A Perfect Calibration ! Now What ? Model Risk for Exotic and Moment Derivatives

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Abstract

We show that several advanced equity option models incorporating stochastic volatility can be calibrated very nicely to a realistic option surface. More specifically, we focus on the Heston Stochastic Volatility model (with and without jumps in the stock price process), the Barndorff- Nielsen-Shephard model and Lévy models with stochastic time. All these models are capable of accurately describing the marginal distribution of stock prices or indices and hence lead to almost identical European vanilla option prices. As such, we can hardly discriminate between the different processes on the basis of their smile-conform pricing characteristics. We therefore are tempted applying them to a range of exotics. However, due to the different structure in path-behaviour and the underlying dependency over time between these models, the resulting exotics prices can be significantly different. It motivates a further study on how to model the fine stochastic behaviour of assets over time.

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Samuelson/Black-Scholes Model (1965-1973)

GEOMETRIC BROWNIAN MOTION MODEL:

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t, \qquad S_0 = x > 0.$$

This SDE has a unique solution:

$$S_t = S_0 \exp\left(\left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t\right).$$

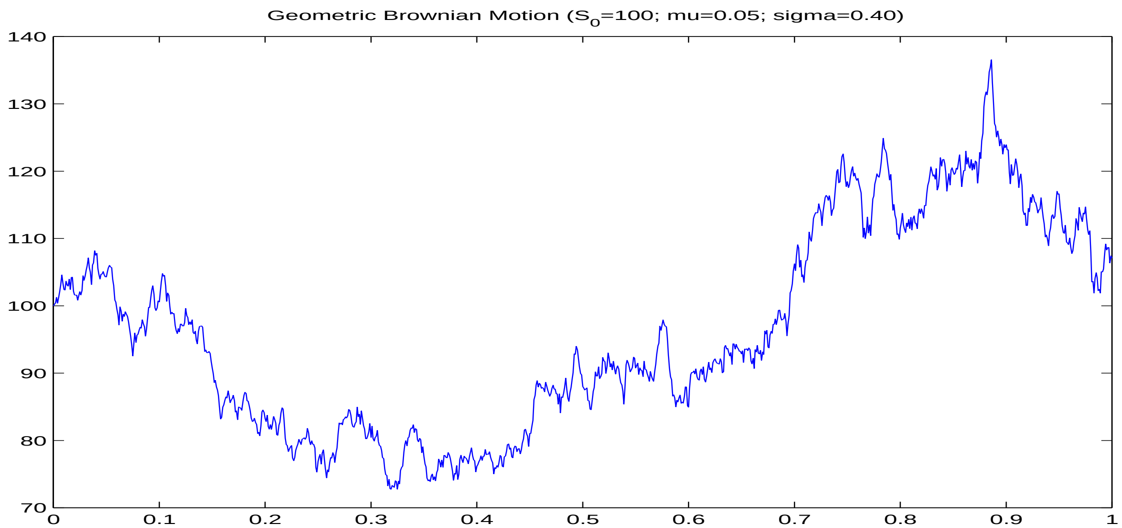

Figure 1: Sample path of a geometric Brownian motion ($S_0 = 100, \mu = 0.05, \sigma = 0.40$)

Note that the log-returns are normally distributed:

$$\log S_t - \log S_{t-1} \sim \text{Normal}\left(\mu - \frac{\sigma^2}{2}, \sigma^2\right).$$

Imperfections of the Black-Scholes Model

The Normal density is not a good approximation of the empirical density.

Empirical Density vs Normal Density

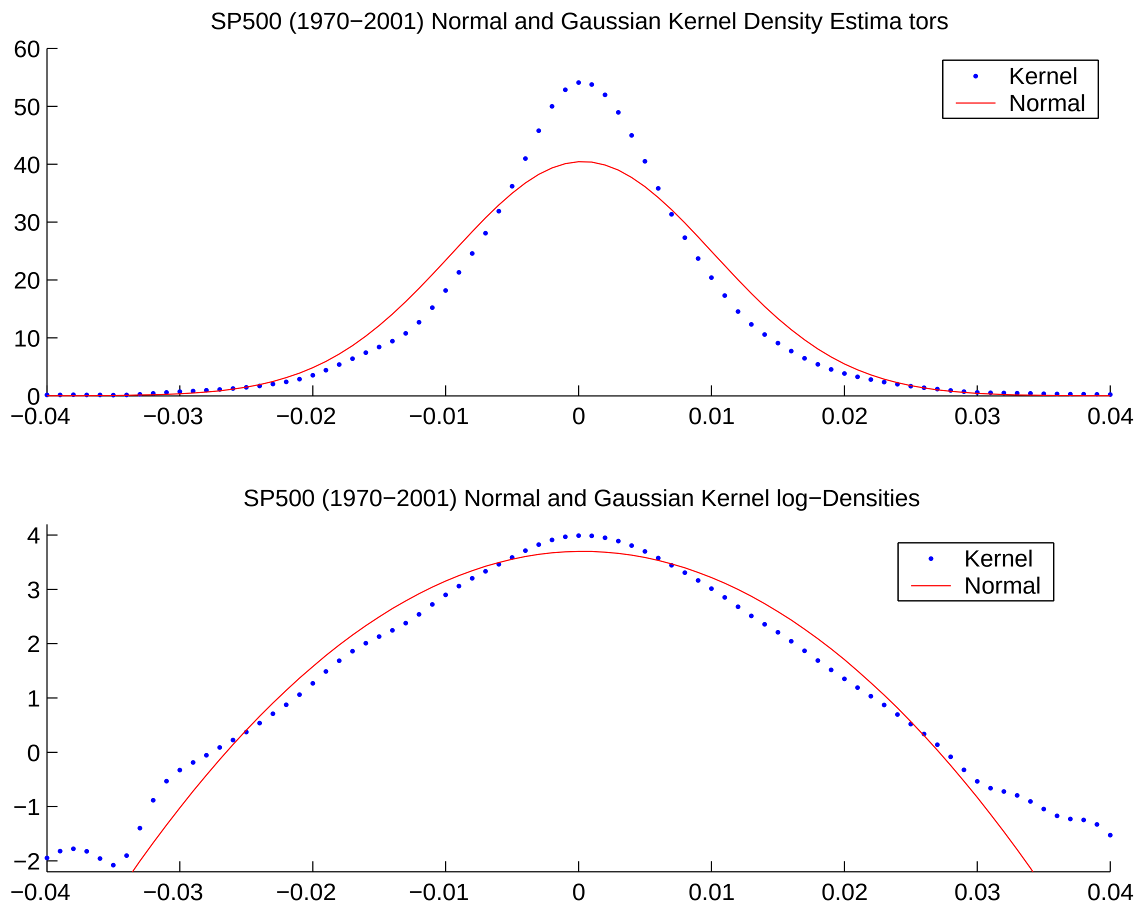

Figure 2: Normal density and Gaussian Kernel estimator of the density of the daily log-returns of the SP500 index

Imperfections of the Black-Scholes Model

The volatility estimated over time typically shown a non-deterministic behaviour.

Historic Volatility

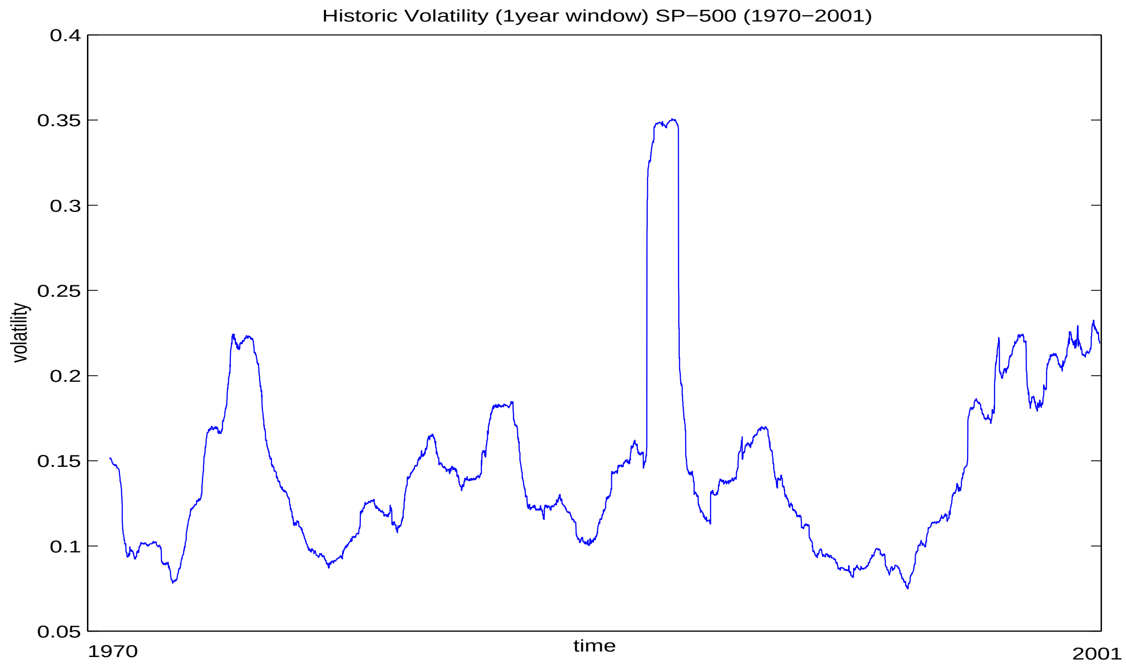

Figure 3: Historic Volatilities on SP-500

Imperfections of the Black-Scholes Model

Black Scholes Option prices

Calibrating the Black-Scholes model prices to the market prices (in the least squared sense) leads to a bad fit of the option surface:

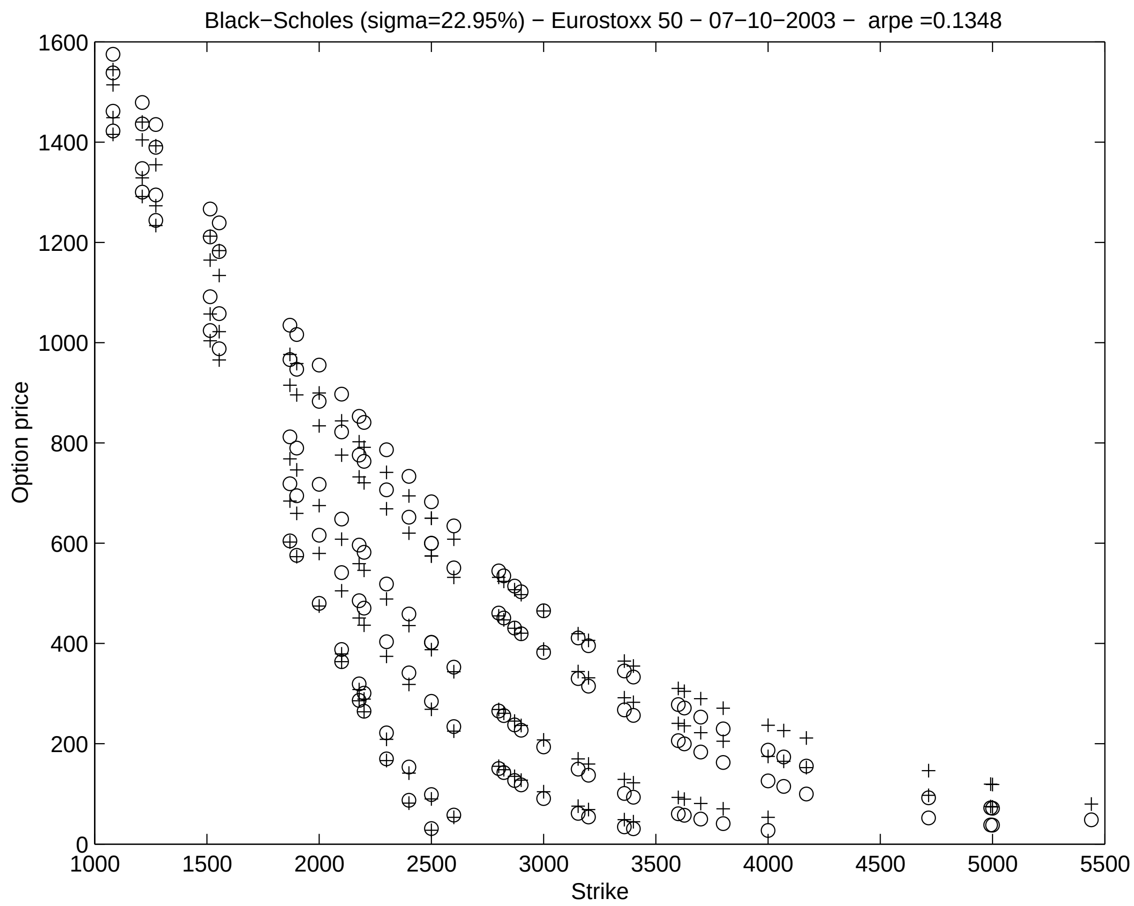

Figure 4: Black Scholes ($\sigma = 0.2295$) calibration on Eurostoxx options (o's are market prices, +'s are model prices)

Lévy Models (1990)

- Instead of taking normally-distributed log-returns, we now model the stock price/index by

$$S_t = S_0 \exp(X_t), \qquad t \geq 0$$

where $\{X_t, t \geq 0\}$ is a Lévy process, i.e..

  - $X_0 = 0$
  - stationary increments
  - independent increments
  - X_t follows an infinitely divisible distribution.

- *Lévy-Khintchine formula:*

$$\log E[\exp(\mathrm{i}uX_1)] = \mathrm{i}\gamma u - \frac{\sigma^2}{2}u^2 + \int_{-\infty}^{+\infty}(\exp(\mathrm{i}ux) - 1 - \mathrm{i}ux 1_{\{|x|<1\}})\nu(\mathrm{d}x),$$

where $\gamma \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a measure on $\mathbb{R}\backslash\{0\}$ with

$$\int_{-\infty}^{+\infty}(1 \wedge x^2)\nu(\mathrm{d}x) < \infty.$$

Lévy Models (1990)

Examples of models

- 1990: Madan and Seneta: Variance Gamma Process
- 1995: Eberlein and Keller: Hyperbolic model
- 1995: Barndorff-Nielsen : Normal Inverse Gaussian model
- 2000: Carr, Madan, Geman and Yor : The CGMY model
- 2000: Grigelionis, Teugels, Schoutens : The Meixner model

Example: The Meixner Process

The characteristic function of the Meixner(α, β, δ) distribution is given by

$$\phi_{Meixner}(u; \alpha, \beta, \delta) = \left(\frac{\cos(\beta/2)}{\cosh \frac{\alpha u - \mathrm{i}\beta}{2}} \right)^{2\delta}$$

The above Lévy process are pure jump process : Jumps of size in some set $A \subset \mathbb{R}$ occur according to a Poisson Process with parameter given by the Lévy measure of A. For the Meixner case:

$$\int_A \nu(\mathrm{d}x) = \int_A d \frac{\exp(bx/a)}{x \sinh(\pi x/a)} \mathrm{d}x,$$

	Meixner(α, β, δ)	**Meixner**$(\alpha, 0, \delta)$
mean	$\alpha\delta \tan(\beta/2)$	0
variance	$\frac{\alpha^2\delta}{2}(\cos^{-2}(\beta/2))$	$\frac{\alpha^2\delta}{2}$
skewness	$\sin(\beta/2)\sqrt{2/\delta}$	0
kurtosis	$3 + \frac{3-2\cos^2(\beta/2)}{\delta}$	$3 + \frac{1}{\delta}$

Empirical Density vs Meixner Density

The Meixner density can be fitted very accurately to the empirical density.

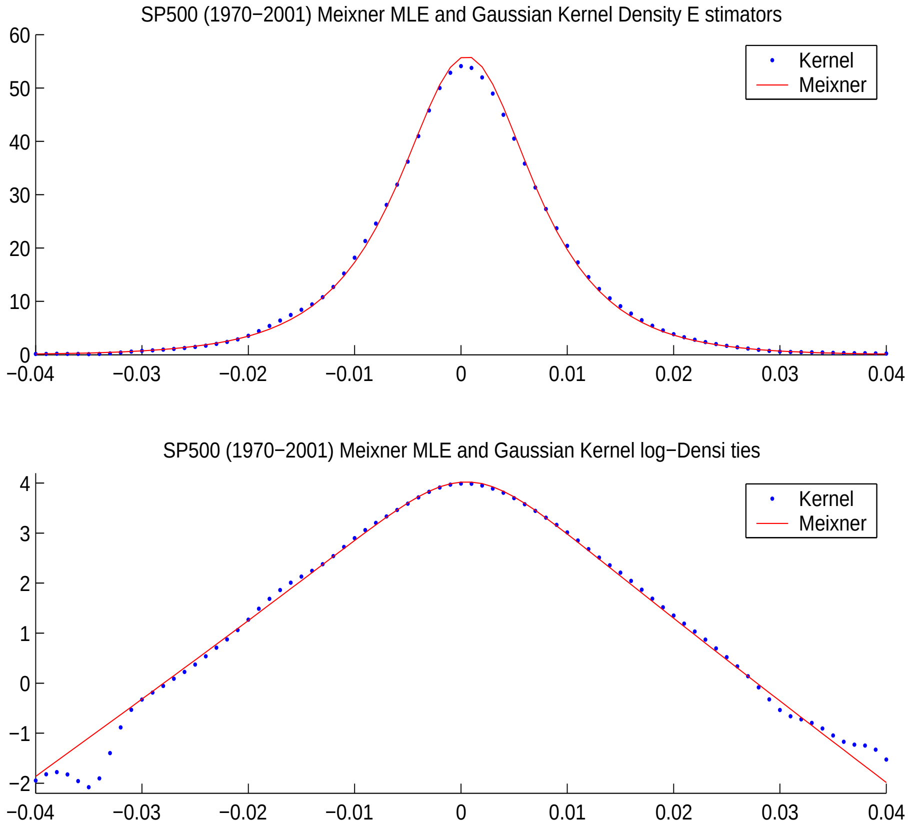

Figure 5: Meixner density and Gaussian Kernel estimator of the density of the daily log-returns of the SP500 index

Pricing of Vanilla Options

Let α be a positive constant such that the αth moment of the stock price exists. Carr and Madan (1998) then showed that

$$C(K,T) = \frac{\exp(-\alpha \log(K))}{\pi} \int_0^{+\infty} \exp(-\mathrm{i}v \log(K)) \varrho(v) \mathrm{d}v,$$

where

$$\begin{aligned} \varrho(v) &= \frac{\exp(-rT) E[\exp(\mathrm{i}(v - (\alpha+1)\mathrm{i}) \log(S_T))]}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v} \\ &= \frac{\exp(-rT)\phi(v - (\alpha+1)\mathrm{i})}{\alpha^2 + \alpha - v^2 + \mathrm{i}(2\alpha+1)v}. \end{aligned}$$

The Fast Fourier Transform can be used to invert the generalized Fourier transform of the call price. Put options can be priced using the put-call parity.

Meixner Option Prices

Calibrating the Meixner model prices to the market prices (in the least squared sense) leads to a better but not satisfactory fit of the option surface:

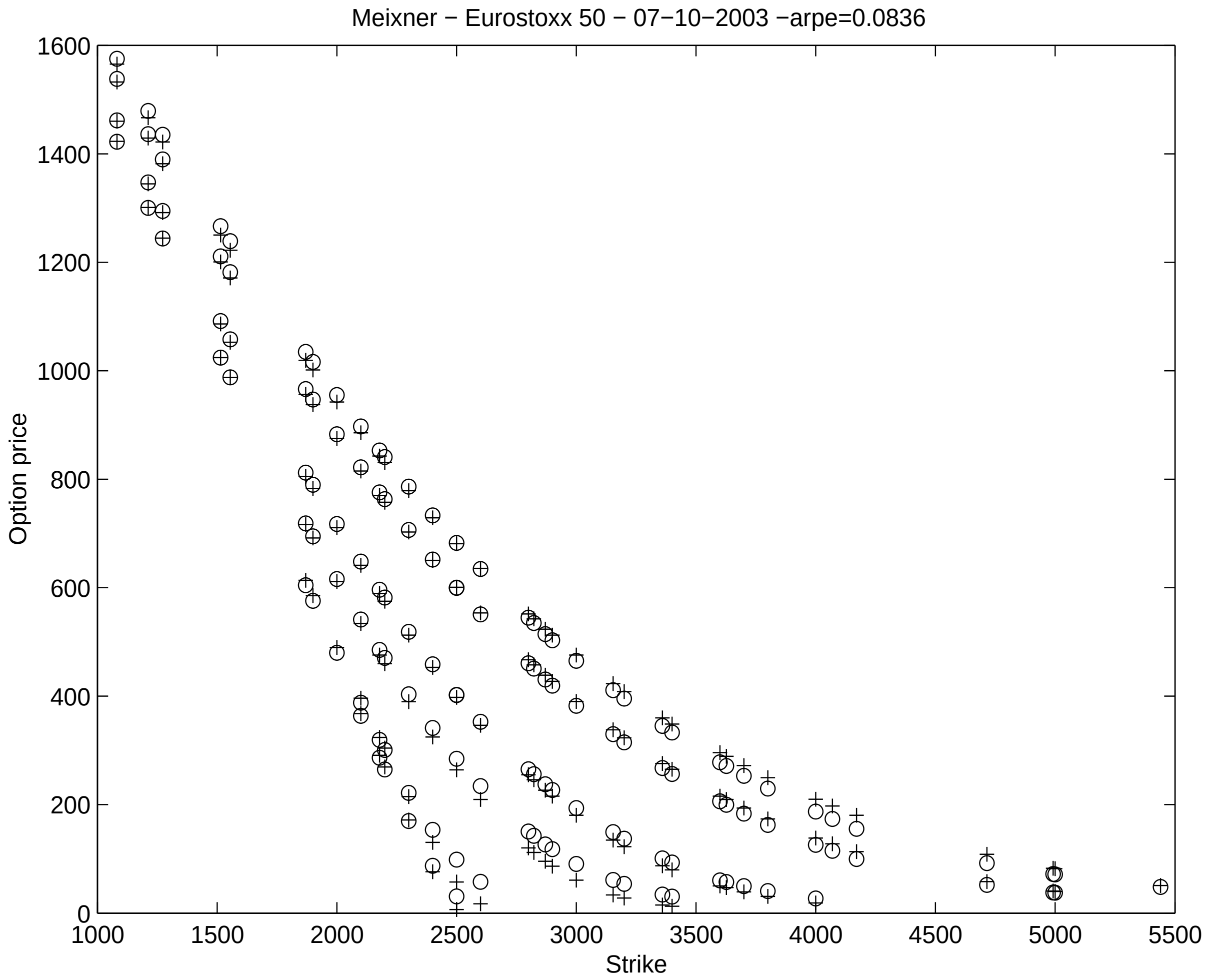

Figure 6: Meixner ($\alpha = 0.3801, \beta = -1.9553, \gamma = 0.3264$) calibration on Eurostoxx options (o's are market prices, +'s are model prices)

Non-Normal Returns and SV

There is a need for models who capture both non-normal returns and stochastic volatility:

- Heston Models (HEST)
- Heston with jumps Model (HESJ)
- Barndorff-Nielsen Shephard Models (BNS)
- Stochastic Volatility Lévy Models

Heston-Stochastic Volatility Model

- In the Heston-Stochastic Volatility model the stock price process follows the following SDE:

$$\frac{\mathrm{d}S_t}{S_t} = r\mathrm{d}t + \sigma_t \mathrm{d}W_t, \quad S_0 \geq 0;$$

  the (squared) volatility process is also assumed to be stochastic and is following the SDE

$$\mathrm{d}\sigma_t^2 = \kappa(\eta - \sigma_t^2)\mathrm{d}t + \lambda\sigma_t \mathrm{d}\tilde{W}_t, \quad \sigma_0 \geq 0,$$

  where $W = \{W_t, t \geq 0\}$ and $\tilde{W} = \{\tilde{W}_t, t \geq 0\}$ are two correlated standard Brownian motions such that $\mathrm{Cov}[\mathrm{d}W_t \mathrm{d}\tilde{W}_t] = \rho \mathrm{d}t$.

- The characteristic function $E[\exp(iu\log(S_t))] = \phi(u,t)$ is in this case given by:

$$\begin{aligned}
\phi(u,t) &= E[\exp(\mathrm{i}u\log(S_t))|S_0, \sigma_0^2] \\
&= \exp(\mathrm{i}u(\log S_0 + rt)) \\
&\quad \times \exp(\eta\kappa\lambda^{-2}((\kappa - \rho\lambda u\mathrm{i} + d)t - 2\log((1 - g\mathrm{e}^{dt})/(1-g)))) \\
&\quad \times \exp(\sigma_0^2\lambda^{-2}(\kappa - \rho\lambda \mathrm{i}u + d)(1 - \mathrm{e}^{dt})/(1 - g\mathrm{e}^{dt})),
\end{aligned}$$

where

$$\begin{aligned}
d &= ((\rho\lambda u\mathrm{i} - \kappa)^2 - \lambda^2(-\mathrm{i}u - u^2))^{1/2} \\
g &= (\kappa - \rho\lambda u\mathrm{i} + d)/(\kappa - \rho\lambda u\mathrm{i} - d)
\end{aligned}$$

HESJ

An extension of HEST introduces jumps in the asset price:

$$\frac{\mathrm{d}S_t}{S_t} = (r - q - \lambda\mu_J)\mathrm{d}t + \sigma_t \mathrm{d}W_t + J_t \mathrm{d}N_t, \quad S_0 \geq 0,$$

$$\mathrm{d}\sigma_t^2 = \kappa(\eta - \sigma_t^2)\mathrm{d}t + \theta\sigma_t \mathrm{d}\tilde{W}_t, \quad \sigma_0 \geq 0,$$

where

- $N = \{N_t, t \geq 0\}$ is an independent Poisson process with intensity parameter $\lambda > 0$
- the percentage jump size J_t follows

$$\log(1 + J_t) \sim \text{Normal}\left(\log(1 + \mu_J) - \frac{\sigma_J^2}{2}, \sigma_J^2\right),$$

- $W = \{W_t, t \geq 0\}$ and $\tilde{W} = \{\tilde{W}_t, t \geq 0\}$ are two correlated standard Brownian motions such that $\mathrm{Cov}[\mathrm{d}W_t \mathrm{d}\tilde{W}_t] = \rho \mathrm{d}t$,
- J_t and N are independent, as well as of W and of $\tilde{W}$.

HESJ

The characteristic function $\phi(u,t)$ is in this case given by:

$$\begin{aligned}\phi(u,t) &= E[\exp(\mathrm{i}u\log(S_t))|S_0,\sigma_0^2] \\ &= \exp(\mathrm{i}u(\log S_0 + (r-q)t)) \\ &\quad \times \exp(\eta\kappa\theta^{-2}((\kappa-\rho\theta u\mathrm{i}-d)t - 2\log((1-g\mathrm{e}^{-dt})/(1-g)))) \\ &\quad \times \exp(\sigma_0^2\theta^{-2}(\kappa-\rho\theta\mathrm{i}u-d)(1-\mathrm{e}^{-dt})/(1-g\mathrm{e}^{-dt})), \\ &\quad \times \exp(-\lambda\mu_J\mathrm{i}ut + \lambda t((1+\mu_J)^{\mathrm{i}u}\exp(\sigma_J^2(\mathrm{i}u/2)(\mathrm{i}u-1))-1)),\end{aligned}$$

where d and g are as in given above.

The BN-S Model (general case)

- In the BN-S model the squared volatility now follows a SDE of the form:

$$\mathrm{d}\sigma_t^2 = -\lambda\sigma_t^2\mathrm{d}t + \mathrm{d}\bar{z}_{\lambda t},$$

  where $\lambda > 0$ and $z = \{\bar{z}_t, t \geq 0\}$ is a Lévy process with only positive increments (a subordinator).

- The risk-neutral dynamics of the log-price $Z_t = \log S_t$ are given by

$$\mathrm{d}Z_t \ = \ (r - q - \lambda k(-\rho) - \sigma_t^2/2)\mathrm{d}t + \sigma_t\mathrm{d}W_t + \rho\mathrm{d}z_{\lambda t},$$

  – where $W = \{W_t, t \geq 0\}$ is a Brownian motion independent of $z = \{z_t, t \geq 0\}$

  – $k(u) = \log E[\exp(-uz_1)]$ is cumulant function of z_1.

- Note that the parameter ρ is introducing a correlation effect between the volatility and the asset price process.

The BN-S Model (Gamma-OU-case)

The Gamma-OU process:

- For this process $z = \{z_t, t \geq 0\}$ is a compound-Poisson process:

$$z_t = \sum_{n=1}^{N_t} x_n,$$

  - where $N = \{N_t, t \geq 0\}$ is a Poisson process with intensity parameter a, i.e. $E[N_t] = at$;
  - $\{x_n, n = 1, 2, \ldots\}$ is an i.i.d. sequence; each x_n follows a Exponential law with mean $1/b$.
- $\sigma^2 = \{\sigma_t^2, t \geq 0\}$ is a stationary process with marginal law that follows a Gamma distribution with mean a and variance a/b: starting the process with an initial value sampled from this Gamma distribution, at each future time point t, σ_t^2 is also following that Gamma distribution.
- $k(u) = \log E[\exp(-uz_1)] = -au(b+u)^{-1}$

The BN-S Model (Gamma-OU-case)

The characteristic function of the log-price process is given by

$$\begin{aligned}
&\phi(u,t)\\
&= E[\exp(\mathrm{i}u\log S_t)|S_0,\sigma_0]\\
&= \exp\left(\mathrm{i}u(\log(S_0)+(r-q-a\lambda\rho(b-\rho)^{-1})t)\right)\\
&\quad\times\exp\left(-\lambda^{-1}(u^2+\mathrm{i}u)(1-\exp(-\lambda t))\sigma_0^2/2\right)\\
&\quad\times\exp\left(a(b-f_2)^{-1}\left(b\log\left(\frac{b-f_1}{b-\mathrm{i}u\rho}\right)+f_2\lambda t\right)\right).
\end{aligned}$$

where

$$\begin{aligned}
f_1 = f_1(u) &= \mathrm{i}u\rho-\lambda^{-1}(u^2+\mathrm{i}u)(1-\exp(-\lambda t))/2,\\
f_2 = f_2(u) &= \mathrm{i}u\rho-\lambda^{-1}(u^2+\mathrm{i}u)/2.
\end{aligned}$$

Stochastic Volatility Lévy Models (CGMY 2001)

To build in stochastic volatility, we can also make time stochastic.

- We increase or decrease the level of uncertainty by speeding up or slowing down the rate at which time passes.
- To build clustering and to keep time going forward we employ a mean-reverting positive process y_t as a measure of the local rate of time change.
- The economic time elapsed in t units of calender time is then given by $Y(t)$ where

$$Y(t) = \int_0^t y_s \mathrm{d}s$$

There are different candidates for the time change:

- the CIR process that solves the SDE:

$$\mathrm{d}y_t = \kappa(\eta - y_t)\mathrm{d}t + \lambda y_t^{1/2}\mathrm{d}W_t$$

- OU-processes , i.e. the stationary process satisfying the SDE:

$$\mathrm{d}y_t = -\lambda y_t \mathrm{d}t + \mathrm{d}z_{\lambda t},$$

where z_t is a subordinator.

- The characteristic function ϕ of $Y(t)$ is explicitly known for:
  - the CIR-process,
  - the Gamma-OU process: $y_t \sim Gamma$,
  - the IG-OU Process: $y_t \sim IG$.
- We now model our (risk-neutral) price process S_t as follows:

$$S_t = S_0 \frac{\exp((r-q)t)}{E[\exp(X_{Y(t)})]} \exp(X_{Y(t)}),$$

where X_t is a Lévy process with

$$E[\exp(iuX_t)] = \exp(t\psi_X(u)).$$

- The characteristic function for the log of our stock price is given by:

$$E[\exp(iu\log(S_t))] = \exp(iu((r-q)t + \log S_0)\frac{\phi(-i\psi_X(u),t)}{\phi(-i\psi_X(-i),t)^{iu}}$$

Calibration: Eurostoxx 50 Option Prices (HEST)

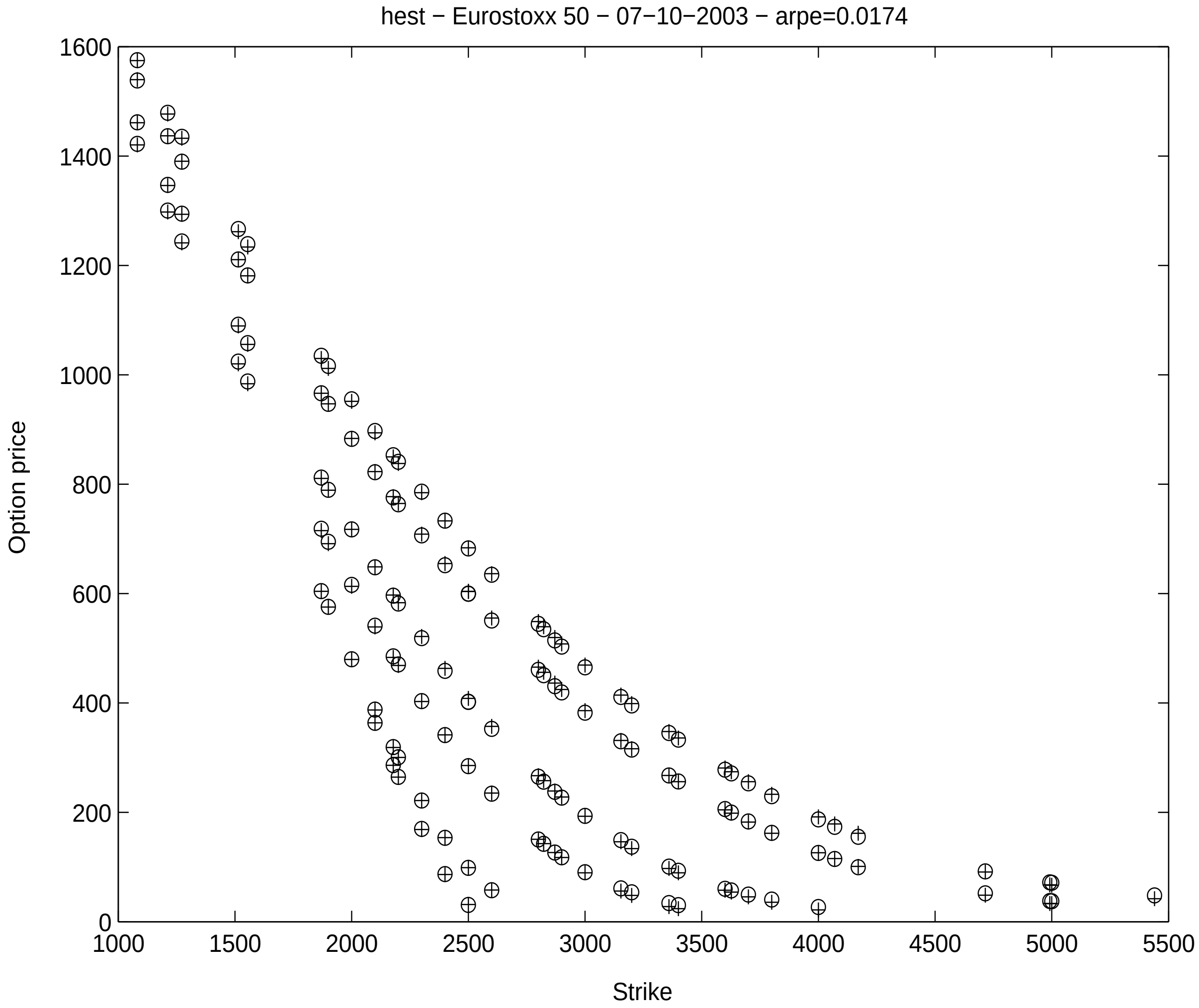

Figure 7: Calibration of Heston-Stochastic Volatility Model

Calibration: Eurostoxx 50 Option Prices (HESJ)

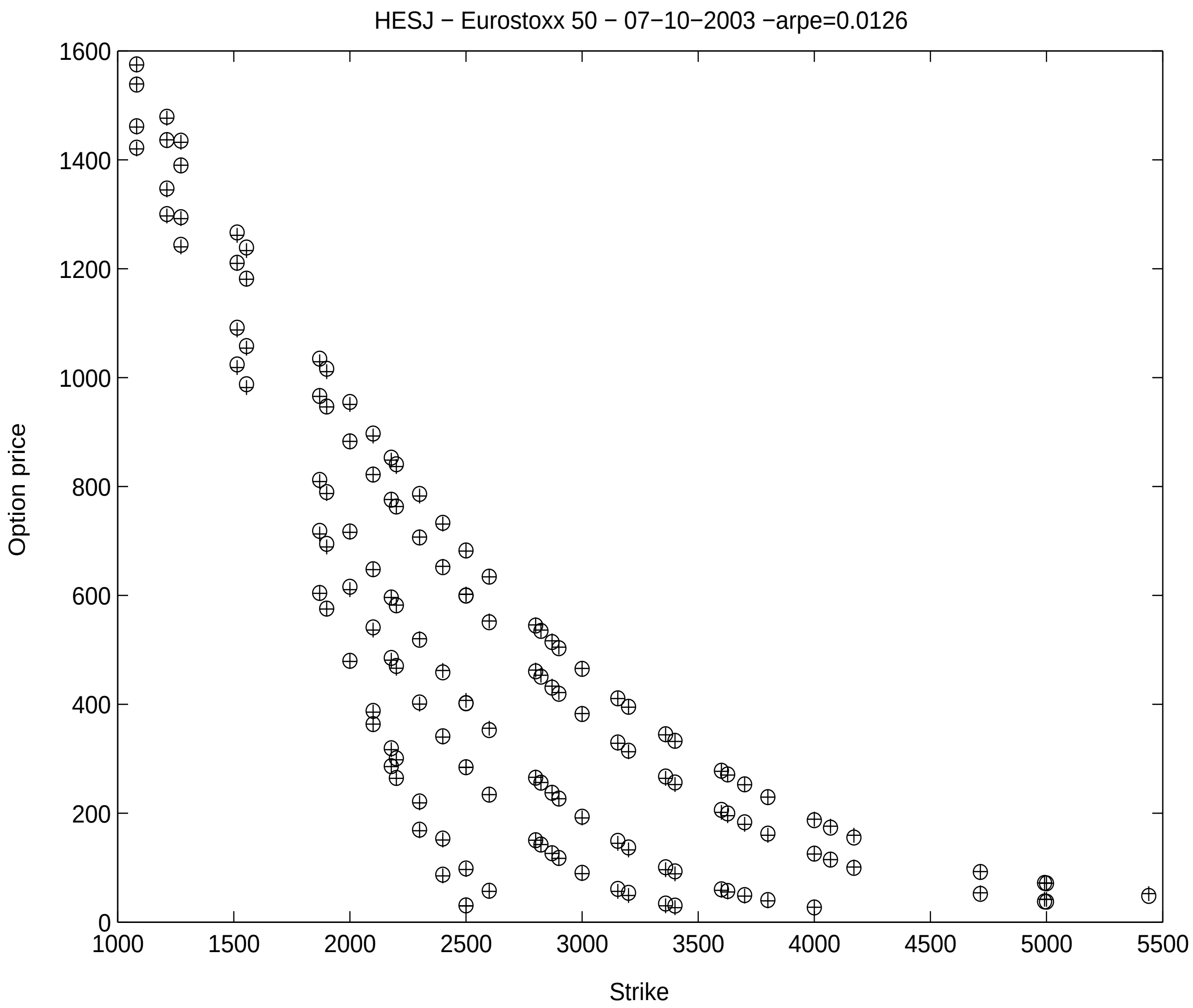

Figure 8: Calibration of Heston with Jumps Stochastic Volatility Model

Calibration: Eurostoxx 50 (BNS-OUGamma)

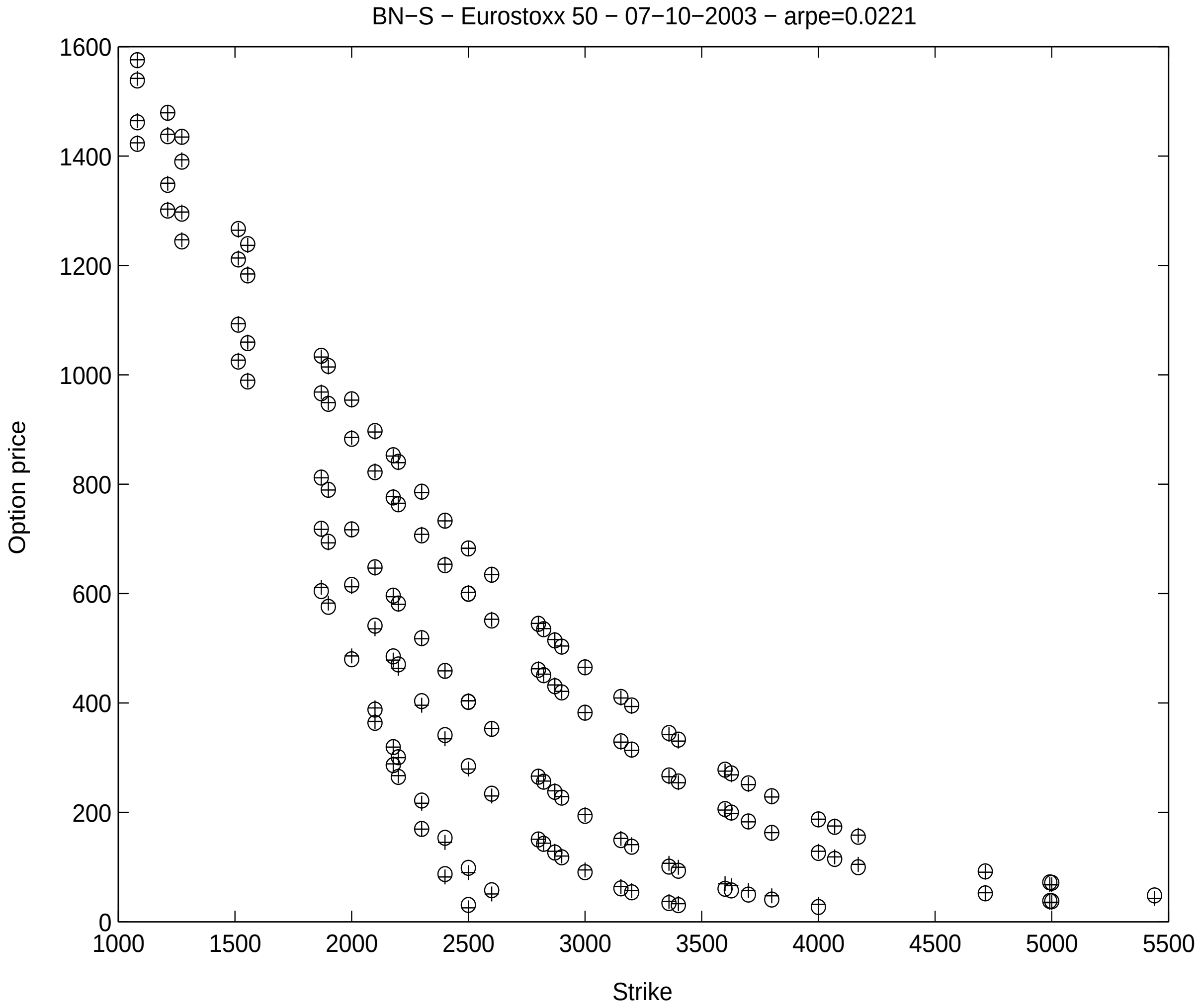

Figure 9: Calibration of Barndorff-Nielsen-Shephard Model

Calibration: Eurostoxx 50 (NIG-OUGamma)

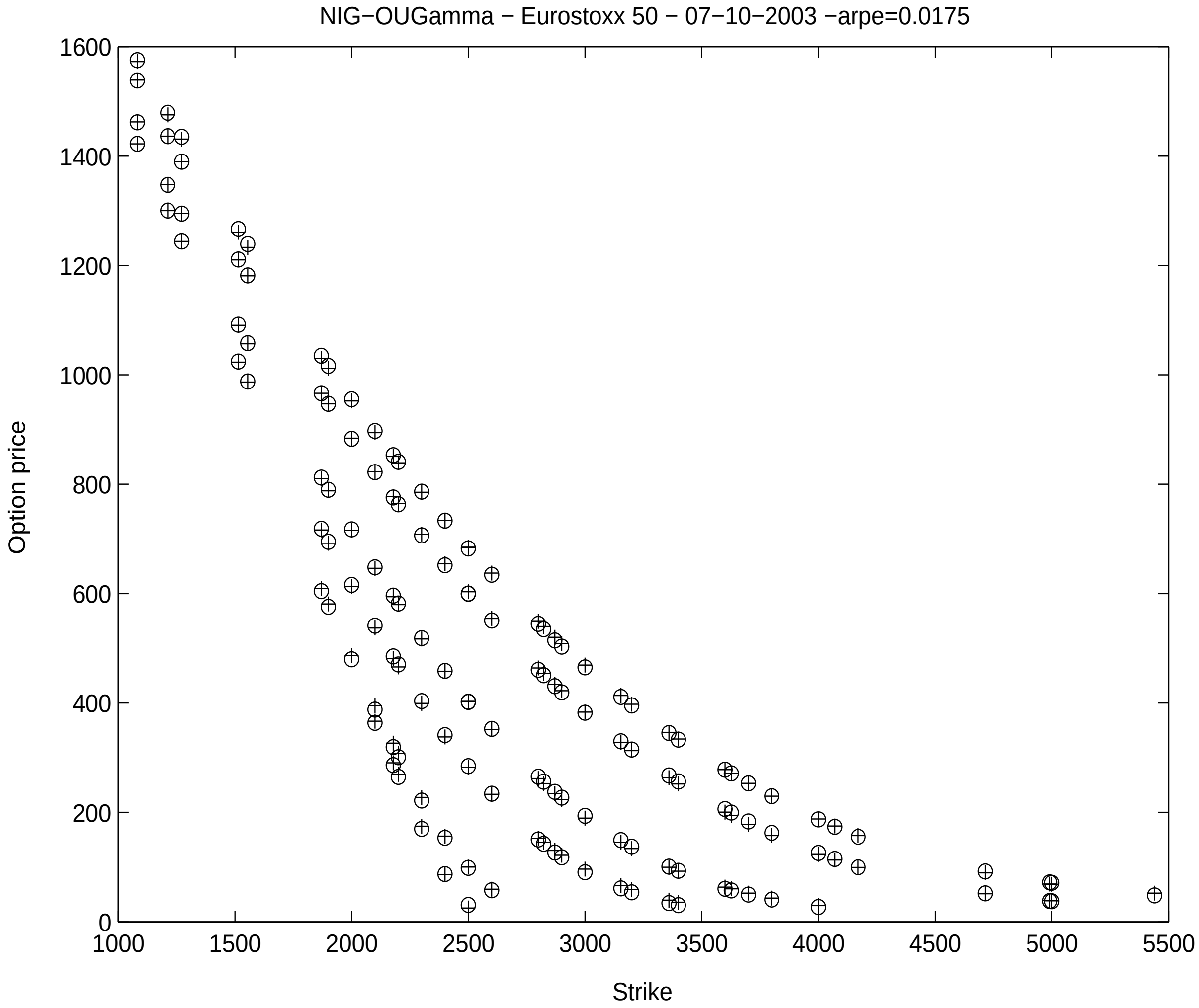

Figure 10: Calibration of NIG-Gamma Model

Calibration: Eurostoxx 50 (NIG-CIR)

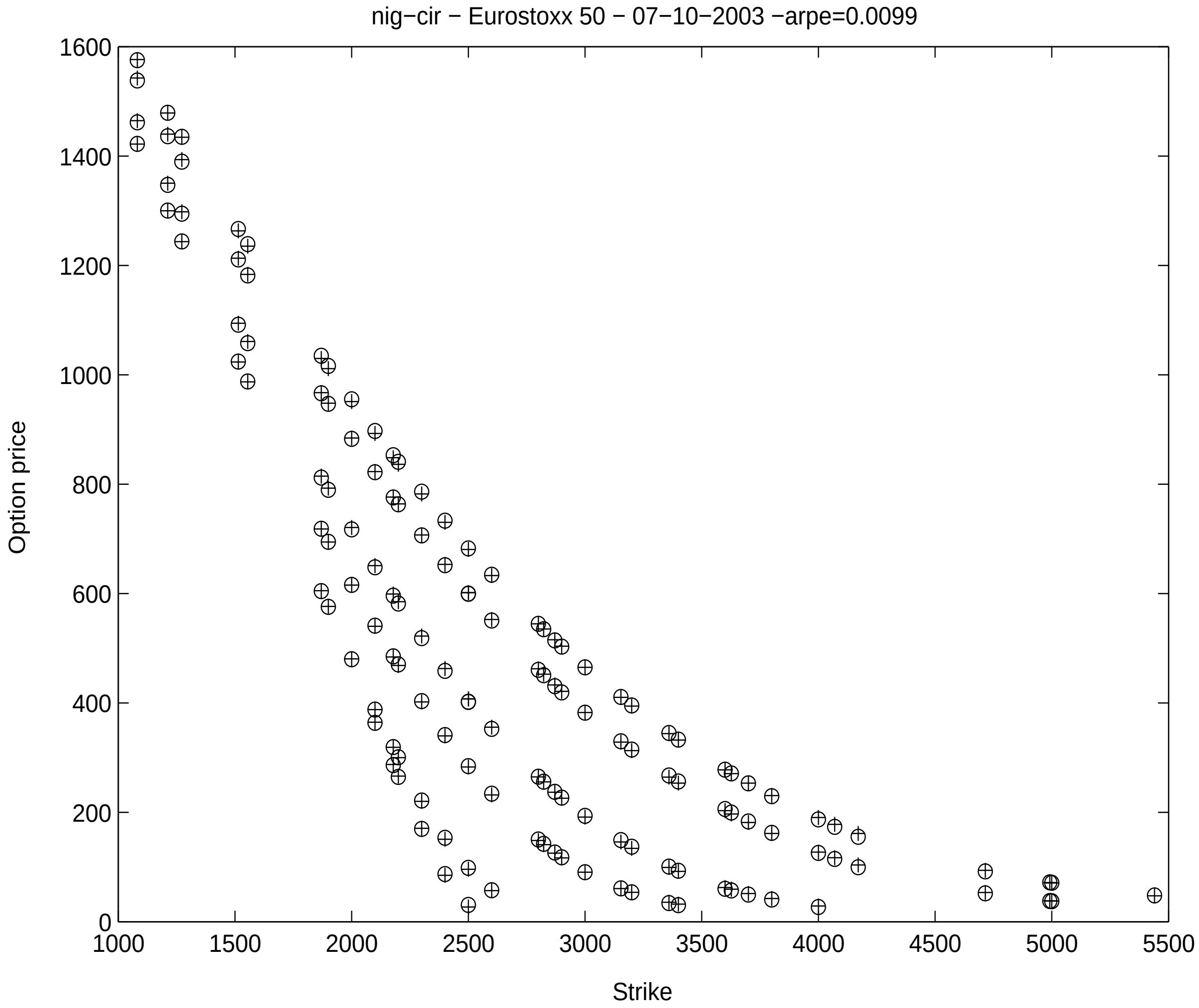

Figure 11: Calibration of NIG-CIR Model

Comparing the Models

In the next table, we compare the average relative pricing error:

$$arpe = \frac{1}{\text{number of options}} \sum_{options} \frac{|\text{Market price} - \text{Model price}|}{\text{Market price}}$$

Model:	arpe
BS	13.48 %
MEIXNER	8.36 %
HEST	1.74 %
HESJ	1.26 %
BN-S	2.21 %
VG-CIR	1.06 %
VG-OUΓ	1.90 %
NIG-CIR	0.99 %
NIG-OUΓ	1.75 %

Table 1: Global fit error measures

Simulation of the CIR Process

We discretize the SDE

$$\mathrm{d}y_t = \kappa(\eta - y_t)\mathrm{d}t + \lambda y_t^{1/2}\mathrm{d}W_t, \quad y_0 \geq 0.$$

The sample path of the CIR process $y = \{y_t, t \geq 0\}$ in the time points $t = n\Delta t$, $n = 0, 1, 2, \ldots$, is then given by

$$y_{n\Delta t} = y_{(n-1)\Delta t} + \kappa(\eta - y_{(n-1)\Delta t})\Delta t + \lambda y_{(n-1)\Delta t}^{1/2}\sqrt{\Delta t}v_n,$$

where $\{v_n, n = 1, 2, \ldots\}$ is a series of independent standard Normal random numbers.

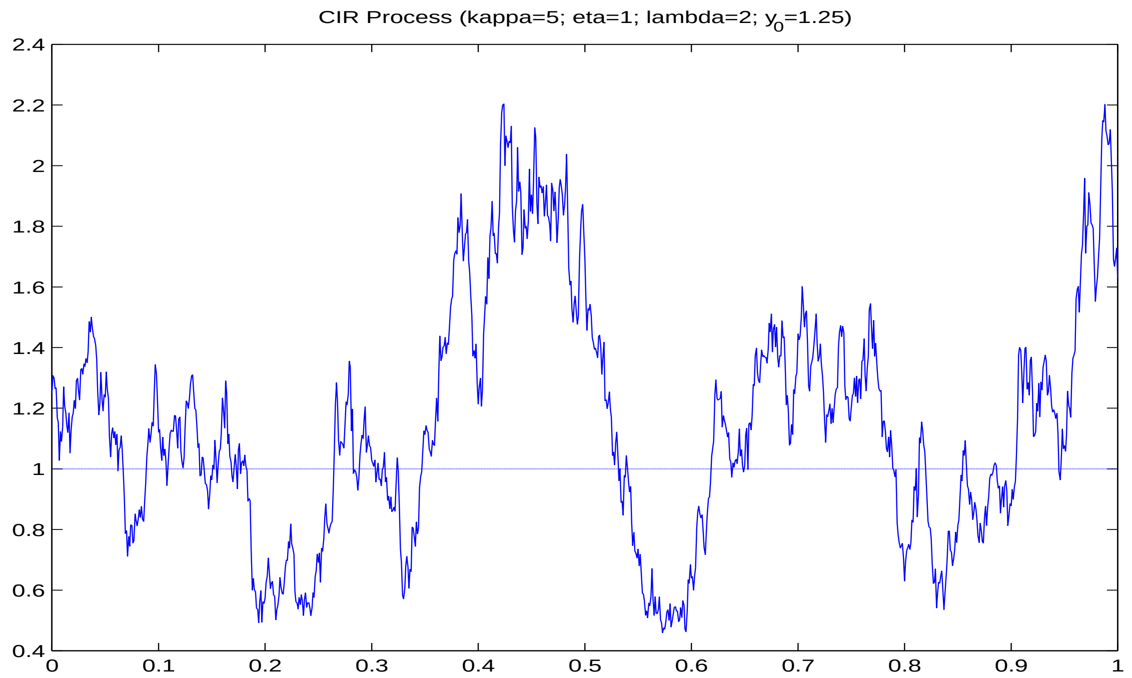

Figure 12: Simulation of CIR-process

Simulation of the Gamma-OU Process

Simulation of a Gamma(a,b)-OU process $y = \{y_t, t \geq 0\}$ in the time points $t = n\Delta t$, $n = 0, 1, 2, \ldots$:

- simulate in the same time points a Poisson process $N = \{N_t, t \geq 0\}$ with intensity parameter $a\lambda$;
- let $\tilde{u}_n$ are a series of independent Uniform random numbers;
- let x_n be exponential random numbers : $x_n = -\log(u_n)/b$, where u_n are a series of independent Uniform random numbers;
- then (with the convention that a empty sum equals zero)

$$y_{n\Delta t} = (1 - \lambda\Delta t) y_{(n-1)\Delta t} + \sum_{n=N_{(n-1)\Delta t}+1}^{N_{n\Delta t}} x_n \exp(-\lambda \Delta t \tilde{u}_n).$$

Simulation of the Gamma-OU Process

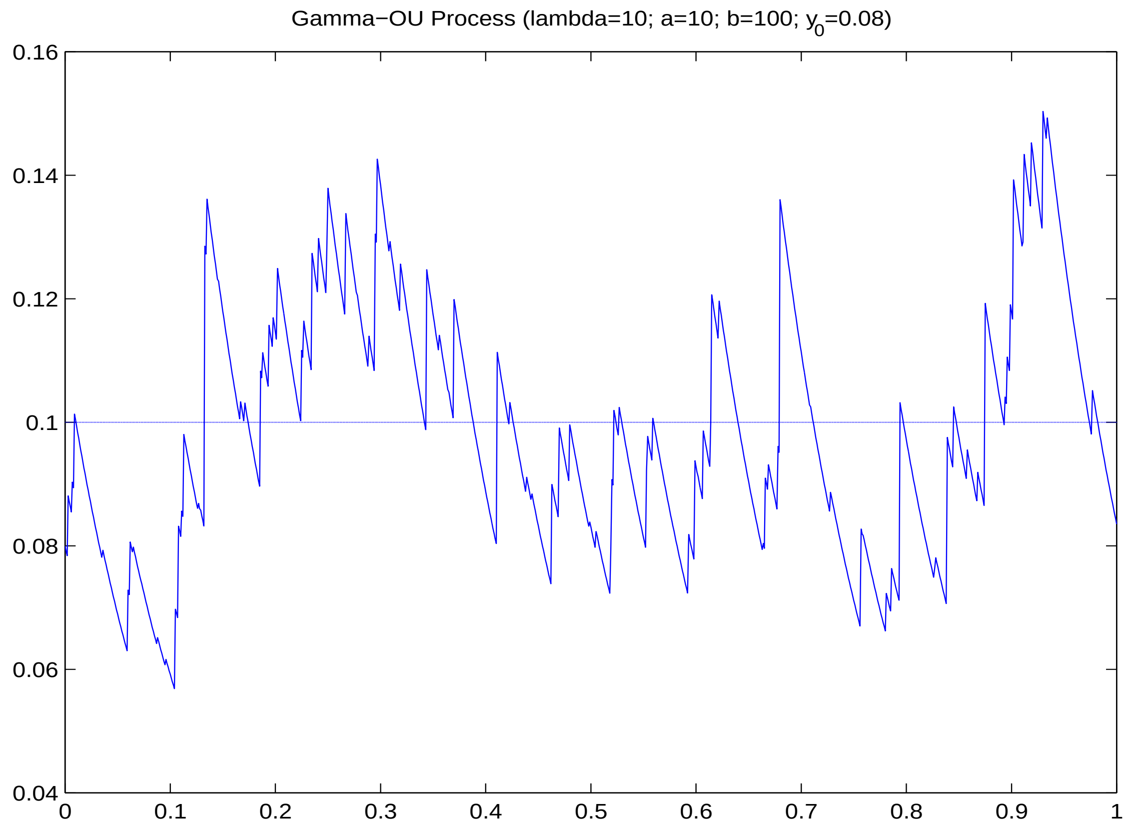

Figure 13: Simulation of Gamma-OU process

Simulation of the NIG-Lévy Process

Simulation of a NIG(α, β, δ) process $X = \{X_t, t \geq 0\}$ in the time points $t = n\Delta t$, $n = 0, 1, 2, \ldots$:

- First simulate IG$(\delta, \sqrt{\alpha^2 - \beta^2})$ random numbers i_k by for example using the Inverse Gaussian generator of Michael, Schucany and Haas.
- Then sample a sequence of standard Normal random variable u_k.
- NIG random numbers n_k are then obtain by:

$$n_k = \beta i_k + \sqrt{i_k} u_k;$$

- Finally, $X_0 = 0, \qquad X_{n\Delta t} = X_{(k-1)\Delta t} + n_k, \quad k \geq 1.$

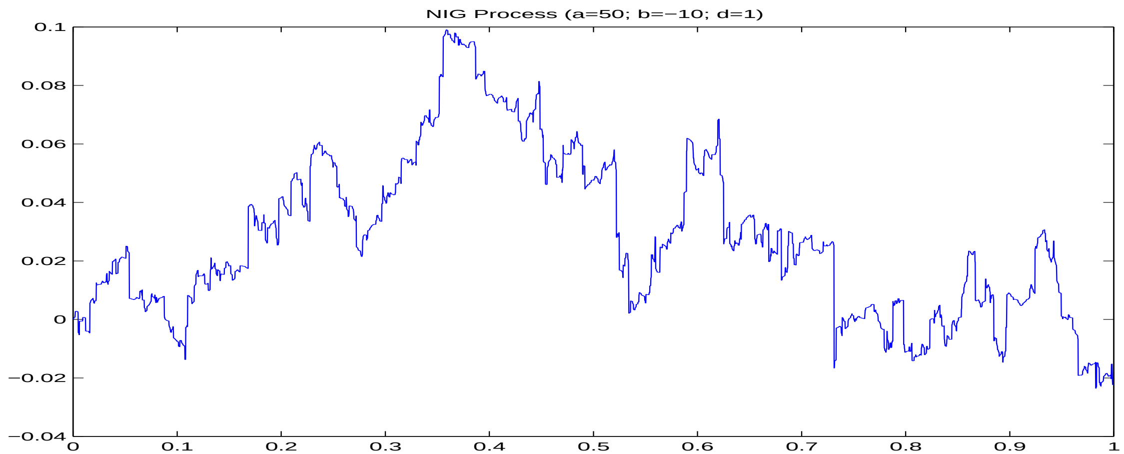

Figure 14: Simulation of NIG process

Simulation of all Ingredients: The NIG-CIR combination

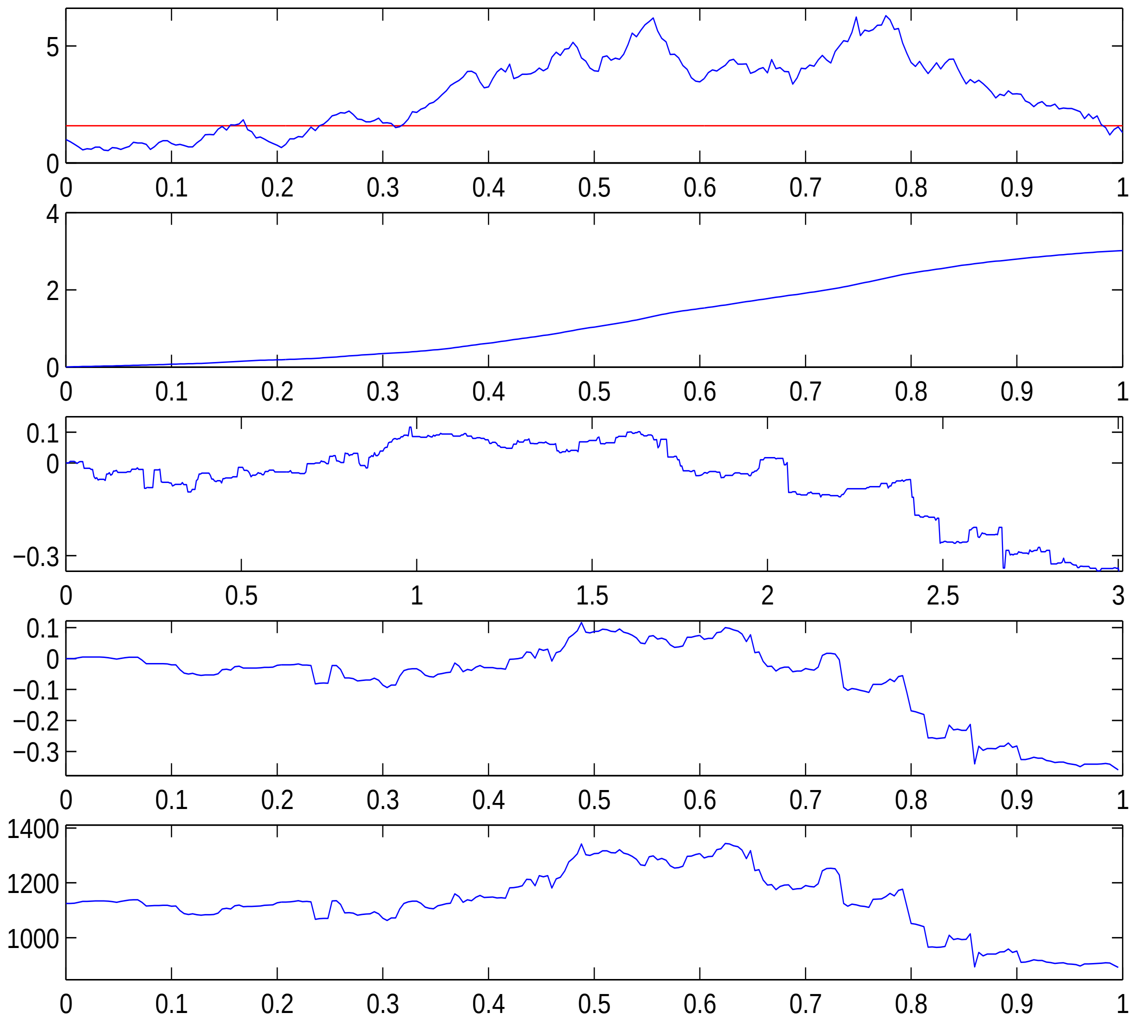

Figure 15: Simulation of y_t, Y_t, X_t, X_{Y_t}, and S_t

Exotic Options

Next, we will price a whole range of exotic options and compare the prices obtained under the different models.

- The down-and-out barrier call option (DOB)
- The up-and-in barrier call option (UIB)
- Lookback Call Option
- Digital Barrier Options
- Cliquets

$$\min\left(cap_{glob}, \max\left(floor_{glob}, \sum_{i=1}^{N} \min\left(cap_{loc}, \max\left(floor_{loc}, \frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)\right)\right)\right)$$

The cliquet option payoff depends on N future stock prices values.

Exotic Monte-Carlo Option Pricing and Back-Testing

- We price all options using 1.000.000 simulated paths.
- For all the time to maturity is $T = 3$ and $K = S_0$.
- In order to check the accuracy of our simulation algorithm we simulated option prices for all European calls available in the calibration set: the price difference were always less than 0.5 percent.

Lookback Prices

HEST	HESJ	BN-S	VG-CIR	VG-OUΓ	NIG-CIR	NIG-OUΓ
838.48	845.19	771.28	724.80	713.49	730.84	722.34

Table 2: Lookback Option Prices

Digital Option Prices

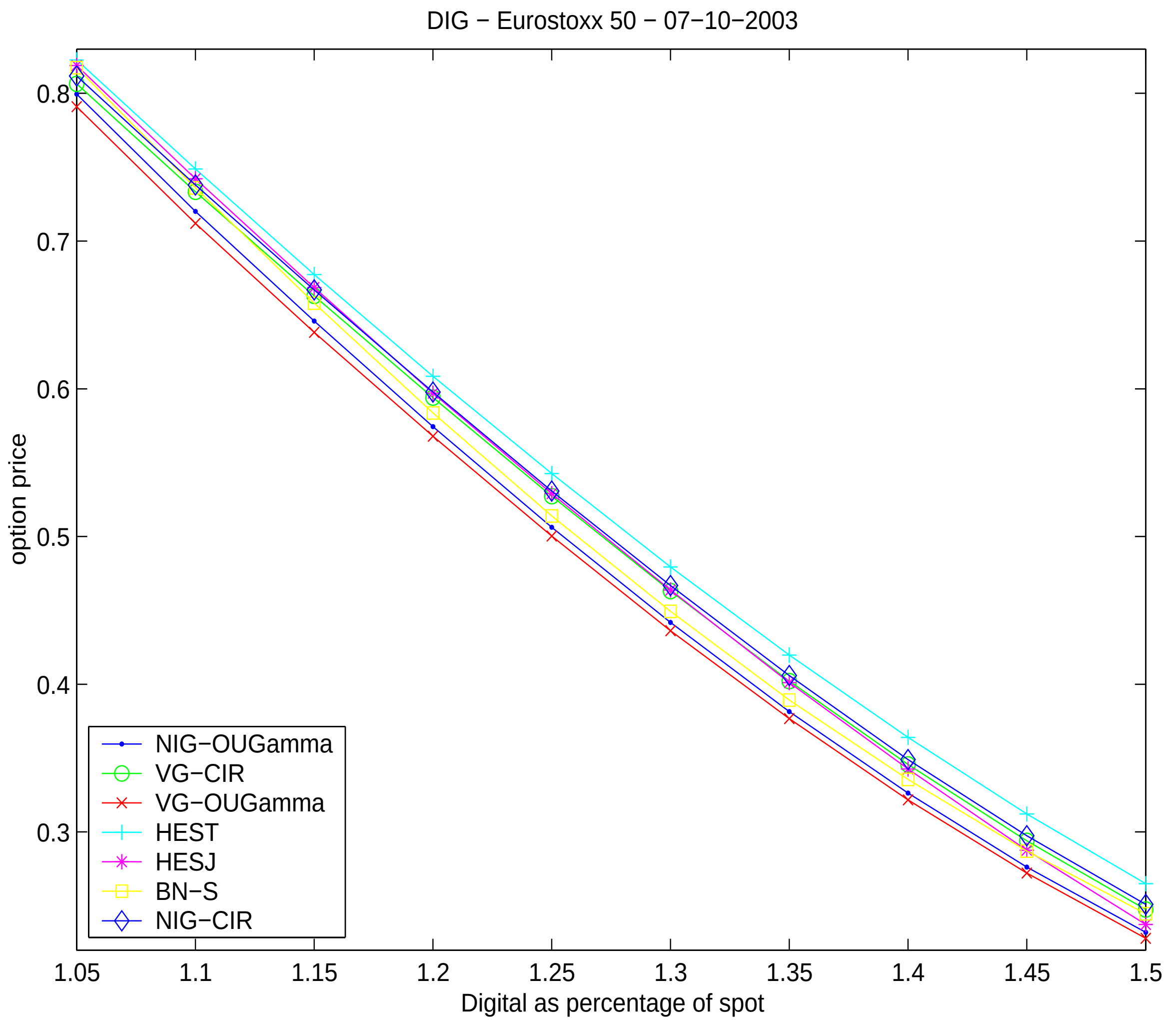

Figure 16: Digital Barrier prices

Barrier Option Prices (DOB)

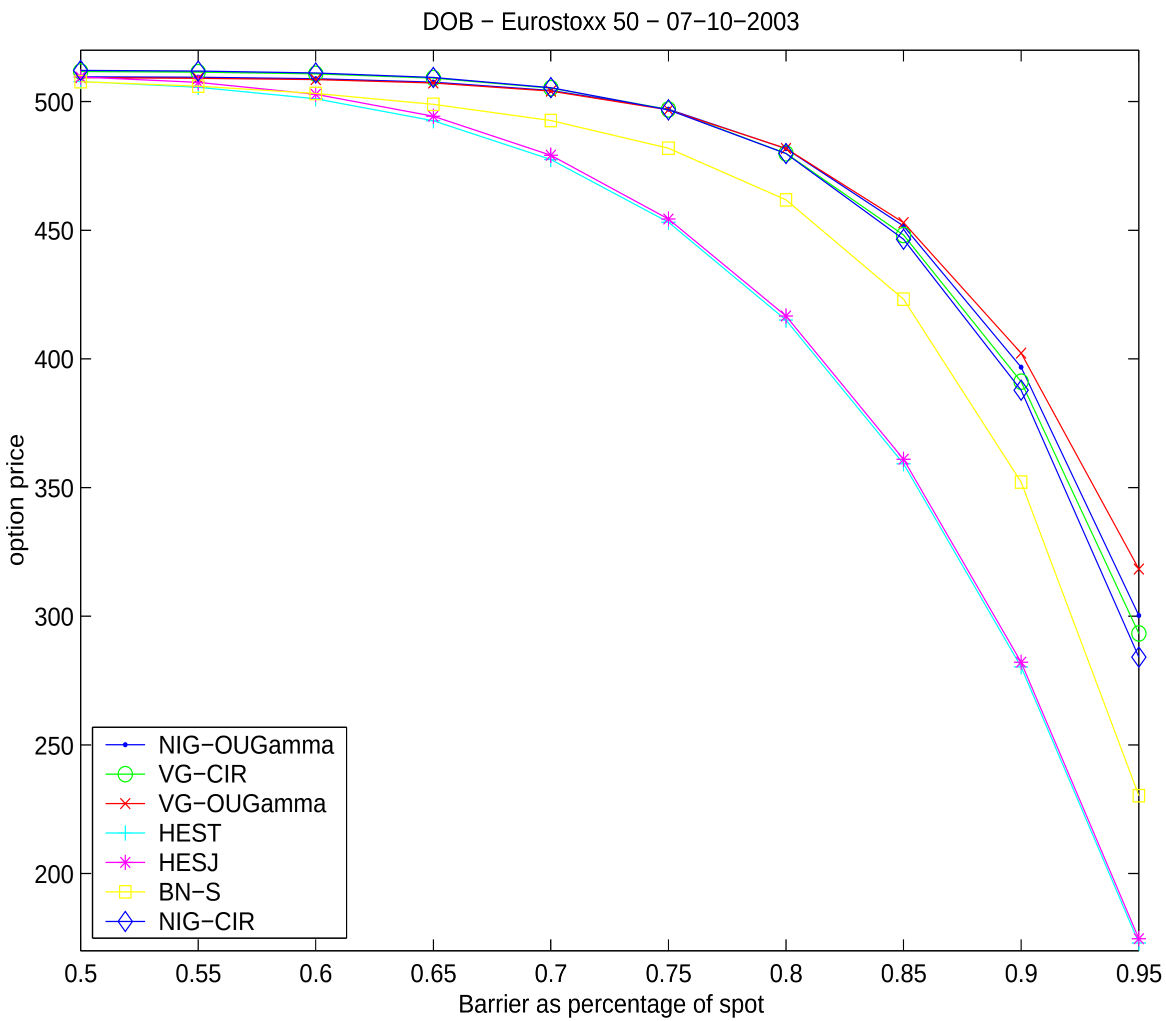

Figure 17: DOB prices

Barrier Option Prices (UIB)

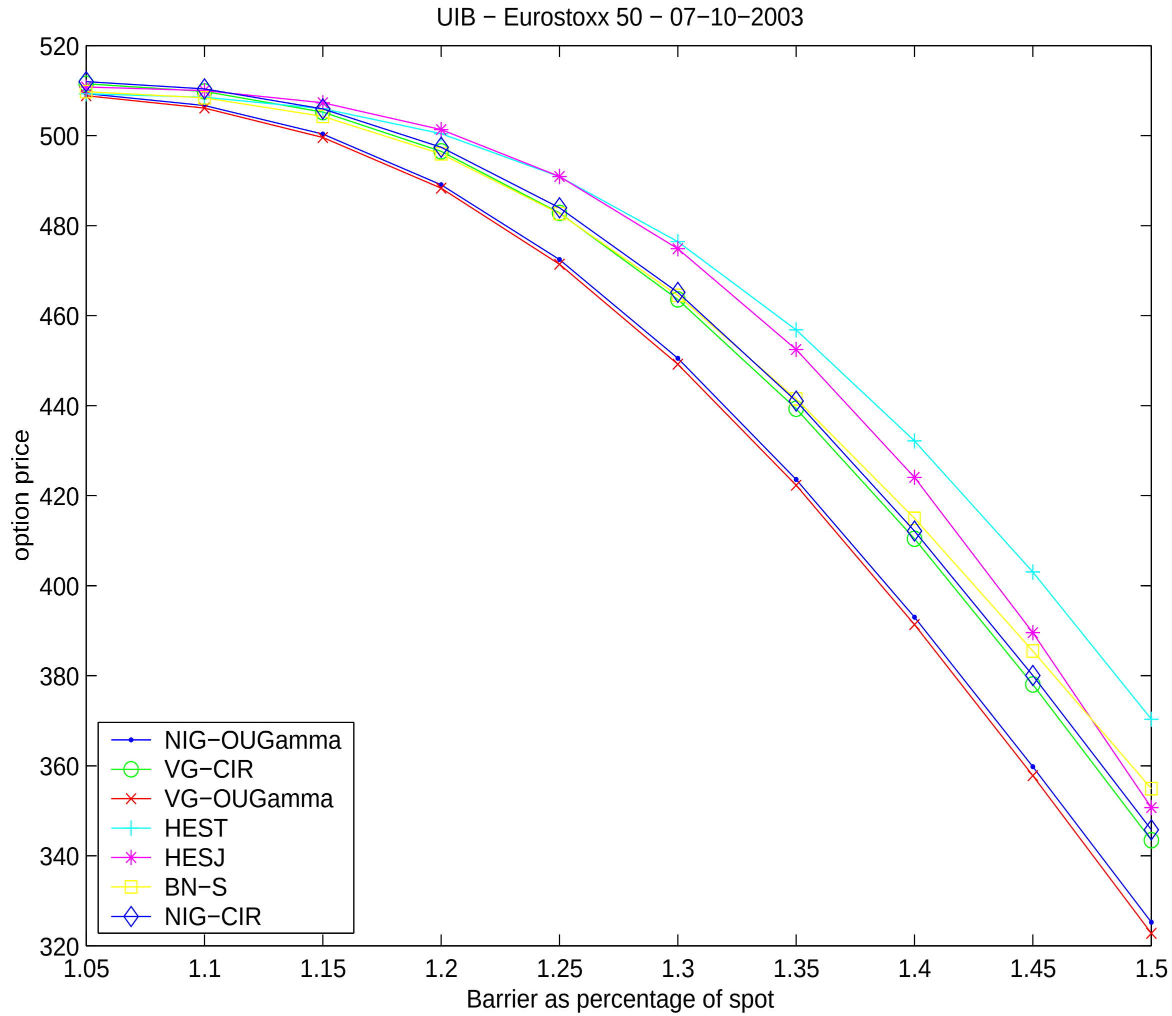

Figure 18: UIB prices

Cliquet Option Prices

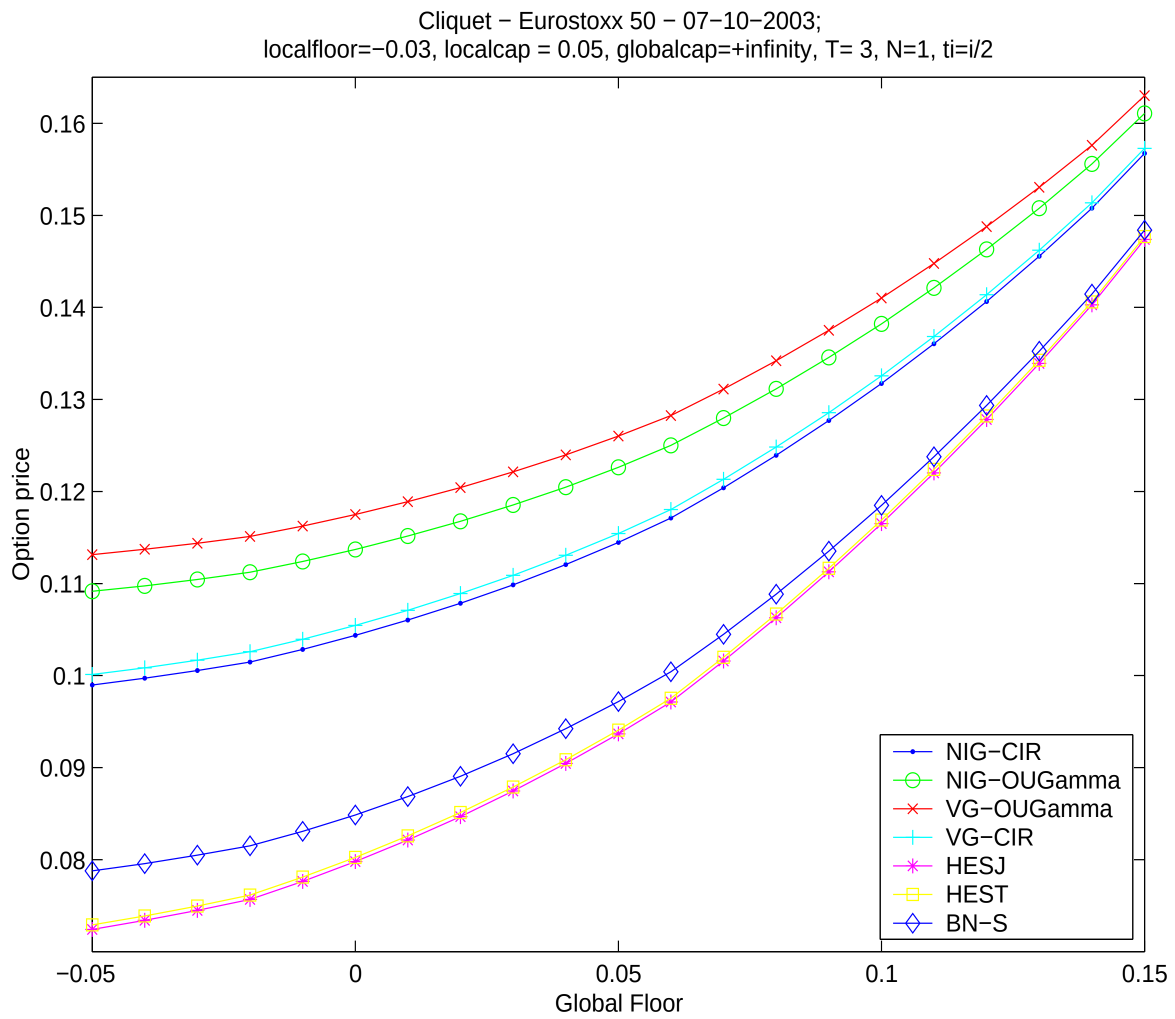

Figure 19: Cliquet Prices: $flo_{loc} = -0.03$, $cap_{loc} = 0.05$, $cap_{glo} = +\infty$, $T = 3$, $N = 6$, $t_i = i/2$

Moment Derivatives

- We have a clear issue of model risk.
- We push this study a little bit further by looking at moment derivatives[1].
- Their payoff is a function of powers of the (daily) log-returns and allow to cover different kinds of market shocks.
- Variance swaps were already created to cover changes in the volatility regime.
- However skewness and kurtosis, also play an important role. To protect against a wrongly estimated skewness or kurtosis, moment derivatives of higher order can be useful.
- Recent theoretical work (Corcuera-Nualart-S, 2004) shows that moment assets are the instruments which can naturally complete a Lévy driven market.

[1] by some people also called Belgian Chocolate Derivatives

Moment Swaps

- Consider a finite set of discrete times $\{t_0 = 0, t_1, \ldots, t_n = T\}$ at which the path of the underlying is monitored.
- Denote the price of the underlying at these points, i.e. S_{t_i}, by S_i for simplicity.
- The kth-moment swap is a contract where the parties agree to exchange at maturity:

$$MOMS^{(k)} = N \times \left(\sum_{i=1}^{n} (\log(S_i) - \log(S_{i-1}))^k \right) = N \times \left(\sum_{i=1}^{n} \left(\log(\frac{S_i}{S_{i-1}}) \right)^k \right),$$

  where N is the nominal amount.

- Variance Swap:

$$VS = N \times \left(\sum_{i=1}^{n} (\log(S_i) - \log(S_{i-1}))^2 \right).$$

- $MOMS^{(3)}$ is related to realized skewness and provides protection against changes in the symmetry of the underlying distribution.
- $MOMS^{(4)}$ derivatives are linked to realized kurtosis and provide protection against the unexpected occurences of very large jumps

Hedging Moment Swaps-1

- We use the following (Taylor-like) expansion of the kth power of the logarithmic function and work with swaps with future prices $F_i = \exp((r-q)(T-t_i))S_i$ as underlying:

$$\frac{(\log(x))^k}{k!} = x - 1 - \log(x) - \frac{(\log(x))^2}{2!} - \frac{(\log(x))^3}{3!} - \ldots - \frac{(\log(x))^{k-1}}{(k-1)!} + \mathcal{O}((x-1)^{k+1}).$$

- Substituting x by F_i/F_{i-1} leads to:

$$(\log(F_i/F_{i-1}))^k = k!\left(\frac{\Delta F_i}{F_{i-1}} - \log(F_i/F_{i-1}) - \sum_{j=2}^{k-1}\frac{(\log(F_i/F_{i-1}))^j}{j!} + \mathcal{O}((\Delta F_i/F_{i-1})^{k+1})\right),$$

where $\Delta F_i = F_i - F_{i-1}$.

- Summing over i gives a decomposition of the $MOMS^{(k)}$ payoff:

$$\begin{aligned} MOMS^{(k)} &= -Nk!\left(\log(F_T) - \log(F_0)\right) \\ &\quad + Nk!\sum_{i=1}^{n}\frac{\Delta F_i}{F_{i-1}} \\ &\quad - \sum_{j=2}^{k-1}\frac{Nk!}{j!}MOMS^{(j)} \\ &\quad + \mathcal{O}\left(\sum_{i=1}^{n}(\Delta F_i/F_{i-1})^{k+1}\right) \end{aligned}$$

Hedging Moment Swaps-2

- Thus up to $(k+1)$th-order terms the sum of the kth powered log-returns decomposes into the payouts from:
  - a log-contract $(-k!\ (\log(F_T) - \log(F_0)))$;
  - a dynamic strategy $(k!\,\Sigma_{i=1}^{n} \frac{\Delta F_i}{F_{i-1}})$;
  - a series of moment contracts of order strictly smaller than k.
- The log-contract itself can be hedged by a static position in the underlying, in a bond and in a (discrete approximation of a continuous) set of European vanilla call and put options maturing at time T (Carr-Lewis).
- Note that

$$\begin{aligned} LOG &= \exp(-rT)E_Q[\log(S_T) - \log(S_0)|\mathcal{F}_0] \\ &= \exp(-rT)\left(-\mathrm{i}\frac{\partial \phi(0,t)}{\partial u} - \log(S_0)\right) \end{aligned}$$

Hence:

$$\begin{aligned} LOG_{hest} &= \exp(-rT)(2\kappa)^{-1}\left(2\kappa rT - \eta\kappa T - \eta \mathrm{e}^{-\kappa T} + \eta - \sigma_0^2 + \sigma_0^2 \mathrm{e}^{-\kappa T}\right) \\ LOG_{LevySV} &= \exp(-rT)\left(rT + E[Y_T]E[X_1] - \log(E[\exp(X_{Y_T})])\right) \\ LOG_{BN-S} &= \exp(-rT)\left(rT + \frac{1}{2b\lambda}\left((a - b\sigma_0^2)(1-\mathrm{e}^{-\lambda T}) - a\lambda T(1 + 2\lambda\rho^2(b-\rho)^{-1})\right)\right) \end{aligned}$$

Moment Options

- Related to the above discussed swaps, we define the associated options on the realized kth moment. More precisely, a *moment option* of order k, pays out at maturity T:

$$\left(\sum_{i=1}^{n}(\log(S_i/S_{i-1}))^k - K\right)^+ .$$

- The price of these options under risk-neutral valuation is given by:

$$MOMO^{(k)}(K,T) = \exp(-rT)E_Q\left[\left(\sum_{i=1}^{n}(\log(S_i/S_{i-1}))^k - K\right)^+\right].$$

- Note that since odd moments can be negative, the strike price for these options can range over the whole real line.

Pricing of Moments Swaps

order	HEST	BN-S	VG-CIR	VG-OUΓ	NIG-CIR	NIG-OUΓ
$e^{-rT}E_Q\left[MOMS^{(2)}\right]$	623.89	804.60	557.55	628.85	557.75	641.71
$e^{-rT}E_Q\left[MOMS^{(3)}\right]$	-0.0807	-312.58	-21.03	-74.91	-21.69	-88.82
$e^{-rT}E_Q\left[MOMS^{(4)}\right]$	0.6366	322.40	7.8698	33.89	8.554	47.99

Table 3: Moment swaps ($N = 10000$)

Pricing of 2nd-Moment Option

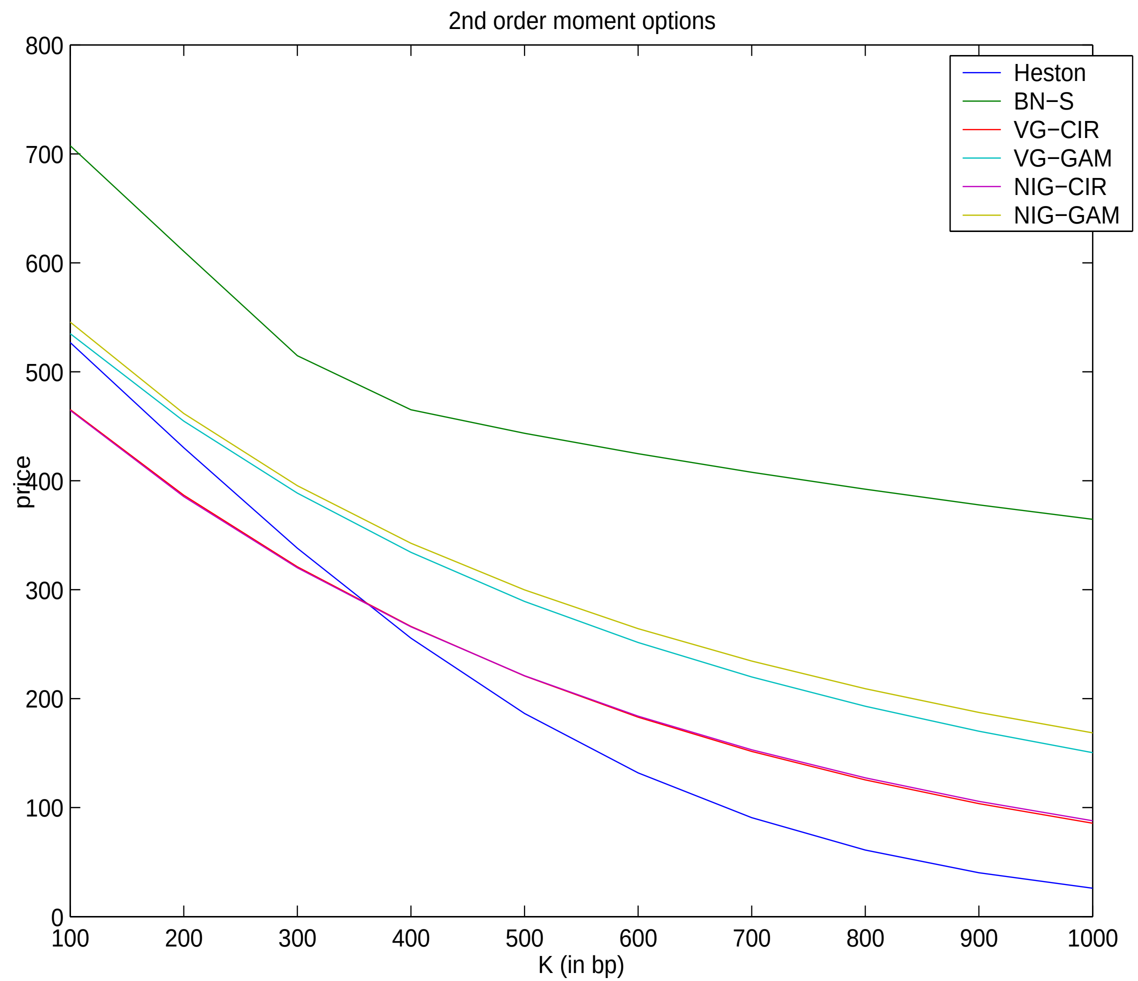

Figure 20: Moment option of 2nd order

Pricing of 3rd-Moment Option

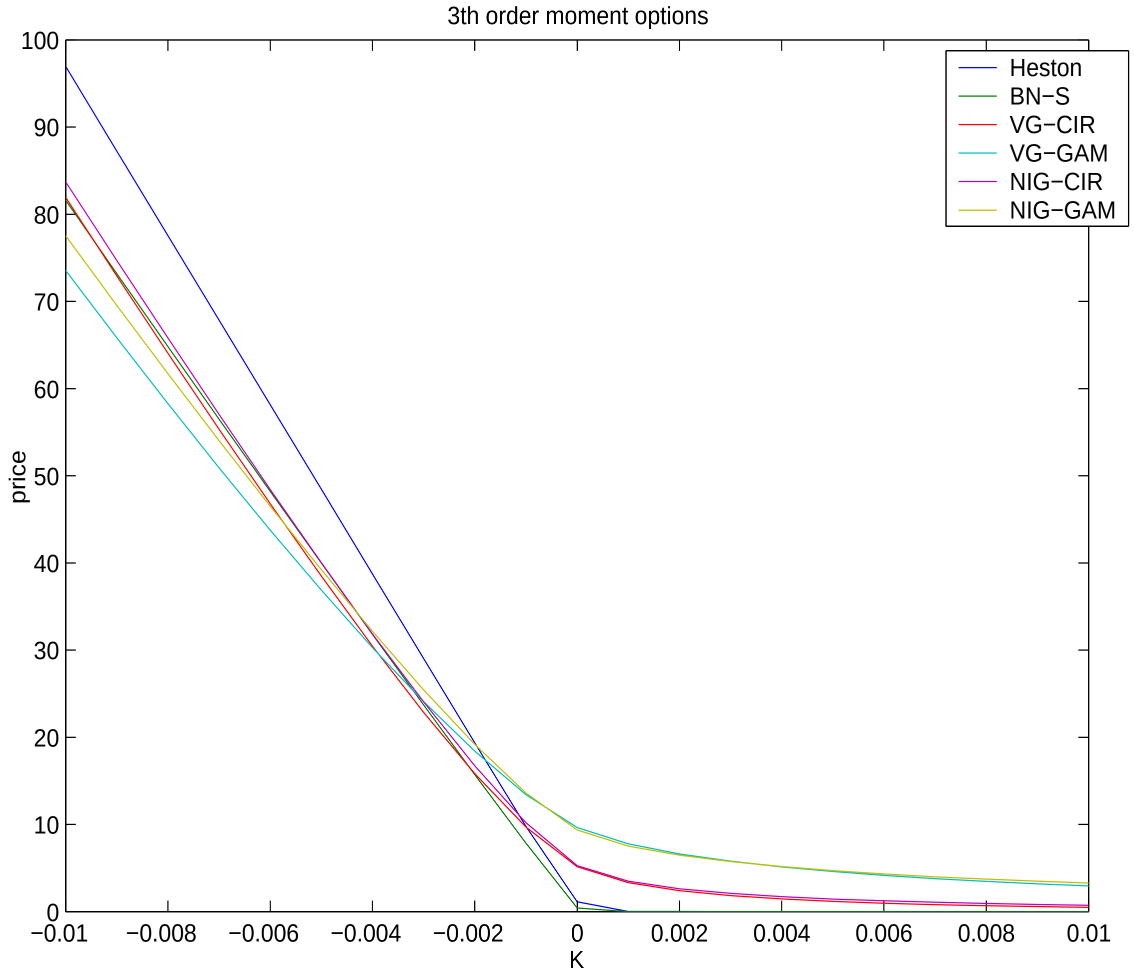

Figure 21: Moment option of 3rd order

Pricing of 4th-Moment Option

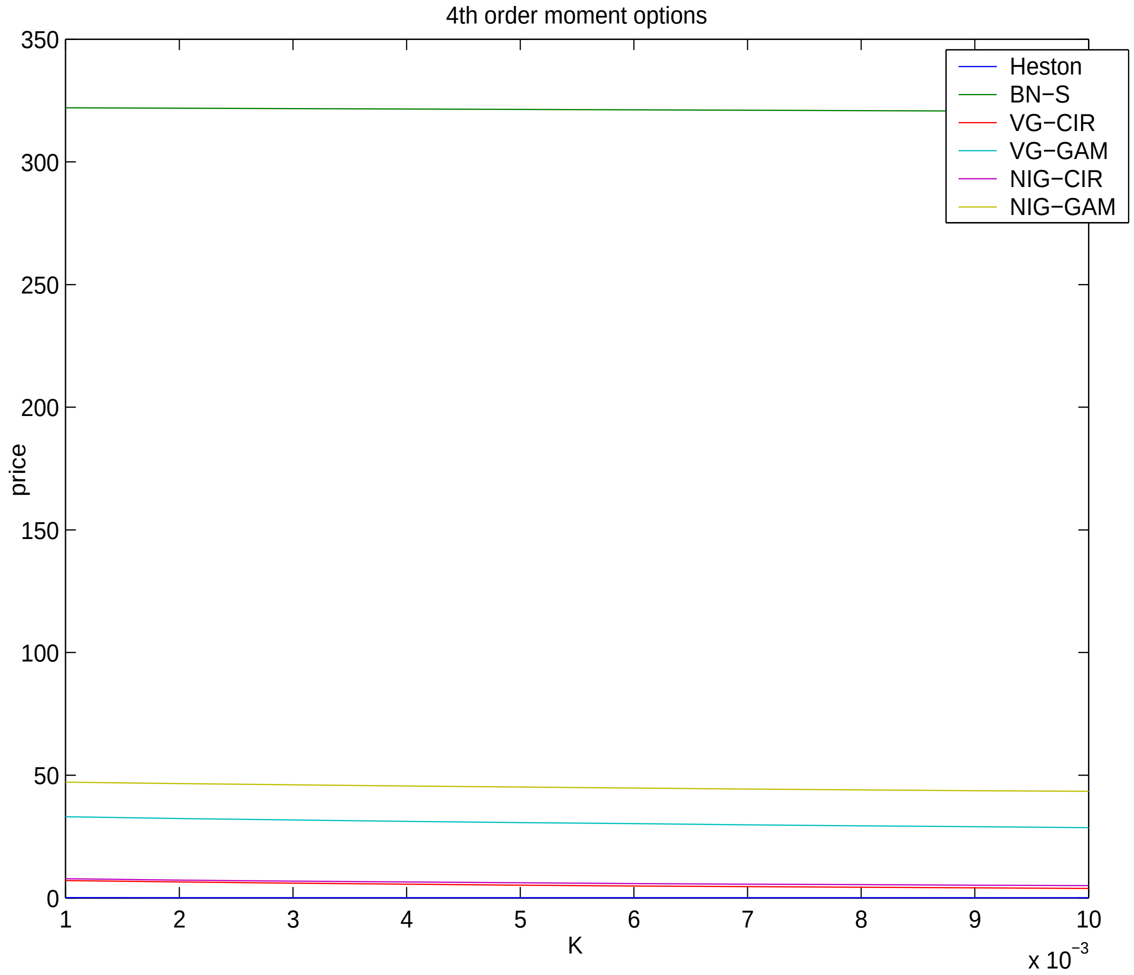

Figure 22: Moment option of 4th order

Conclusion

- The Black-Scholes model does not correspond with reality.
- Moreover, Black-Scholes- exotic option prices depend heavily one the choice of the volatility parameter and it is not clear which value to take.
- More advanced Lévy process-based models give a very accurate fit to real market option data.
- Monte-Carlo simulation for these models is possible.
- Prices of exotics however can differ significantly over different attractive models.
- We have a clear issue of model risk.
- We have push thius study further by pricing moment derivatives.
- Moment derivatives are related to realized higher moments.
- Hedging aspects of moment swaps were discussed.
- By pricing moment swaps and moment options, we have shown that the disparity between the models is amplified.
- More detailed studies are needed in order to differentiate between the models.

THE END