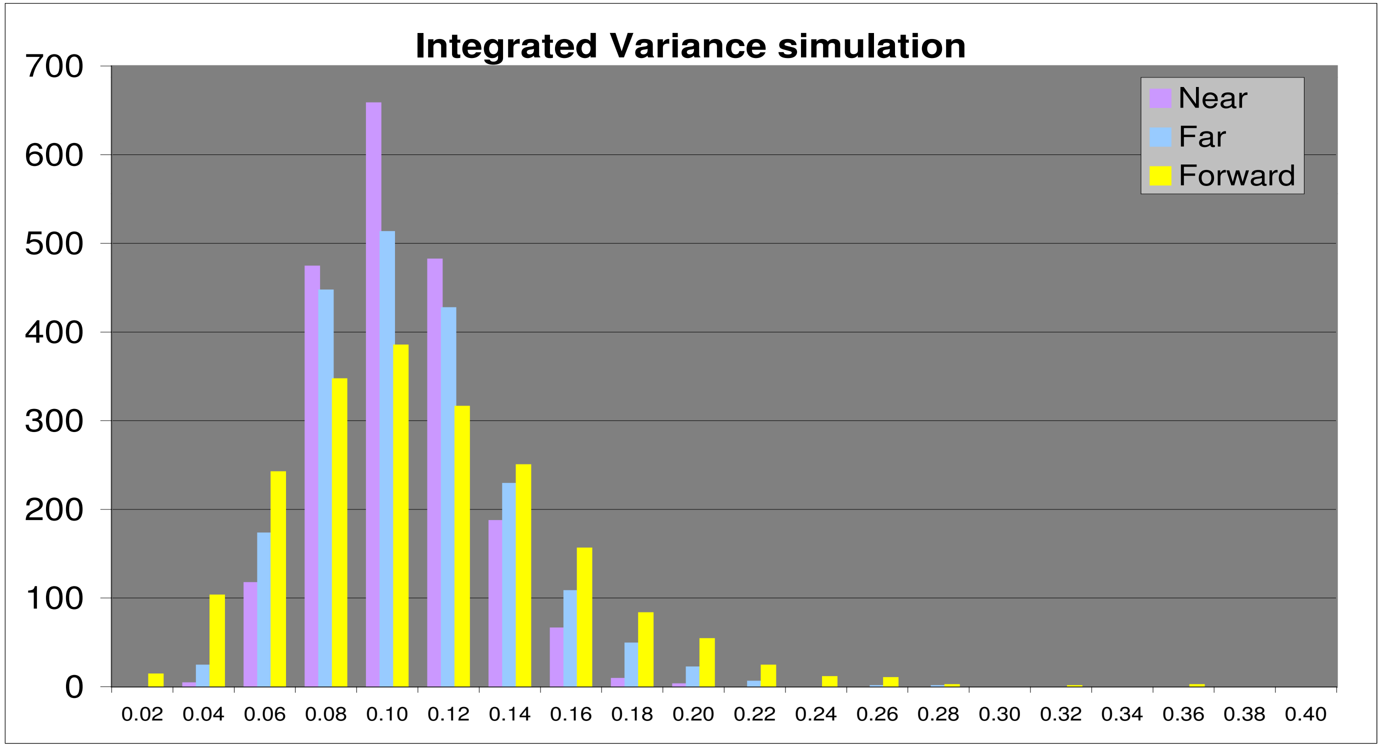
OTM and ITM option prices are convex in vol;
Net effect of a up vol move and a down vol move is on the upside (Jensen's inequality);
As the forward-start time goes to infinity, the distribution of the stochastic vol process reaches an equalibrium, forward smile converges to a optimal smile;

Think of the "Coin-tossing" stochastic vol model



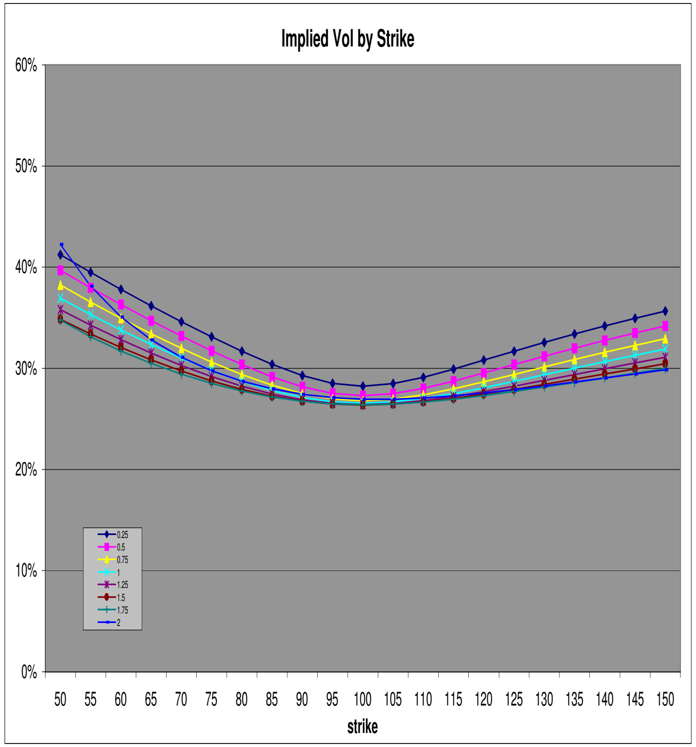
ATMF (At-the-money-foward) Call is nearly linear in vol (slightly convex); OTM options are concave;

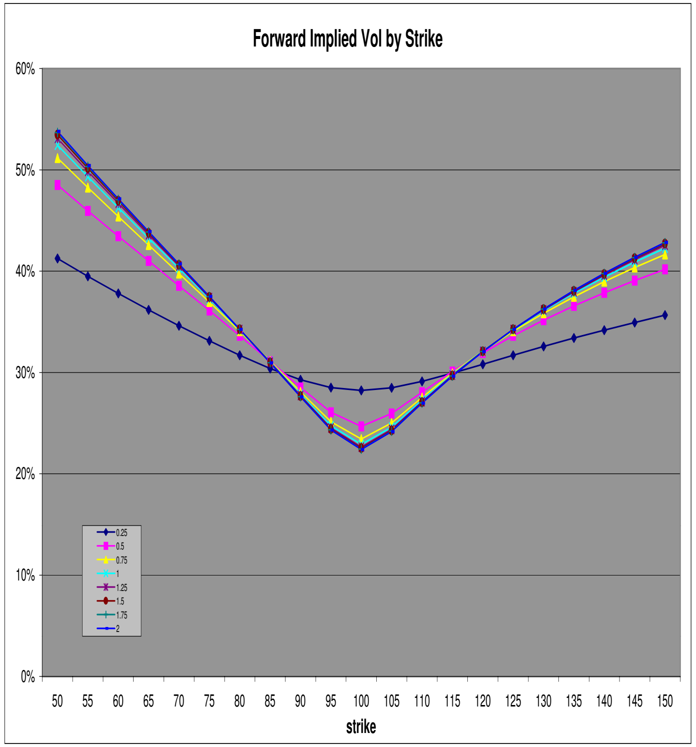
Let's toss some coin



2000 simulations (vol-of-vol $= 0.4$, $v_0 = \mu = 0.09$, $\kappa = 1$)

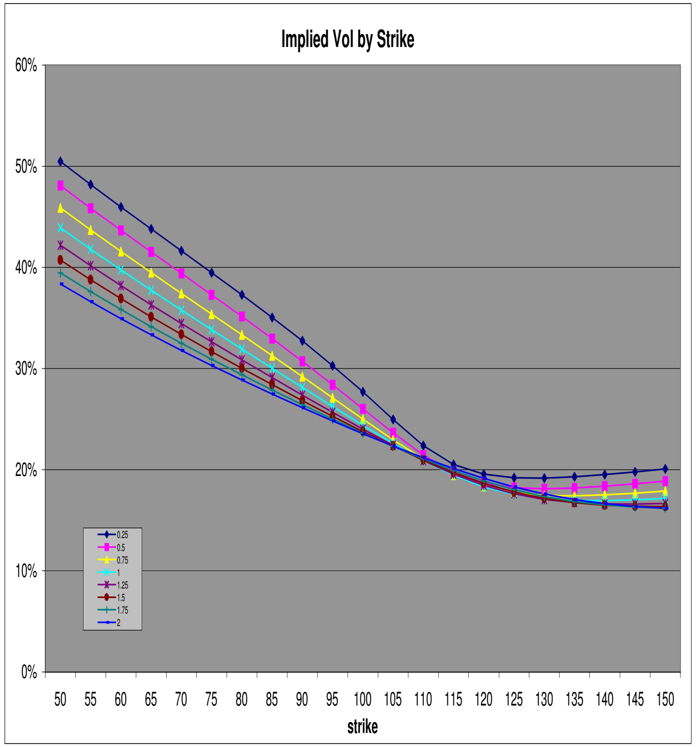
Zero Correlation

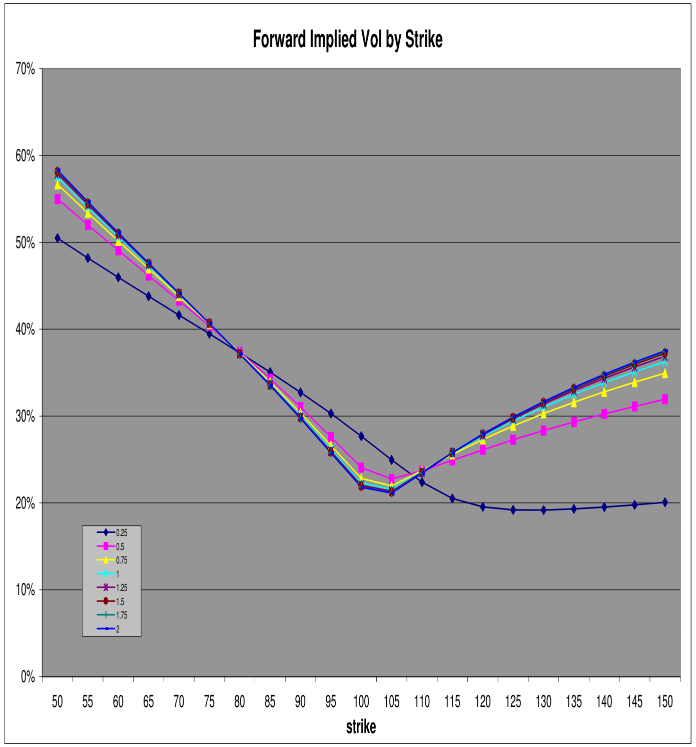




$$\sigma_v = 0.8, v_0 = \mu = 0.09,\ \kappa = 1, \rho = 0$$

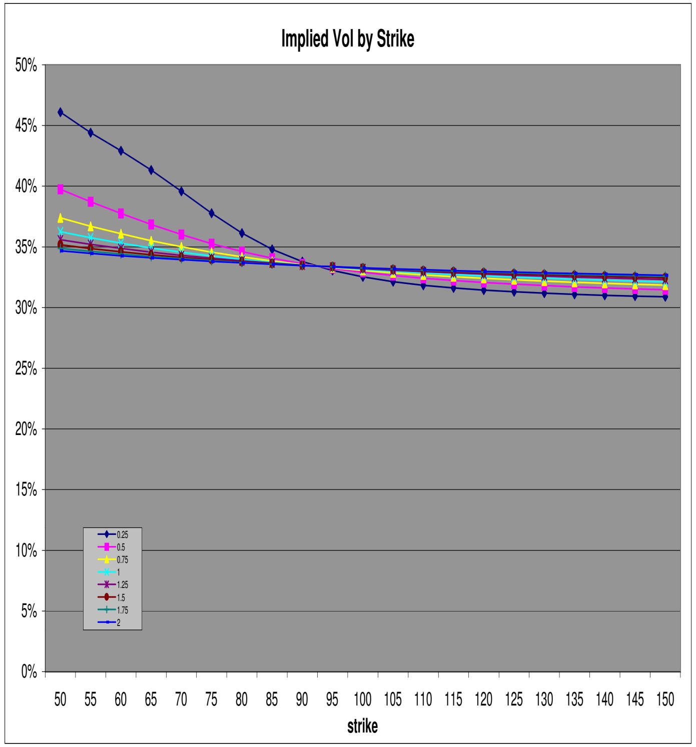
Skew (Correlation = -0.8)

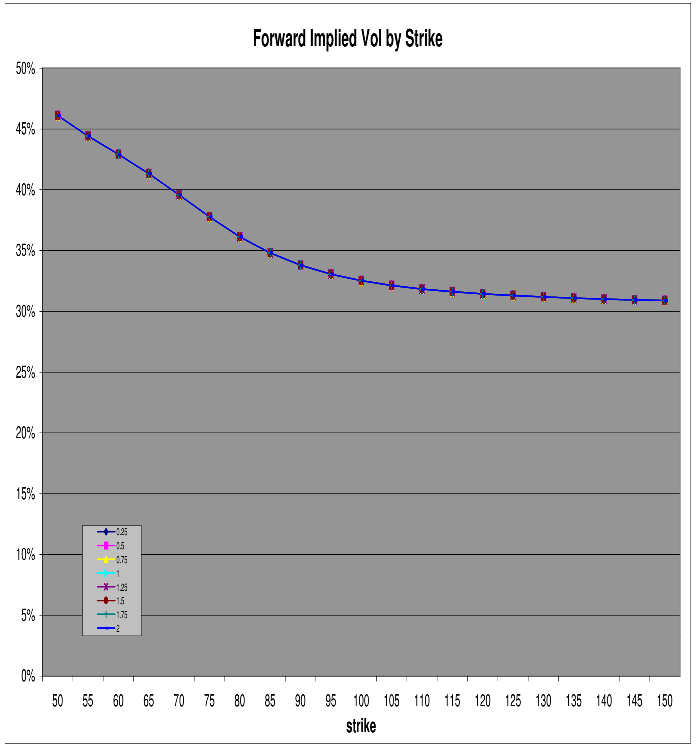




$$\sigma_v = 0.8, v_0 = \mu = 0.09,\ \kappa = 1, \rho = -0.8$$

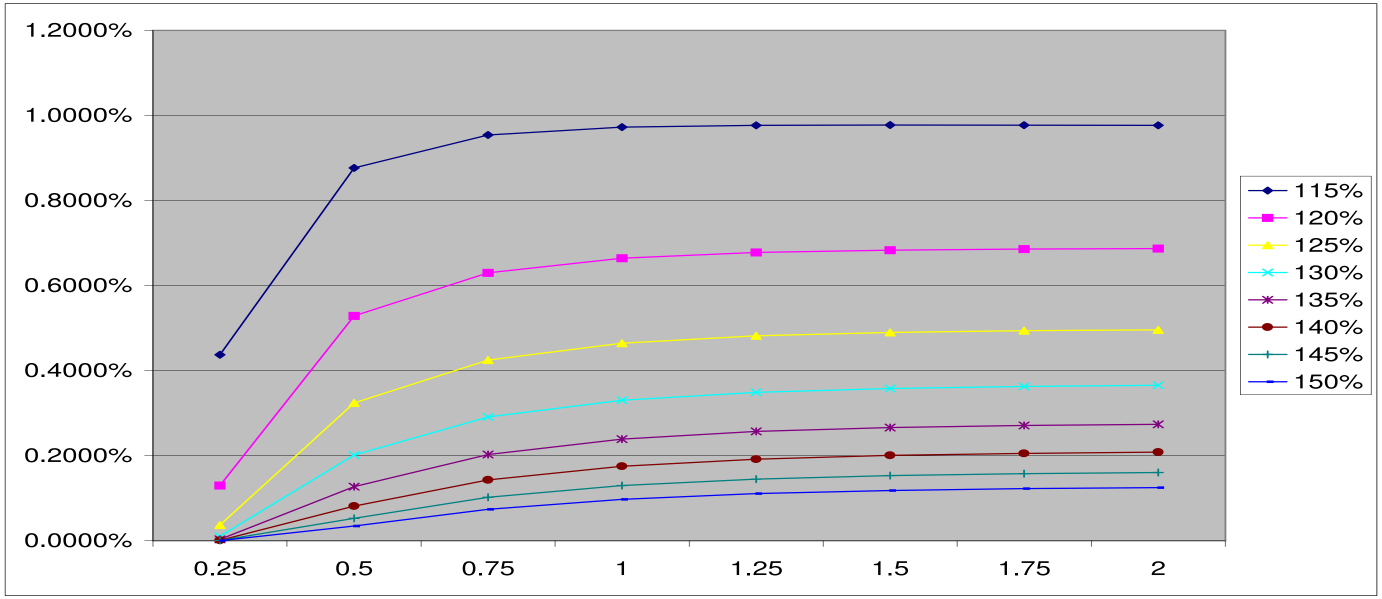
Jumps





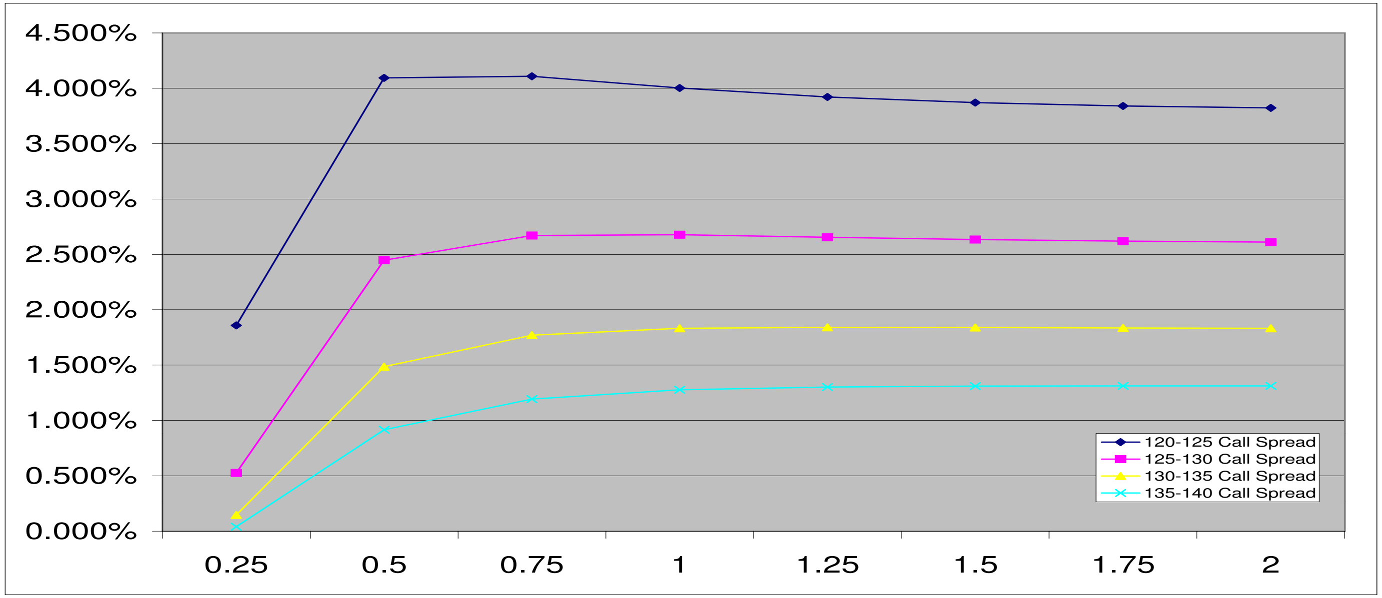
$\lambda = 0.2, \mu_J = -0.3, \ \sigma_J = 0.04$

OTM call



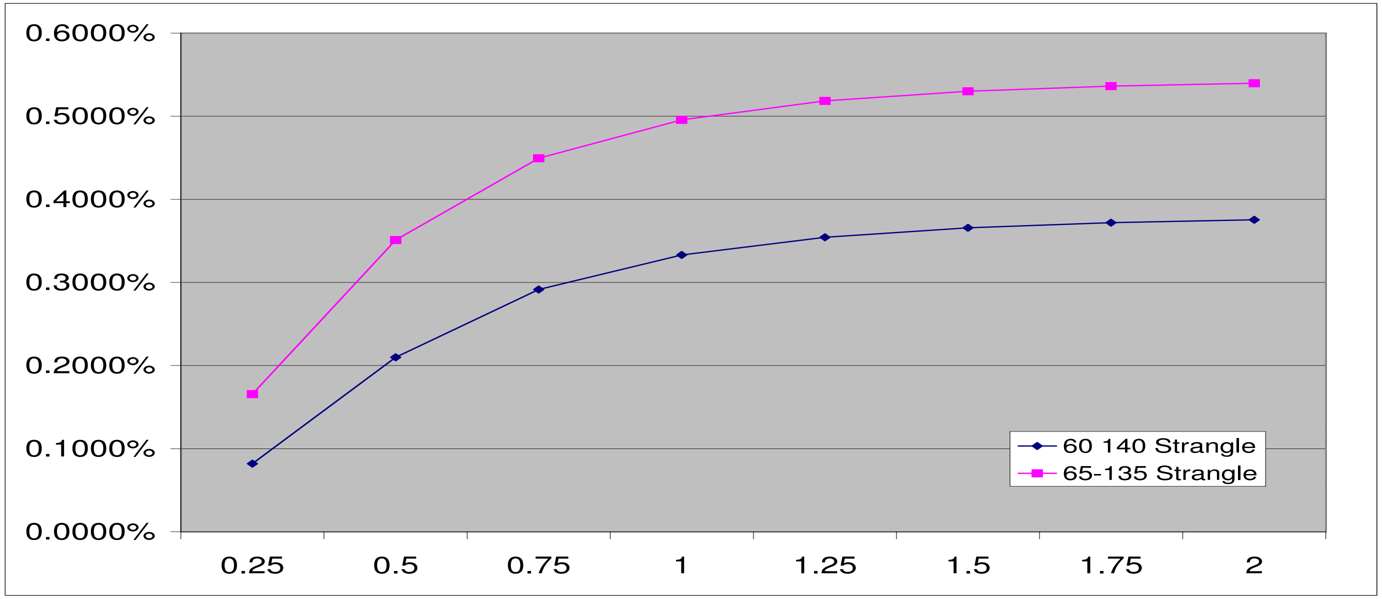
Three month 115 Call: 44 penny, 2 year forward-start about $ 1

OTM call spread



Three month 120-125 Call Spread: \$ 1.86, 2 year forward-start \$ 3.82 .

OTM strangle



Three month 55-145 Strangle: 3.8 penny, 2 year forward-start about 26 penny.

A Canonical example: the SVJ model

$$
\begin{aligned}
dS/S &= (r - \delta - \lambda\,\mu_J)dt + \sqrt{\nu}\,dW_S + J\,dq \\
d\nu &= \kappa(\mu - \nu)dt + \sigma_\nu \sqrt{\nu}\,dW_\nu
\end{aligned}
$$

The price of trying to fit both the short end and long end skew = 8 parameters;

ρ - correlation between W_S and W_ν.
σ_ν - vol of vol
κ - mean-reversion speed
ν_0 - initial vol^2
μ - long-run vol^2
λ - jump intensity
μ_J - mean of jump
σ_J - vol of jump
(there are worse...)

Sub-models

Though simple, these models serve as good examples to think about price dynamics;

SV (Heston)

$$\begin{aligned} dS/S &= (r-\delta)dt + \sqrt{\nu}\, dW_S \\ d\nu &= \kappa(\mu - \nu)dt + \sigma_\nu \sqrt{\nu}\, dW_\nu \end{aligned}$$

BSJ (Merton)

$$dS/S = (r - \delta - \lambda\, \mu_J)dt + \sigma_S\, dW_S + J\, dq$$

(Actually another reason is we can do them in closed-form...)

The FFT method (Carr & Madan '89)

Characteristic function:

$$\phi(u) := \mathbb{E}_{\mathbb{Q}}\left[\exp(iu \cdot s_T)\middle|(s_0, \nu_0)\right]$$

Scaled call ($\alpha > 0$):

$$c_T(k) := \exp(\alpha k)\mathbb{E}_{\mathbb{Q}}\left[e^{-rT}(e^{s_T} - e^k)_+\right],$$

Transform of the call:

$$\begin{aligned}\psi_T(v) &:= \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk \\ &= \frac{e^{-rT}\phi(v - (\alpha+1)i)}{(\alpha^2 + \alpha - v^2) + i(2\alpha+1)v}\end{aligned}$$

Inverse fast Fourier transform (FFT) gives option prices across strikes

$$C(T, K) = \frac{\exp(-\alpha k)}{2\pi} \int_{-\infty}^{\infty} e^{-ivk} \psi_T(v) dv.$$

Characteristic Functions

$$
\begin{aligned}
\phi_{\mathrm{BS}}(u) &:= \exp\Big[\, iu \cdot \big(s_0 + (r-\delta)T\big) - \tfrac{1}{2}(u^2 + iu)T \cdot \nu\Big], \\
\phi_{\mathrm{BSJ}}(u) &:= \phi_{\mathrm{BS}}(u) \cdot \varphi_{\,\mathrm{J}}(u), \\
\phi_{\mathrm{SV}}(u) &:= \exp\Bigg[\, iu \cdot \big(s_0 + (r-\delta)T\big) + \left(\frac{2\zeta(1-e^{-\theta T})}{2\theta - (\theta-\gamma)(1-e^{-\theta T})}\right) \cdot \nu_0 \\
&\qquad\qquad - \frac{\kappa\mu}{\sigma_\nu^2}\left[2\,\log\left(\frac{2\theta - (\theta-\gamma)(1-e^{-\theta T})}{2\theta}\right) + (\theta-\gamma)T\right]\Bigg], \\
\phi_{\mathrm{SVJ}}(u) &:= \phi_{\mathrm{SV}}(u) \cdot \varphi_{\,\mathrm{J}}(u),
\end{aligned}
$$

where

$$
\begin{aligned}
&\zeta \equiv -\tfrac{1}{2}\big(u^2 + iu\big), \qquad \gamma \equiv \kappa - i\,\rho\sigma_\nu\, u, \qquad \theta \equiv \sqrt{\gamma^2 - 2\sigma_\nu^2\zeta}, \\
&\varphi_{\mathrm{J}}(u) \equiv \exp\Big[\lambda T\Big[(1+\mu_{\mathrm{J}})^{iu} \exp\big(\zeta \cdot \sigma_J^2\big) - 1 - iu \cdot \mu_{\mathrm{J}}\Big]\Big],
\end{aligned}
$$

Pricing Forward-starts (now the fun starts)

Key Idea: Forward characteristic function

Following Carr & Madan, consider the forward log return, $s_{t,T} := s_T - s_t \equiv \log(\frac{S_T}{S_t})$, we want

$$C_{t,T}(k) := \mathbb{E}_{\mathbb{Q}}\big[e^{-rT}\big(e^{s_{t,T}} - e^k\big)_+ \big| s_0, \nu_0\big].$$

Consider the $\mathbb{Q}$-density, $q_{t,T}(\cdot)$, of $s_{t,T}$ conditional on the time-0 state,

$$q_{t,T}(s)ds := \mathbb{Q}\Big(s_{t,T} \in [s, s+ds), \Big| s_0, \nu_0\Big),$$

and the modified option price with an exponential scaling term

$$c_{t,T}(k) := \exp(\alpha k)\cdot C_{t,T}(k) \equiv e^{\alpha k} \int_{-\infty}^{\infty} e^{-rT}\big(e^s - e^k\big)_+ q_{t,T}(s)ds, \quad \text{some } \alpha > 0.$$

Fourier-transforming it gives:

$$
\begin{aligned}
\psi_{t,T}(v) &:= \int_{-\infty}^{\infty} e^{ivk} c_{t,T}(k) dk \\
&= \int_{-\infty}^{\infty} e^{(\alpha+iv)k} \int_{k}^{\infty} e^{-rT} \big(e^{s} - e^{k}\big) q_{t,T}(s) ds dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_{t,T}(s) \int_{-\infty}^{s} e^{(\alpha+iv)k} \big(e^{s} - e^{k}\big) dk ds \\
&= \int_{-\infty}^{\infty} \frac{e^{-rT} q_{t,T}(s)\ e^{(\alpha+1+iv)s}}{(\alpha+iv)(\alpha+1+iv)} ds \\
&= e^{-rT} \frac{\phi_{t,T}\big(v - (\alpha+1)i\big)}{(\alpha+iv)(\alpha+1+iv)}.
\end{aligned}
$$

So we are done if we can find the *forward characteristic function*:

$$
\phi_{t,T}(u) := \mathbb{E}_{\mathbb{Q}}\big[\exp(iu \cdot s_{t,T}) | s_0, \nu_0\big] \equiv \int_{-\infty}^{\infty} e^{ius} q_{t,T}(s) ds.
$$

Forward Characteristic function in Heston

$$
\begin{aligned}
\phi_{t,T}(u) &\equiv \mathbb{E}_{\mathbb{Q}}\big[\exp(iu(s_{\mathrm{T}} - s_{\mathrm{t}}))|s_0, \nu_0\big] \\
&\equiv \mathbb{E}_{\mathbb{Q}}\Big[\ \mathbb{E}_{\mathbb{Q}}\big[\exp(iu(s_{\mathrm{T}} - s_{\mathrm{t}}))|s_t, \nu_t\big]\Big|s_0, \nu_0\Big] \\
&= \mathbb{E}_{\mathbb{Q}}\Bigg[\exp\Bigg\{\ iu\cdot(r-\delta)\tau - \frac{\kappa\mu}{\sigma_\nu^2}\Big[2\ \log\Bigg(\frac{2\theta - (\theta-\gamma)(1-e^{-\theta\tau})}{2\theta}\Bigg) + (\theta-\gamma)\tau\Big] \\
&\qquad\qquad + \Bigg(\frac{2\zeta(1-e^{-\theta\tau})}{2\theta - (\theta-\gamma)(1-e^{-\theta\tau})}\Bigg)\cdot\nu_t\Bigg\}\Bigg|s_0, \nu_0\Bigg] \\
&=: \mathbb{E}_{\mathbb{Q}}\Big[\exp\big[A(u,\tau) + B(u,\tau)\ \nu_t\big]\Big|s_0, \nu_0\Big] = e^{A(u,\tau)}\mathbb{E}_{\mathbb{Q}}\Big[e^{B(u,\tau)\ \nu_t}\Big|\nu_0\Big] \\
&= \exp\Bigg[A(u,\tau) + \Bigg(\frac{B(u,\tau)\ e^{-\kappa t}}{1 - B(u,\tau)/c_t}\Bigg)\cdot\nu_0\Bigg]\big(1 - B(u,\tau)/c_t\big)^{-\frac{2\kappa\mu}{\sigma_\nu^2}}
\end{aligned}
$$

where the $\mathbb{C}$-valued functions $A(\cdot,\cdot), B(\cdot,\cdot)$ are defined in the obvious way with parameters

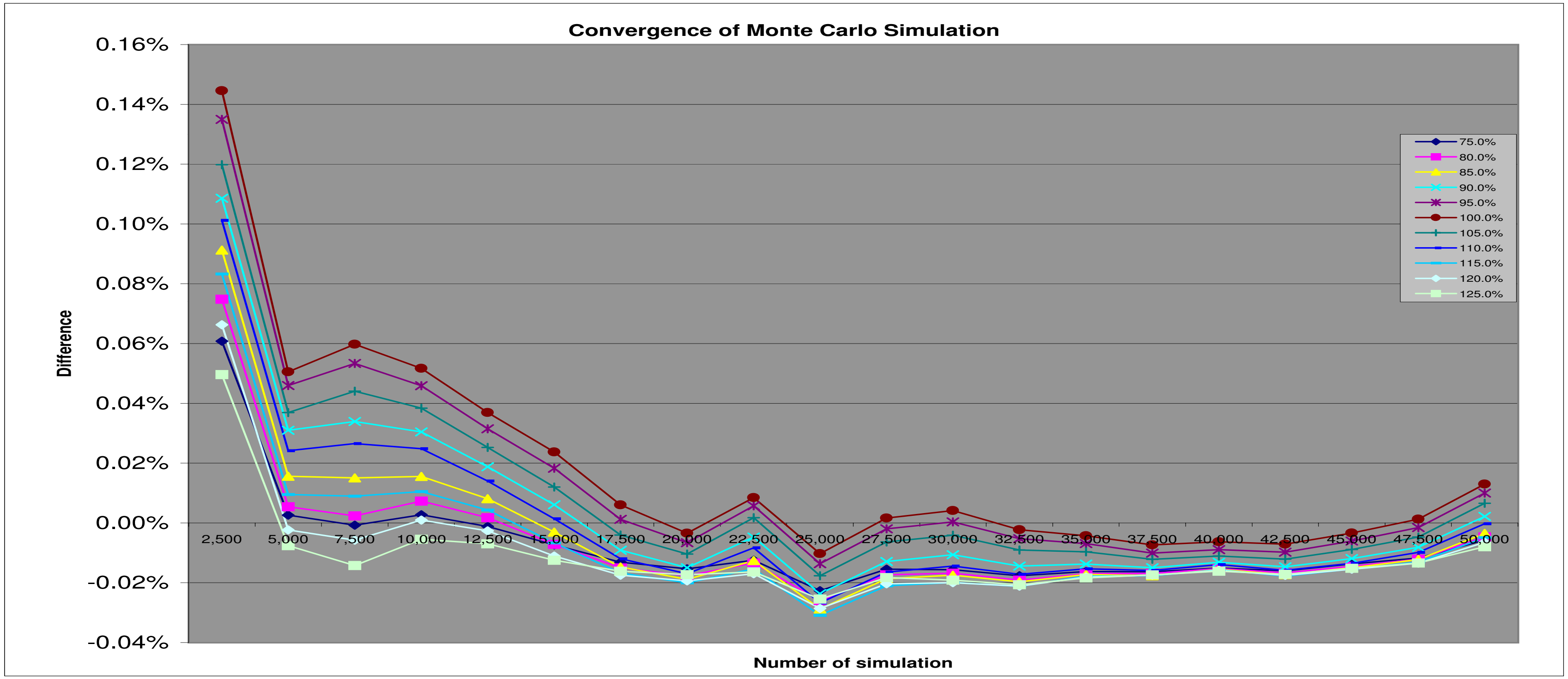
$$\zeta \equiv -\tfrac{1}{2}\big(u^2 + iu\big), \qquad \gamma \equiv \kappa - i\,\rho\sigma_\nu\,u,$$

$$\theta \equiv \sqrt{\gamma^2 - 2\sigma_\nu^2\zeta}, \qquad c_t = \frac{2\kappa}{\sigma_\nu^2(1 - e^{-\kappa t})}.$$

The last line in the identity is the Laplace transform of the squared root diffusion process of ν.

Alternatively, one can give a less mathematical proof...

A Trader-style proof



Hit F9 a few more times and we are done.

Adding Jumps

Since the jump component is independent of the diffusion bit, and has itself got stationary and independent increment, we can simply multiply the forward charateristic functions in the diffusion setting by an extra jump part.

$$\phi^{\text{JD}}_{t,T}(u) = \phi^{\text{BS}}_{t,T}(u) \cdot \varphi^{\text{J}}_{T-t}(u),$$
$$\phi^{\text{SVJ}}_{t,T}(u) = \phi^{\text{SV}}_{t,T}(u) \cdot \varphi^{\text{J}}_{T-t}(u),$$

where

$$\begin{aligned} \phi^{\text{BS}}_{t,T}(u) &= \phi^{\text{BS}}_{T}(u)/\phi^{\text{BS}}_{t}(u) \\ &= \exp\Big[\, iu(r-\delta)(T-t) - \tfrac{1}{2}(u^2+iu)(T-t)\cdot\nu\Big], \\ \varphi^{\text{J}}_{\tau}(u) &:= \exp\Big[\lambda\tau\Big[(1+\mu_{\text{J}})^{iu}\exp\left(\zeta\cdot\sigma_J^2\right) - 1 - iu\cdot\mu_{\text{J}}\Big]\Big]. \end{aligned}$$

Performance

• same efficiency as the vanillas; < 0.05 seconds for all strikes with one forward-start date and maturity ($N = 4096$);

• The very same transform also draws you the vol smile, gives the skew, kurtosis;

• switching models = switching forward characteristic functions, as long as you can derive them!

• these forward characteristic functions have other applications....

Variance Swap

A *Variance Swap* is a contract paying, at maturity T, an amount of

$$\text{Payoff}_T = U \times (V_T - K_{\text{var}}),$$

where U is the swap notional, K_{var} the *variance strike*, T the maturity in years, and the *realised variance* defined as

$$V_T := \frac{1}{T}\sum_{i=1}^{n}\left[\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^2 \equiv \frac{A}{n}\sum_{i=1}^{n} r_i^2$$

by the n returns, r_i, of the underlying price S_t discretely sampled on fixings $0 = t_0 < t_1 < \cdots < t_n = T$. The frequency of the sampling is determined by $\Delta t := \frac{T}{n}$ or the annualisation factor $A := \frac{n}{T} \equiv \frac{1}{\Delta t}$, defined as the number of sampling/observation points per year. For example, $A = 252$ for a daily variance swap.

***The* variance swap theory**

There is currently one market standard theory (Carr, Derman ...) to the price and hedge a variance swap, assuming:

A1 no jumps

- Asset price S_t evolves as a continuous diffuion,

$$dS/S = \mu_t dt + \sigma_t dW$$

where σ_t denotes the (possibly stochastic) volatility process;

A2 perfect static hedge

- A continum of calls and puts across all strikes can be used as hedging vehicles;

A3 continuous sampling

- A fixing scheme which is frequent enough to permit the approximation of variance

$$V_T \approx V := \frac{1}{T} \int_0^T \sigma_t^2 dt$$

A4 perfect dynamic hedge

- Possibility of continuous rebalancing an account of the underlying S

We don't assume any of these!

(but there is a caveat...)

Discrete Variance Swap

Key Idea: view the variance swap as a portfolio of n forward starting contracts with squared log-payoff:

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}[V_n] &\equiv \mathbb{E}_{\mathbb{Q}}\left[\frac{1}{T}\sum_{i=1}^{n}\left[\log\left(\frac{S_{t_i}}{S_{t_{i-1}}}\right)\right]^2\right] \\
&\equiv \frac{1}{T}\sum_{i=1}^{n}\mathbb{E}_{\mathbb{Q}}\left[(s_{t_i}-s_{t_{i-1}})^2\right] \\
&\equiv \frac{1}{T}\sum_{i=1}^{n}\mathbb{E}_{\mathbb{Q}}\left[r_i^2\right]
\end{aligned}$$

One can relate the n bits to the second moments of the forward characteristic functions $\phi_i(u)$:

$$\mathbb{E}_{\mathbb{Q}}\left[r_i^2\right] \equiv -\frac{\partial^2\phi_i(u)}{\partial u^2}\Big|_{u=0}$$

$$\phi_i(u) := \mathbb{E}_{\mathbb{Q}}\Big[\exp(iu \cdot r_i)|\mathcal{F}_0\Big] \equiv \int_{-\infty}^{\infty} e^{ius} q_{t_{i-1},t_i}(s)ds$$

where $q_{t_{i-1},t_i}(\cdot)$ is the *unconditional* risk-neutral density of the $[t_{i-1}, t_i]$-period return, $r_i \equiv s_{t_i} - s_{t_{i-1}}$. Thus if $\phi_i(\cdot)$, $\forall i = 1, \ldots, n$, are known in closed form, interchanging the summation and the differential operators gives

$$\mathbb{E}_{\mathbb{Q}}[V_n] = -\frac{1}{T}\frac{\partial^2}{\partial u^2} \sum_{i=1}^{n} \phi_i(u)\Big|_{u=0}$$

Summing n such functions and evaluate the second derivative at zero by finite differencing gives the risk-neutral expectation of V_n. Alternativly symbolic differentiation using software like MATLAB or manual to obtain a closed-form expression.

This can be done again for all models with forward characteristic functions. (the ones mentioned and more...)

Weekly versus Daily Variance Swaps

Before jumping to the results, consider two variance swaps with different frequency of fixings:

Daily Variance Swap: 262 fixings for one year;

Weekly Variance Swap: 52 fixings for one year;

Same price.

Which one do you want to buy (or sell)?

Toy example: bi-daily vs daily

Daily: lots of $(r_1^2 + r_2^2)$ pairs

Bi-daily: lots of $(r_1 + r_2)^2$

Difference $= 2 \times r_1 \times r_2$

Positive on average if you believe price is trending and negative if you believe it oscilates within a range;

Same logic applies for other frequencies;

Will jump add/reduce value to discrete sampling?

Levy processess: iid increments implies $\mathbb{E}_{\mathbb{Q}}[r_i r_j] = 0$ for $i \neq j$

i.e. on average all jumps cancel out, either finite or infinite variations

Stochastic volatility with non-zero correlation (to spot) has neither independent nor identically distributed returns;

For negative correlation(skew), down(up) moves are more likely to be followed by down(up) moves;

It is not a jump trade

Long weekly and short weekly if you believe in stochastic vol and negative skew;

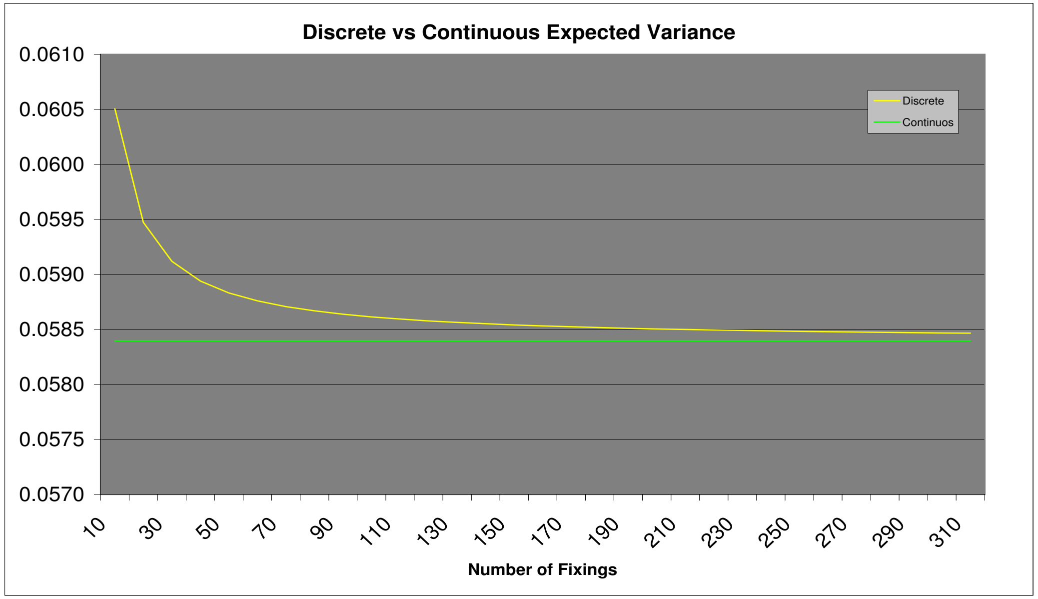
It's an autocorrelation trade;

Makes about 50bps on average for a 1 year swap in a Heston world ($\rho = -0.8$, $\sigma_v = 0.8$);

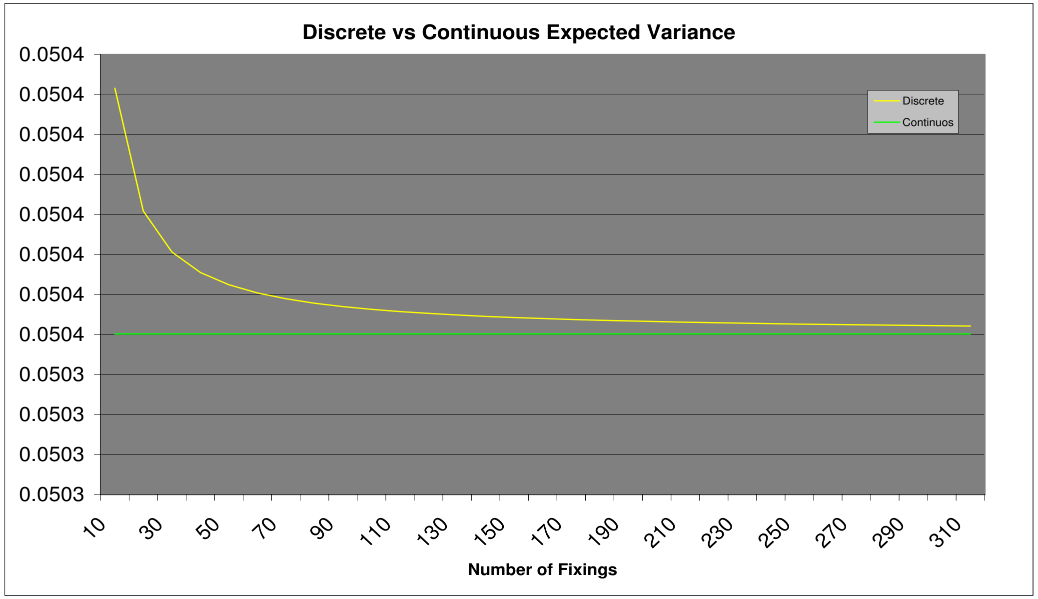
Ok, that's not a lot... ($\sim 0.07\%$ in vol points; 0.6% for a monthly vs daily);

But the difference will be more significant in models later.

Expected discrete variance under Heston



Looks the same in a pure Poison jump model?



But check out the Y-axis! (with zero drift it will exactly zero)

Generalise...

We can go further than the stochastic vol + jump model;

If fact, we can generalise the methods above to virtually all models falling under the framework of time-changed Levy processes;

We just need the forward characteristic functions!

The elegant approach of unifying stochastic volatility models and Jump type models via a ”stochastic clock”; (Carr & Wu)

Intuitively

Price process runs under ”trading” time, not real time; Things seem to move faster when there is more activity in the market;

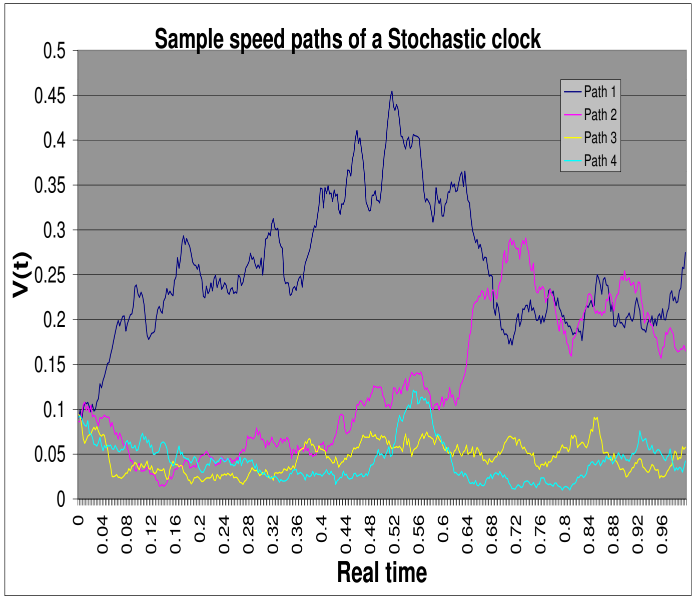
The distribution of price assuming a flat activity rate is modelled by a Levy procee;

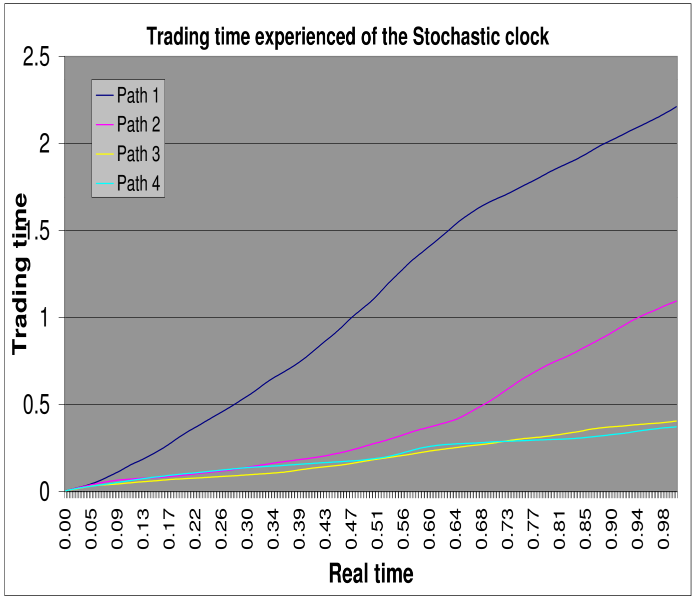
The distribution of the activity rate is modelled by a correlated mean-reverting process;

The former is completely specified by the characteristic exponent of the Levy process while the latter by a Laplace transform;

Combine the two gives the distribution of the underlying run under the random clock.

Some CIR clock runs





$$\sigma_v = 0.4, v_0 = 0.09, \mu = 0.16, \kappa = 1$$

Some copy-and-pasting...

Levy process X and its generalised characteristic function:

$$\phi_X(u) := \mathbb{E}_{\mathbb{Q}}\big[e^{iuX_T}\big] = e^{-T\Psi_x(u)}, u \in D \subset \mathbb{C}$$

Random time, a non-decreasing semi-martingale,

$$\mathcal{T}_t := \int_0^t v(s)ds,$$

with the instantaneous activity rate $v(t)$. Laplace transform of $\mathcal{T}_t$ as

$$\mathcal{L}_{\mathcal{T}_t}(\lambda) := \mathbb{E}_{\mathbb{Q}}\big[e^{-\lambda \mathcal{T}_t}\big] = \mathbb{E}_{\mathbb{Q}}\Big[\exp\big(-\lambda \int_0^t v(s)ds\big)\Big].$$

Note that this looks like a bond...

Let's skip the math...

The time-changed process $Y_t := X_{\mathcal{T}_t}$ has the characteristic function

$$\phi_{Y_t}(u) = \mathbb{E}_{\mathbb{Q}}\left[e^{iuX_{\mathcal{T}_t}}\right] = \mathbb{E}_{\mathbb{Q}(u)}\left[e^{-\mathcal{T}_t\Psi_x(u)}\right] = \mathcal{L}^u_{\mathcal{T}_t}\Big(\Psi_x(u)\Big),$$

where the expectation and the Laplace transform are computed under a complex-valued measure $\mathbb{Q}(u)$:

$$\left.\frac{d\mathbb{Q}(u)}{d\mathbb{Q}}\right|_t = \exp\left(iuY_t + \mathcal{T}_t\Psi_x(u)\right)$$

(Theorem 1 in Carr & Wu) So that if both the Laplace transform and the Levy exponent are known in closed-form, the characteristic function of the time-changed process is also available in closed-form.

Pick & Mix

Since we have a large class of analytic short rate models and a large class of analytic Levy process models, combining them yield a multitude of models which possess analytically tractable characteristic functions.

E.g. Heston = Brownian motion run under a CIR clock.

One can have two Levy processes and two random clocks, one for each processes, and model the underlying as the sum of the two time-changed processes. E.g. a normal stochastic volatility component and a jump component with random arrival rate, or two jump components, one with infinite variation and the other finite variation, run on two structurally distinct clocks.

E.g. Stochastic Skew model (Carr & Wu) is the latest such model;

Forward Characteristic Function: Second Time

Recall all we need is the forward characteristic function:

$$\begin{aligned} \phi_{t,T}(u) &\equiv \mathbb{E}_{\mathbb{Q}}\left[e^{iu(s_T - s_t)} | \mathcal{F}_0\right] \\ &= e^{iu(r-\delta)(T-t)} \mathbb{E}_{\mathbb{Q}}\left[e^{iu(Y_T - Y_t)} | \mathcal{F}_0\right] \end{aligned}$$

where $t < T$. For ease of notation I shall assume a single time-changed process but it is clear that the result extend to the vector version. Denote the Radon-Nikodym derivative of the leverage neutral measure with respect to the risk neutral measure up to the time horizon T by

$$M_T(u) := \frac{d\mathbb{Q}(u)}{d\mathbb{Q}}\bigg|_T = \exp\left(iuY_T + \mathcal{T}_T \Psi_x(u)\right)$$

Optimal stopping theorem ensures

$$M_t(u) = \mathbb{E}_{\mathbb{Q}}\left[M_T(u)\big|\mathcal{F}_t\right] = \exp\left(iuY_t + \mathcal{T}_t \Psi_x(u)\right)$$

is a $\mathbb{Q}$-martingale and that

$$\mathbb{E}_{\mathbb{Q}(u)}\left[\Xi_T\middle|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}}\left[\tfrac{M_T(u)}{M_t(u)}\Xi_T\middle|\mathcal{F}_t\right]$$

for all $\mathcal{F}_T$ random variable Ξ_T. Hence we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[e^{iu(Y_T-Y_t)}\right] &= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{iu(Y_T-Y_t)}\middle|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[e^{iu(Y_T-Y_t)+(\mathcal{T}_T-\mathcal{T}_t)\Psi_x(u)-(\mathcal{T}_T-\mathcal{T}_t)\Psi_x(u)}\middle|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}}\left[\tfrac{M_T(u)}{M_t(u)}e^{-(\mathcal{T}_T-\mathcal{T}_t)\Psi_x(u)}\middle|\mathcal{F}_t\right]\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[\mathbb{E}_{\mathbb{Q}(u)}\left[e^{-(\mathcal{T}_T-\mathcal{T}_t)\Psi_x(u)}\middle|\mathcal{F}_t\right]\right]
\end{aligned}$$

For Makovian arrival rates v, the inner expectation will be a function of $v(t)$ only. If we further restrict our attention to time-homogeneous processes, such as a CIR process with time-independent parameters, we have

$$\mathbb{E}_{\mathbb{Q}(u)}\left[e^{-(\mathcal{T}_T-\mathcal{T}_t)\lambda}\Big|\mathcal{F}_t\right] = \mathbb{E}_{\mathbb{Q}(u)}\left[e^{-(\int_t^T v(s)ds)\lambda}\Big|v_t\right] = L^u(\lambda, T-t, v_t)$$

for some function L^u. Therefore, for all the arrival rate which is affine, we will have an exponential affine function in v_t:

$$L^u(\lambda,\tau,v) = \exp\left[\alpha(\lambda,\tau)+\beta(\lambda,\tau)v\right]$$

Hence we have

$$\begin{aligned}
\mathbb{E}_{\mathbb{Q}}\left[e^{iu(Y_T-Y_t)}\right] &= \mathbb{E}_{\mathbb{Q}}\left[L^u(\Psi_x(u),T-t,v_t)\right] \\
&= \mathbb{E}_{\mathbb{Q}}\left[e^{\alpha(\Psi_x(u),\tau)+\beta(\Psi_x(u),T-t)v_t}\right] \\
&= e^{\alpha(\Psi_x(u),\tau)}\mathbb{E}_{\mathbb{Q}}\left[e^{\beta(\Psi_x(u),T-t)v_t}\right] \\
&= e^{\alpha(\Psi_x(u),\tau)}\phi_{v_t}\big(-i\beta(\Psi_x(u),T-t)\big)
\end{aligned}$$

where $\phi_{v_t}(\cdot)$ represents the (generalised) characteristic function of v_t under $\mathbb{Q}$.

Conclusion

- A new method for calculating forward-starts and forward implied vols;
- Even simple forward-starting vanillas can be valued very differently while calibrating to the same implied vol surface;
- Discrete sampling effect in variance swaps;
- Techniques generalise to a wide class of models;

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