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Pricing Forward Start Options under the CEV Model

With Applications in Financial Engineering

Abstract

Index-linked securities are offered by banks, financial institutions and building societies to investors looking for downside risk protection whilst still providing upside equity index participation. This article explores how reverse cliquet options can be integrated into the structure of a guaranteed principal bond.

Pricing problems are discussed under the constant-elasticity-of-variance model. Forward start options are the main element of this structure and new closed formulae are obtained for these options under the square-root process model. Risk management issues are also discussed. An example is described showing how this structure can be implemented and how the financial engineer may forecast the coupon payment that will be made to investors buying this product.

Key words: cliquet options, structured products, constant-elasticity-of-variance model, forward-start options,

JEL codes: G13, G24.

I. INTRODUCTION

From an investor's point of view traditional equity-linked instruments provide an opportunity to participate indirectly in the performance of a single share. For the last two decades increasingly complex, customised structures have been created in a way that enables, in many cases, regulatory constraints on the use of derivative securities, such as forwards, futures and options, to be by-passed. Convertible bonds provide a good example of an instrument that customarily has a pay out profile of a call option and that have been available to investors for many years. Liquid Yield Option Notes™ ® (LYONs™ ®) evolved as a variation on the convertible bond theme. These securities were structured to provide investors with equity performance with a strong element of built-in price stability and are described and analysed in McConnell and Schwarz [1986,1992]. The evolution of single stock LYONs™ ® led to the development of many variations in single stock linked notes and in the late 1980s equity index-linked instruments began to appear, for example, equity linked certificates of deposits explained in Gastineau and Purcell [1993].

The growth of derivative markets globally, coupled with more informed investor understanding of the risk and return characteristics of structured investment opportunities, has led to an enormous growth in the number and variety of equity index-linked securities being offered by banks, mortgage banks, and building societies. The recent decline in the level of the major international equity indexes worldwide has further stimulated investor demand for financial products that limit downside risk whilst still offering upside equity index participation. Recent guaranteed bond and note issues, for example, can be found which draw on the performance of the EuroSTOXX50 index and offer investors a callable certificate issued at a price above par, which guarantees a minimum return of par plus the full positive return on the underlying benchmark index. In the case of the bond not being called by the issuer the maturity redemption value of the bond can be expressed as:

$$B_{mat} = P \cdot Max\left[1, 1 + \left(\frac{I_T - I_0}{I_0}\right)\right]$$

where B_{mat} is the bond's redemption value, P the guaranteed amount (par), I_{T} the index level at the bond's maturity date, I_0 the initial index level or strike price.

A second example issues a bond at par and offers a minimum redemption value above par over a specified time period but with a reduced participation level in the underlying equity index. At maturity the bond's redemption value can be expressed as:

$$B_{mat} = P \cdot Max\left[1 + y, 1 + x\left(\frac{I_T - I_0}{I_0}\right)\right]$$

where: y represents guaranteed return above par expressed as a proportion, and x represents the benchmark index participation level as a proportion.

The pricing and hedging of these types of structures is well-known (see e.g. Eales [2000]; Das [2001]). The financial institution offering the instrument will, ideally, invest in a zero coupon bond for a price less than the sum invested and use the residual to purchase the appropriate quantity of call options on the index. This approach to structuring a hedged investment instrument is most effective in a low volatility high interest rate economic climate.

A variation on this can be found in equity index-linked cliquet participation notes. These instruments make use of cliquet which are well-established instruments. They were first introduced in France using the CAC 40 equity index as the underlying security. Cliquets are also called ratchet options in the literature because they are based on resetting the strike of a derivative structure to the last fixing of the reference underlying. Ratchets can be regular as described by Howard [1995] or compound as discussed by Buetow [1999]. For the latter type there are no intermediary payments, all gains being used to increase the volume of the derivative that is used as a vehicle for the ratchet. A wide range of ratchet caps and floors in an interest rate context described in Martellini et al. [2003].

In an equity context a similar example of the use of ratchets can be found in a note which offers a minimum redemption value set above par and whose redemption yield is related to the monthly percentage changes in a specified index over a defined period of time. To manage the risk of large index movements the monthly percentage returns are collared in a tight band around the periodically reset index strike price.

$$B_{mat} = P \cdot Max\left[1 + y, 1 + \sum_{t=0}^{T} max\left[-1.5, min\left(1.5, \left(\frac{I_{t+1} - I_t}{I_t}\right)\right)\right]\right]$$

A similar approach can be adopted when seeking to price and hedge this structure as that described in the guaranteed instruments introduced earlier. Following the purchase of a zero coupon bond residual funds can be used to buy a set of cliquet call and put options with monthly expirations extending to the bond's maturity date. The portfolio of options required to create this position will be long ATM calls combined with short OTM calls and Short ATM puts combined with long OTM puts. Clearly the availability of any residual funds derived from the portfolio of options will help determine the feasibility, the attractiveness and the competitiveness of the instrument. A mirror image instrument could be constructed which links coupon to the percentage changes in an index to falls rather than rises index.

The pricing of a cliquet option typically proceeds by regarding it as a portfolio of at-the-money (ATM) forward start options. A cliquet bestows on the holder the right to buy a regular at-the-money call with time to maturity T at some future specified date t_1. Thus,

$\tau_1 = t_1 - t$ is the length of time that elapses before the forward start option comes into existence and $\tau = T - t$ is the length of time to maturity. An early approach used in the pricing of a forward start option is presented by Rubinstein [1991]. This method bases the risk-neutral value of an ATM forward start call option on the expected value of the underlying security at time t_1 and results in the option value reducing to that of a regular ATM call *where the time to maturity is the effective time* $\tau - \tau_1$, Zhang [1998]. This implies that the Black and Scholes pricing formula can be used to obtain the cliquet option's price (call or put). If the tenors are defined by the partition $t_1 < t_2 < ... < t_{n+1} \equiv T$ then

$$Cliquet_{put}(t) = \sum_{i=1}^{n} S(t)\left[e^{-r(t_{i+1}-t_i)-\delta(t_i-t)} N\left(-d_2^{ATM}(i)\right) - e^{-\delta(t_{i+1}-t)} N\left(-d_1^{ATM}(i)\right)\right] \tag{1}$$

where $S(t)$ represents the underlying asset at time t, r the risk free rate of interest, δ is the dividend yield, σ represents volatility, and

$$\begin{aligned} d_1^{ATM}(i) &= \frac{\left(r - \delta + 0.5\sigma^2\right)}{\sigma}\sqrt{t_{i+1} - t_i} \\ d_2^{ATM}(i) &= d_1 - \sigma\sqrt{t_{i+1} - t_i} \end{aligned} \tag{2}$$

Pricing forward start options is the key to pricing cliquets. A forward start option is a particular case of multi-stage options, which are derivatives allowing decisions to be made via conditions evaluated at intermediate time points during the life of the contingent claim (see Etheridge [2002]). Multistage options can be priced similarly to options on stocks paying discrete dividends at intermediate points over the life of the option. Under general common assumptions, the pricing equation of multistage options in a risk-less world is the well-known Black-Scholes PDE

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S(t)^2\frac{\partial^2 V}{\partial S^2}+rS(t)\frac{\partial V}{\partial S}-rV=0 \tag{3}$$

with some final condition such as $V(T,S) = G(S)$.

The Feynman-Kac solution of the above equation is

$$V(S(t),t)=e^{-r(T-t)}\widetilde{E}[G(S(T))|S(t)] \tag{4}$$

where the expectation operator is taken under the risk-neutral measure.

The forward start option is an option that comes into existence at time T_1 and has maturity T. The following backward procedure can be used to calculate the price of this option:

(a) Calculate the final payoff of the option at time T.

(b) Calculate the value of the payoff from (a) at time T_1; this is given as the solution of the above PDE (3) with $t = T_1$.

(c) Check the conditions and calculate the terminal value of the option at T_1 and for $t < T_1$ use again the pricing PDE (3) to get the solution

$$V(S(t),t)=e^{-r(T_1-t)}\widetilde{E}[V(S(T_1),T_1)|S(t)] \tag{5}$$

Out-of-the-money (OTM),in-the-money cliquets (ITM), and more exotic structures can also be handled in the same partial differential equation (PDE) pricing framework.

In the same vein Monte Carlo simulation (MCS) and quasi-MCS can be used to price cliquets taking into account the element of path dependency ignored by the standard Black and Scholes formula. Buetow [1999] suggests that pricing this type of instrument accurately is best undertaken using different methods and comparing the results obtained.

These pricing methods, however, all suffer from the assumption of constant volatility. Wilmott [2002] highlights the problems associated with this assumption and illustrates the dangers faced by writers of cliquets when ignoring volatility risk. It can be shown that the gamma of a cliquet option is the sum of gamma values for regular options because the gamma of a forward start option is zero before the starting time. This may create the impression that risk management is easy in this case. However, for this type of option, hedging can be quite complex because the delta, vega and theta have discontinuities around reset times.

This article explores how reverse cliquet options can be integrated into the structure of a guaranteed principal bond. Pricing is developed and discussed further under the constant-elasticity-of-variance model. Forward start options are the main element of this structure and under the latter model the pricing of these important options is not easy. *En passant* new closed formulae are derived for forward start options under the CEV model. Furthermore some theoretical results from mathematical finance show that it is very important to consider carefully the underlying when doing financial engineering.

II. FINANCIAL ENGINEERING WITH REVERSE CLIQUETS

Unlike the structures discussed so far, reverse cliquet options are best employed when volatility levels are substantially higher than historically observed volatilities and are expected to revert back to normal or when investors hold the view that the markets are likely to become more bullish (puts) or bearish (calls). Reverse cliquets rely on the creation of a pool of funds derived from, for example, investors augmenting their investment funds by writing forward start options. The fund starts with a value of greater than 100% and is drawn on over time if and when the written options expire in-the-money (ITM). A reverse cliquet can be integrated into the structure of a guaranteed principle bond. In this construction the bond may guarantee full return of principal invested and offer a higher than market coupon which declines as the underlying asset to which the bond is linked declines in value (put) or rises in value (calls) as measured on pre-specified future dates. Coupons could be paid on defined intermediate dates or as a single payment at the instrument's maturity.

If it is assumed that investor's views are bullish concerning equity market performance and that volatilities are high, a bond could be offered which pays out an amount determined by the total initial option net income fund less the sum of the declines in the benchmark index either at maturity or on intermediate coupon dates t_1 t_2 , ...$t_{n+1.}$ = T.

From the issuing institution's perspective one way in which the structure could be engineered would be to combine a zero coupon bond, purchased using the investor's deposit, together with a portfolio of income generating forward start written put options. The put option premia represents an additional pool of funds that will need to be drawn on should the underlying asset decline in value in any period.

There is clearly a real risk in the structure that needs to be addressed. Large falls or a series of falls in the asset's value may result in the additional funds being exhausted and the investor's investment principal being used to meet settlement obligations. In such situations, to ensure that the principal return guarantee is met the institution offering the product will need to meet the cost from their own funds. To avoid this potentially expensive problem each cliquet in the portfolio will need to have a floor in place to ensure that potential losses are capped. Exhibit 1 suggests the instrument's construction.

Insert Exhibit 1 Here

A possible course of action that would create a series of appropriate floors would be for the institution to purchase offsetting OTM forward start put options for each of the short forward start put options held in the portfolio. This introduces a conflict. The long OTM options will act as a drain on the funds which are being used to enable the offering of a higher than market coupon as an incentive to the investor. On the one hand the product requires a coupon high enough to attract investors on the other the risk of severe market index falls must be capped, achieving this by buying OTM cliquet options will exert a downward pull on the coupon.

III. RISK CONTROL ISSUES

The way in which the guaranteed principal instrument has been created by Financial Institution A. Falls in the equity index result in sums being drawn down from the fund that is prevented from becoming negative by the protective long puts forming the caps. Three market scenarios can be considered for each period: (i) the equity index rises by η%, (ii) the equity index remains at its current level, (iii) the equity index falls by η%. On reaching maturity in cases (i) and (ii) the investor’s achieved coupon will be the maximum offered in the bond’s indenture $C_{\max}$. Under the third scenario, the most realistic case, the achieved coupon will be determined by:

$$0 \le \left(C_{max} + \left(\sum_{t=1}^{T} \left(-1 \cdot max\left[Strike_{ATM_t} - S_{t-1}, 0 \right] + max\left[Strike_{OTM_t} - S_{t-1}, 0 \right] \right) \right) \right) \le C_{max} \qquad (6)$$

In the case of the institution providing the cliquet options the pay out will be the mirror image of those generated by the investor. Under scenarios (1) and (2) the institution will meet the coupon pay out from the funds made up of the original investment plus the net income generated by the collar. Under scenario (iii) the coupon paid to the investor will be reduced by an amount reflecting the downside protected fall in the index.

For simplicity we shall assume that the guaranteed amount to the investor is 100%. In other words the structured investment product guarantees the return in full of the sum invested at maturity *T*. Let *H* denote the price, at time 0, of a zero coupon risk free bond with maturity *T*. Obviously 0< *H* < 100 and 100-*H* is available for using in the reverse cliquet structure. Over each period of time $[t_{i-1}, t_i]$ of constant length $\Delta i = t_i - t_{i-1}$, with $i = 1,2,...,n+1$ the financial institution will sell ATM forward start put options and buy OTM forward start put options. Let $S(i)$ be the price of the index at time t_i and let 0<η<1 be a factor defining the OTM strike price as $\eta S(t_{i-1})$ for the period $[t_{i-1}, t_i]$.

The payoff of the short ATM forward start put at t_i is $-\max[S(t_{i-1}) - S(t_i),0]$ and the payoff of the long OTM forward start put at the same time is $\max[\eta S(t_{i-1}) - S(t_i),0]$. This forward start spread has the combined value

$$\max[\eta S(t_{i-1}) - S(t_i),0] - \max[S(t_{i-1}) - S(t_i),0]. \qquad (7)$$

At time 0 this can be priced as a portfolio of options using risk-neutral valuation in the framework developed by Harrison and Kreps [1979]. Using the formulae[1] for forward start put options provided in Zhang [1998] the premium of the forward spread at time 0 is

$$\begin{aligned} &S(0)[e^{-r\Delta i-\delta t_{i-1}} N(-d_2^{ATM}(i)) - e^{-\delta t_i} N(-d_1^{ATM}(i))] - S(0)[\eta e^{-r\Delta i-\delta t_{i-1}} N(-d_2(i)) - e^{-\delta t_i} N(-d_1(i))] \\ &= S(0)e^{-r\Delta i-\delta t_{i-1}}[N(-d_2^{ATM}(i)) - \eta N(-d_2(i))] + S(0)e^{-\delta t_i}[\eta N(-d_1(i)) - N(-d_1^{ATM}(i))] \end{aligned} \qquad (8)$$

Recall that for the OTM forward start option

[1] We have corrected some typos that appear in Zhang [1998]

$$d_1(i) = \frac{r-\delta+\sigma^2/2}{\sigma}\sqrt{\Delta i} - \frac{\ln\eta}{\sigma\sqrt{\Delta i}} \quad and \quad d_2(i) = d_1(i) - \sigma\sqrt{\Delta i}$$

The total revenue at time 0 from the forward start engineered structure is

$$Q = \sum_{i=1}^{T} S(0)e^{-r\Delta i-\delta t_{i-1}}[N(-d_2^{ATM}(i)) - \eta N(-d_2(i))] + S(0)e^{-\delta t_i}[\eta N(-d_1(i)) - N(-d_1^{ATM}(i))] \qquad (9)$$

so the seller of the reverse cliquet has 1-*H*+*Q* at their disposal.

A common practice is to provide investors with a variable coupon that pays at each reset time or in one payment at maturity the difference between a fixed coupon rate *x* (%) and the level of percentage decline in the index over the ending period. For period $[t_{i-1}, t_i]$ the decline in the index is $\max[S(i-1)-S(i),0]$ so when all payments are settled at maturity T the coupon paid is

$$\Pi = \sum_{i=1}^{T} \max\left(x - \max[\frac{S(i-1)-S(i)}{S(i-1)},0],0\right). \qquad (10)$$

As in the previous section, considering the worst case scenario that for each period the ATM put options will be exercised due to a decline of the index at or below the floor provided by the OTM options, the financial engineer must make sure that

$$1-H+Q \geq \Pi e^{-rT} + \sum_{i=1}^{T} \min\left(\max[S(i-1)-S(i),0],(1-\eta)S(i-1)\right)e^{-ri} \qquad (11)$$

otherwise payments may be missed or losses will be made.

IV. APPLICATION

In order to examine how this type of product can be engineered consider the following example:

A non-callable bond is issued offering a minimum return of full principal invested at the end of three years or full principal plus 100% - the sum of the monthly declines in a defined equity index.

Recall that the financial engineer has to establish at what level *x* can be set and this will in turn be determined by the amount available from the sale of ATM puts less the cost of the OTM puts needed to create the cap. To illustrate how the structure can be replicated we will price in a Black and Scholes framework both long and short forward start put options that comprise the cliquet option collar initially. This, of course, ignores volatility stochasticity and any volatility smile. Proceeding with this approach we assume that the discount rate is 2.35% this implies that the institution will today pay 93.27% for a zero coupon bond with a three year maturity.

$$H = \frac{FV}{(1 + dr)^T} = \frac{100}{(1 + 0.0235)^3} = 93.27\% \qquad (12)$$

Thus (1 – H) = (1 - 0.9327) = 0.0673%, implies that a 6.73% residual is immediately available to invest in the fund that will be used to make payments to the put holders if and when required. To price the forward start options we assume that the yield curve is flat and that risk free rate for all maturities is 2.52%; dividend yield is 1.58%; volatility is 25% p.a. and the life span of each option is 30-days, and that a 1% fee is charged by the issuing institution. Using the formulae presented in equation (5) above the price of each ATM forward start option in this regime is 2.815% and since 1 regular put and 35 forward start puts are needed to cover the maturity of the bond and the number of resets the total income from ATM options will be 101.352%.

In order to cover the period-by-period downside investor risk, the institution will need to buy 35 OTM forward start put options and 1 OTM regular put option for the first month of the structured product life. In order to achieve a total 101% fund including the institutional fee and simultaneously hedge against large falls in the index value the appropriate OTM strike price will need to be established. Following a search to find a strike that satisfies the Fund's total requirement the OTM strike is found 10.15% below the ATM strike and, using this strike, the required 36 options can be secured at a cost of 7.083%, ignoring transactions costs. The net contribution of the put option transactions to the fund will be 94.269%, combining this with the 6.731% residual from the zero coupon bond purchase, generates a fund of 101%. This fund provides an indication of the maximum coupon that the investor can expect to receive when there are no payouts from the fund at any of the reset dates. Should the index fall to a level below the relevant floor in each period the long OTM put options will be exercised ensuring that the investor receives the minimum return on the instrument, namely the original investment principal.

To consider the risk control aspects of this product, possible paths for the index can be simulated in order to calculate the amounts that will be paid by the financial institution to the counterparty under each scenario. Continuing with the same data provided above in this section, the Monte Carlo simulation exercise suggests that the average present value of total payment made by the seller of the structured product is approximately 25%. There is, however, a 4% chance that over the three-year period market declines will exceed the Fund's capacity to meet the obligation. More informative views are described in Exhibits 2. and 3. Exhibit 2 presents a bar chart of the simulated pay outs to the bond holder. There is clearly a high probability that the bond holder will receive a high return and that the issuing financial institution will be able to meet the coupon obligation without drawing on its own funds. There is, however, also evidence that market states will arise which will place significant demands on the financial institution. In this simple simulation exercise the highest potential loss faced by the bond issuer is over £7,000,000.

Insert Exhibit 2 here

Exhibit 3. illustrates a histogram of the distribution of pay outs to the bond holder. Although the maturity payout distribution demonstrates the expected replication of a normal distribution it disguises the fact that it is the sequence of monthly pay outs that play a crucial role in determining the effectiveness of the hedge strategy.

Insert Exhibit 3 here

A sequence of strong positive market moves followed by a series of falls will be enough to defeat the carefully constructed hedge structure as illustrated in Exhibit 4.

Insert Exhibit 4 here

V. PRICING UNDER CEV MODEL

The standard pricing mechanism for reverse cliquets falls under the Black-Scholes umbrella. The essential step is pricing forward start options as described by Zhang [1998] or Etheridge [2002]. The key point is the factorization of the value of the option, at the time point where the option comes into existence, as the product of the underlying stock and a multiplicative factor that does not depend on the underlying.

The cliquet is a very liquid over-the-counter derivative in equity markets. The assumption of constant variance or volatility is contradicted by the empirical evidence showing that volatility changes with stock price[2].

In this section we take a step further and we model the underlying with a constant-elasticity-of-variance (CEV) process and derive the price of the forward start options that are the building block for the reverse cliquets. Once this is realised everything else regarding financial engineering with reverse cliquets follows more or less the same methodology as above.

The CEV model for an asset S is described by the following SDE

$$dS(t) = \mu S(t)dt + \sigma S(t)^{\alpha} dZ(t) \tag{13}$$

where μ is the drift parameter, $\alpha > 0$ is a constant parameter and everything else is exactly as for a geometric Brownian motion. This alternative stochastic process has been proposed by Cox & Ross [1976] for pricing options and they provided closed-formulae for European vanilla options when $\alpha < 1$. CEV models are now applied in almost all areas of quantitative finance, LIBOR models Andersen & Andreasen (1998), credit derivatives models Andreasen (2000), barrier and lookback options Boyle & Tian (1999) and Davydov & Linetsky (2001) .

Empirical evidence shows that the CEV model in general outperforms the Black-Scholes model. MacBeth and Merville [1980] and Emanuel and MacBeth [1982] found empirical evidence supporting this conclusion on stock options markets while Hauser and Bagley [1986] showed similar results on the currency options markets. For the particular case of square-root process, that is for $\alpha = 0.5$, Beckers [1980] revealed that Black-Scholes ITM call and OTM put

[2] Schmalensee & Trippi (1978) found evidence of negative relationship between stock price changes and changes in implied volatility while Black (1976) discovered on ten years of data of six stocks that a proportional increase, respectively decrease, in the stock price is associated with a larger proportional increase, respectively decrease in the standard deviation of the stock.

prices evaluated at implicit volatilities of at-the-money options are lower than those counterparts calculated with the CEV model. The larger the $|\alpha|$ the greater the price difference between CEV and Black-Scholes option prices. Jackwerth & Rubinstein (1998) estimate the CEV parameters implicit in the 6-month S&P500 options for 1986-1994, daily data. Before 1987 crash sample values of α were close to 1 as needed for a lognormal model. After the crash the market shifted to a new regime with values for α in the range -2 to -3. Reiner (1994) and Jackwerth & Rubinstein (1998) find that typical values of α implicit in the S&P500 option prices are strongly negative and as low as -3. Values of $\alpha > 1$ are empirically valid for some commodity futures options with upward sloping implied volatility.

The CEV model implies a smile pattern that is frequently encountered in equity, index and currency options markets. However, the CEV model still leaves some Black-Scholes smile effects unexplained such as underpricing of ITM puts and OTM calls. Fortunately, for the structured product presented here the OTM puts are important.

Emanuel and MacBeth [1982] determined the formula for the case when $\alpha > 1$, which for technical mathematical reasons and different boundary behaviour is different than the formula for $\alpha < 1$. The CEV vanilla call option formula involves an infinite series of Gamma-functions, difficult to evaluate in Mathematica or C/C++ directly . The values of vanilla options can be found directly in terms of integrals of Bessel functions; this can be coded in C++ using standard routines Press et al. (1992). Schroder [1989] showed how to express the CEV option pricing formulae in terms of the noncentral chi-square distribution. This is recovered here when pricing forward start options, although it is not mentioned in the text explicitly.

For the sake of clarity we focus in this section on pricing an ATM forward start call option that kicks in at time T_1 and matures at T. Similar calculations can be made for OTM or ITM forward start options. Employing risk-neutral valuation we get the value of the option at time T_1 as

$$V(S(T_1);T-T_1) = e^{-r(T-T_1)}\widetilde{E}_t\left[(S(T)-S(T_1))^+\right] \tag{14}$$

For simplicity, and without loss of generality, we restrict to the case $\alpha = 0.5$ which is the case most investigated in the literature. Denoting by $F_t(u) = P(S(T) \le u \mid S(t))$ Cox and Ross [1976] employed the following useful result due to Feller [1951]

$$\begin{aligned} &dF_t(u) = \sqrt{\theta_t \vartheta_t}\, e^{-\vartheta_t - \theta_t u} u^{-1/2} I_1(2\sqrt{\theta_t \vartheta_t u}) \cdot du, \quad \textit{for any } u > 0 \\ &F_t(0) = G(\mu;1) \\ &F_t(u) = 0, \textit{ for any } u < 0 \end{aligned} \tag{15}$$

where $\theta_t = \dfrac{2\mu}{\sigma_0^2[e^{\mu(T-t)}-1]}$, $\vartheta_t = S(t)\theta_t e^{\mu(T-t)}$, $G(x;a) = \int_x^\infty \dfrac{1}{\Gamma(a)} y^{a-1} e^{-y} dy$ (16)

and $I_1(\cdot)$ is the modified Bessel function of the first kind of order one.

For risk-neutral martingale pricing one sets either $\mu = r$ or $\mu = r - \delta$ if dividends are paid continuously at rate δ. For a general strike price X and maturity T the price of a European call at time t is

$$V(S(t),T-t) = e^{-r(T-t)}\widetilde{E}_t[(S(T)-X)^+] = e^{-r(T-t)}\int_X^\infty (s-X)dF_t(s) \tag{17}$$

and using Feller's result given above it follows that

$$V(S(t),T-t) = e^{-r(T-t)}\int_X^\infty (s-X)\left(\sqrt{\theta_t\vartheta_t}\, e^{-\vartheta_t-\theta_t s} s^{-1/2}\right) I_1(2\sqrt{\theta_t\vartheta_t s})ds \tag{18}$$

However, the modified Bessel function can be approximated using the following series

$$I_1(z) = \frac{z}{2}\sum_{k=0}^{\infty}\frac{\left(\frac{z}{2}\right)^{2k}}{k!\Gamma(k+2)} \tag{19}$$

Replacing this in equation (18) leads to

$$V(S(t), T-t) = e^{-r(T-t)}\int_X^{\infty}(s-X)\left(\sqrt{\theta_t \vartheta_t}\, e^{-\vartheta_t - \theta_t s} s^{-1/2}\right)\sum_{k=0}^{\infty}\frac{\theta_t^k \vartheta_t^k s^k}{k!(k+1)!}ds \tag{20}$$

In the Appendix it is shown that after rearrangement we get to

$$V(S(T_1); T-T_1) = S(T_1)\left\{\sum_{k=1}^{\infty} g(\vartheta_{T_1}; k) G(\theta_{T_1} S(T_1); k+1) - e^{-r(T-T_1)}\sum_{k=1}^{\infty} g(\vartheta_{T_1}; k+1) G(\theta_{T_1} S(T_1); k)\right\} \tag{21}$$

where $g(x;m) = \frac{x^{m-1}}{\Gamma(m)}e^{-x}$ is the probability density function for a gamma distribution with mean and variance equal to *m*.

The second factor delimited by the large brackets is a function $\psi(r,\sigma,T_1,T,S(T_1))$ so that we can write

$$V(S(T_1); T-T_1) = S(T_1)\psi(r,\sigma,T_1,T,S(T_1)) \tag{22}$$

and unfortunately, under a CEV model, we cannot continue as described above when using a Black-Scholes model because the second factor is *not* independent of the underlying. This will complicate the calculation of the value of the forward start option at time $t = 0$. All is not lost, however, since we can still apply risk-neutral pricing. Thus,

$$V(S(0); T) = e^{-rT_1}\widetilde{E}\left[S(T_1)\psi(r,\sigma,T_1,T,S(T_1))\right] \tag{23}$$

In the Appendix it is shown that

$$V(S(0); T) = \Omega_1 - \Omega_2 \tag{24}$$

where

$$\Omega_1 = \theta_0^2 \vartheta_0^2 e^{-\vartheta_0 - rT_1} \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} \sum_{k=1}^{\infty} \frac{b^{k-1}}{(k-1)!} \sum_{i=0}^{k} \frac{\theta_{T_1}^{k-i}}{(k-i)!} \frac{(2k+j-i)!}{(\theta_0 + \theta_{T_1} + b)^{2k+j-i+1}} \qquad (25a)$$

$$\Omega_2 = \theta_0^2 \vartheta_0^2 e^{-\vartheta_0 - rT} \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} \sum_{k=1}^{\infty} \frac{b^{k}}{k!} \sum_{i=0}^{k-1} \frac{\theta_{T_1}^{k-i-1}}{(k-i-1)!} \frac{(2k+j-i-1)!}{(\theta_0 + \theta_{T_1} + b)^{2k+j-i}} \qquad (25b)$$

where $b = \theta_{T_1} e^{rT_1}$.

For $\alpha < 1$ the risk-neutral probability of absorption at zero (bankruptcy) is (see Cox, 1975)

$$Q_{S_0}(S_T = 0) = G\left(\frac{1}{2|\alpha - 1|}, \frac{\mu S_0^{-2(\alpha-1)}}{2\sigma^2(\alpha-1)(e^{2\mu(\alpha-1)T} - 1)}\right) \qquad (26)$$

Can then one use a CEV model with $\alpha < 1$ for equity *index*-linked products?

For $\alpha > 1$ the discounted underlying process is not a martingale, but only a local martingale. There is no equivalent martingale measure and this problem is avoided in practice by taking a very large fixed number N and considering the volatility specification

$$\sigma_N(S) = \sigma \min[N^{\alpha-1}, S^{\alpha-1}] \qquad (27)$$

This transformation is called limited CEV (LCEV) by Andersen & Andreasen (1998) and it helps because infinity is now a natural boundary and the discounted process becomes a martingale in any finite time interval.

As pointed out above, on equity index markets the range for the key parameter α is in the negative region between -2 and -3. The seminal "no-arbitrage" formula for call pricing given by Cox (1975,1996) was derived with risk-neutral valuation although it had not been proved that there is a *unique* equivalent measure for the CEV model. However, the risk-neutral method requires only the *local* arbitrage free property and that is not equivalent to the arbitrage free property (Delbaen & Schachermayer 1994 and 1995). Moreover, Delbaen & Shirakawa (1996) prove that there is a unique equivalent martingale measure and derive the law of the stock price process for the CEV model. This seems to cover for pricing derivatives under the CEV model. The problem for financial engineers is that the same authors show that, when the stock price is conditioned to be strictly positive, the CEV model allows for *arbitrage opportunities*.

One cannot eliminate the possibility that a CEV process with $\alpha < 1$ will hit 0 and be absorbed into that state. As with the initial real-world CEV process there is strict positive probability that the risk-neutral stock price process is absorbed at 0. Thus, one cannot use the CEV model to price derivatives (e.g. forward start options) contingent on price movements of an equity index but one *can* use it when the derivative is contingent on price movements of a single stock[3].

VI. CONCLUSION

Structured products are establishing themselves as a class of instruments in modern finance. Here we have investigated a product underpinned by reverse cliquet options. We provided an approach to price and implement this type of structure under the standard Black-Scholes model**.** The financial engineer is able to perform calculations to determine the size of the hedged fund that will determine the maximum possible coupon payment offered to the bond holder. Looking at possible scenarios we illustrated that the structure still poses potentially serious financial risk to the issuing institution under certain market conditions. The main difficulty in pricing the components of this structure is a forward start option. Hedging this type of option remains a difficult and an open area. Here we circumvented this difficult task by flooring the downside movements of the underlying. In this way the improvement is

[3] The reason for this is that a corporate can go bankrupt and thus its equity value drop to zero.

sought in the direction of better pricing models that take account of well known empirical facts. Thus, we have used the CEV model as a starting point and derived a new option pricing formula for forward start options.

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Appendix

First we show how to calculate the following integral

$$V(S(t),T-t)=e^{-r(T-t)}\int_X^{\infty}(s-X)\left(\sqrt{\theta_t\vartheta_t}e^{-\vartheta_t-\theta_t s}s^{-1/2}\right)\sum_{k=0}^{\infty}\frac{\theta_t^k\vartheta_t^k s^k}{k!(k+1)!}ds$$

We are going to separate the integral into two integrals. Thus

$$V(S(t),T-t)=e^{-r(T-t)}\theta_t\vartheta_t e^{-\vartheta_t}\left\{\int_X^{\infty}se^{-\theta_t s}\sum_{k=0}^{\infty}\frac{\theta_t^k\vartheta_t^k s^k}{k!(k+1)!}ds-X\int_X^{\infty}e^{-\theta_t s}\sum_{k=0}^{\infty}\frac{\theta_t^k\vartheta_t^k s^k}{k!(k+1)!}ds\right\}$$

Making the change of variable $\theta_t s=y$ we get

$$\begin{aligned}V(S(t),T-t)&=e^{-r(T-t)}\vartheta_t e^{-\vartheta_t}\left\{\int_{\theta_t X}^{\infty}ye^{-y}\sum_{k=0}^{\infty}\frac{\vartheta_t^k y^k}{k!(k+1)!}\frac{1}{\theta_t}dy-X\int_{\theta_t X}^{\infty}e^{-y}\sum_{k=0}^{\infty}\frac{\vartheta_t^k y^k}{k!(k+1)!}dy\right\}\\&=S(t)\sum_{k=0}^{\infty}\frac{\vartheta_t^k}{k!}e^{-\vartheta_t}\left(\int_{\theta_t X}^{\infty}\frac{y^{k+1}}{(k+1)!}e^{-y}dy\right)-e^{-r(T-t)}X\sum_{k=0}^{\infty}\frac{\vartheta_t^{k+1}}{(k+1)!}e^{-\vartheta_t}\left(\int_{\theta_t X}^{\infty}\frac{y^k}{k!}e^{-y}dy\right)\\&=S(t)\sum_{k=1}^{\infty}\frac{\vartheta_t^{k-1}}{(k-1)!}e^{-\vartheta_t}\left(\int_{\theta_t X}^{\infty}\frac{y^k}{k!}e^{-y}dy\right)-e^{-r(T-t)}X\sum_{k=1}^{\infty}\frac{\vartheta_t^k}{k!}e^{-\vartheta_t}\left(\int_{\theta_t X}^{\infty}\frac{y^{k-1}}{(k-1)!}e^{-y}dy\right)\end{aligned}$$

If $g(x;m)=\dfrac{x^{m-1}}{\Gamma(m)}e^{-x}$ is the probability density function for a gamma distribution with mean and variance equal to *m* and the incomplete gamma function is defined as in the text in formula (9) it follows then that

$$V(S(t),T-t)=S(t)\sum_{k=1}^{\infty}g(\vartheta_t;k)G(\theta_t X;k+1)-e^{-r(T-t)}X\sum_{k=1}^{\infty}g(\vartheta_t;k+1)G(\theta_t X;k)$$

Hence, at time $t=T_1$, the value of the ATM forward start option is

$$V(S(T_1);T-T_1)=S(T_1)\left\{\sum_{k=1}^{\infty}g(\vartheta_{T_1};k)G(\theta_{T_1}S(T_1);k+1)-e^{-r(T-T_1)}\sum_{k=1}^{\infty}g(\vartheta_{T_1};k+1)G(\theta_{T_1}S(T_1);k)\right\}$$

The second calculation detailed here is

$$V(S(0);T) = e^{-rT_1} \widetilde{E}\left[S(T_1)\psi(r,\sigma,T_1,T,S(T_1))\right]$$

where $\psi(r,\sigma,T_1,T,S(T_1)) = \left\{ \sum_{k=1}^{\infty} g(\vartheta_{T_1};k) G(\theta_{T_1} S(T_1);k+1) - e^{-r(T-T_1)} \sum_{k=1}^{\infty} g(\vartheta_{T_1};k+1) G(\theta_{T_1} S(T_1);k) \right\}$

Evidently $V(S(0);T) = e^{-rT_1} \int_0^{\infty} s\psi(r,\sigma,T_1,T,s) dF_0(s)$ which after replacement of ψ and the other known expressions becomes

$$e^{-rT_1} \sum_{k=1}^{\infty} \frac{b^{k-1}}{(k-1)!} \theta_0^2 \vartheta_0^2 e^{-\vartheta_0} \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} \int_0^{\infty} s^{k+j} e^{-s(b+\theta_0)} G(\theta_{T_1} s;k+1) ds -$$
$$e^{-rT} \sum_{k=1}^{\infty} \frac{b^{k-1}}{k!} \theta_0^2 \vartheta_0^2 e^{-\vartheta_0} \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} \int_0^{\infty} s^{k+j+1} e^{-s(b+\theta_0)} G(\theta_{T_1} s;k) ds$$

Let's denote the first term by Ω_1 and the second term by Ω_2. The key element in the subsequent calculations is the integral $I_n(A) = \int_A^{\infty} y^n e^{-y} dy$ that can be shown after some integration by parts to be equal to

$$I_n(A) = \int_A^{\infty} y^n e^{-y} dy = e^{-A} \sum_{i=0}^{n} \frac{n!}{(n-i)!} A^{n-i}$$

In order to calculate Ω_1 we need to calculate first the integral

$$\int_0^{\infty} s^{k+j} e^{-s(b+\theta_0)} G(\theta_{T_1} s;k+1) ds = \int_0^{\infty} s^{k+j} e^{-s(b+\theta_0)} \frac{1}{k!} I_k(\theta_{T_1} s) ds$$
$$= \frac{1}{k!} \sum_{i=0}^{k} \frac{k!}{(k-i)!} \theta_{T_1}^{k-i} \int_0^{\infty} s^{2k+j-i} e^{-s(\theta_0+\theta_{T_1}+b)} ds$$
$$= \sum_{i=0}^{k} \frac{\theta_{T_1}^{k-i}}{(k-i)!} \int_0^{\infty} \frac{u^{2k+j-i}}{(\theta_0+\theta_{T_1}+b)^{2k+j-i+1}} e^{-u} du$$
$$= \sum_{i=0}^{k} \frac{\theta_{T_1}^{k-i}}{(k-i)!} \frac{(2k+j-i)!}{(\theta_0+\theta_{T_1}+b)^{2k+j-i+1}}$$

Thus

$$\Omega_1 = \theta_0^2 \vartheta_0^2 e^{-\vartheta_0} e^{-rT_1} \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} \sum_{k=1}^{\infty} \frac{b^{k-1}}{(k-1)!} \sum_{i=0}^{k} \frac{\theta_{T_1}^{k-i}}{(k-i)!} \frac{(2k+j-i)!}{(\theta_0+\theta_{T_1}+b)^{2k+j-i+1}}$$

Similarly

$$\Omega_2 = \theta_0^2 \vartheta_0^2 e^{-\vartheta_0} e^{-rT} \sum_{j=0}^{\infty} \frac{1}{j!(j+1)!} \sum_{k=1}^{\infty} \frac{b^k}{k!} \sum_{i=0}^{k-1} \frac{\theta_{T_1}^{k-i-1}}{(k-i-1)!} \frac{(2k+j-i-1)!}{(\theta_0+\theta_{T_1}+b)^{2k+j-i}}$$

List of Exhibits

Exhibit 1.

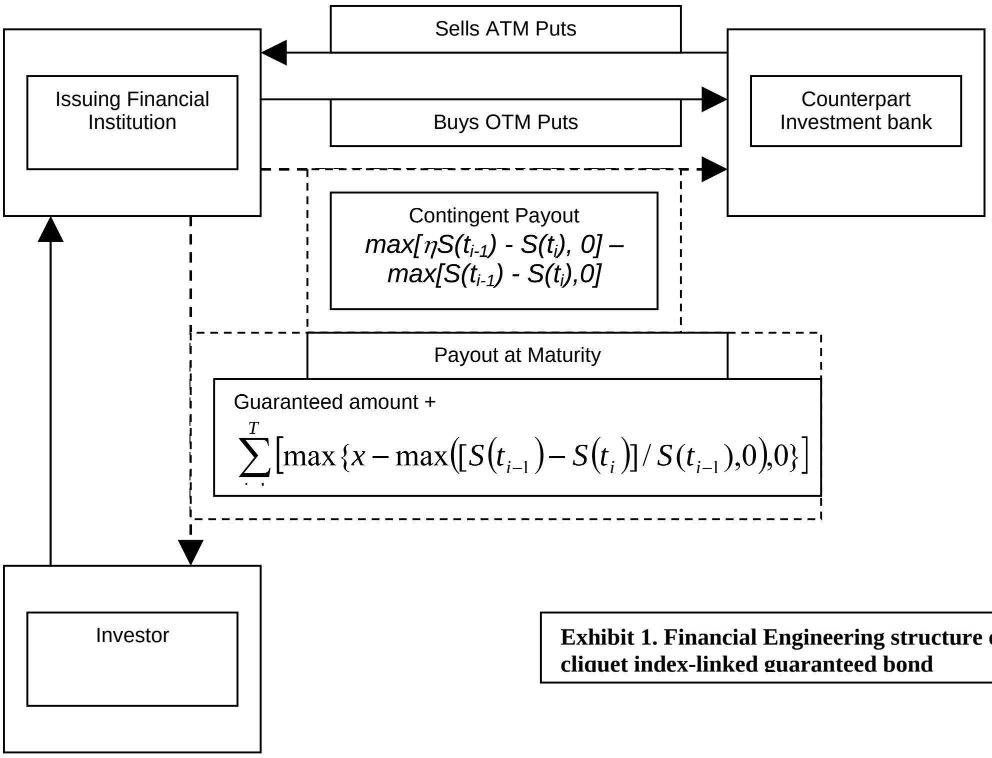


Exhibit 1. Financial Engineering structure of a reverse cliquet index-linked guaranteed bond

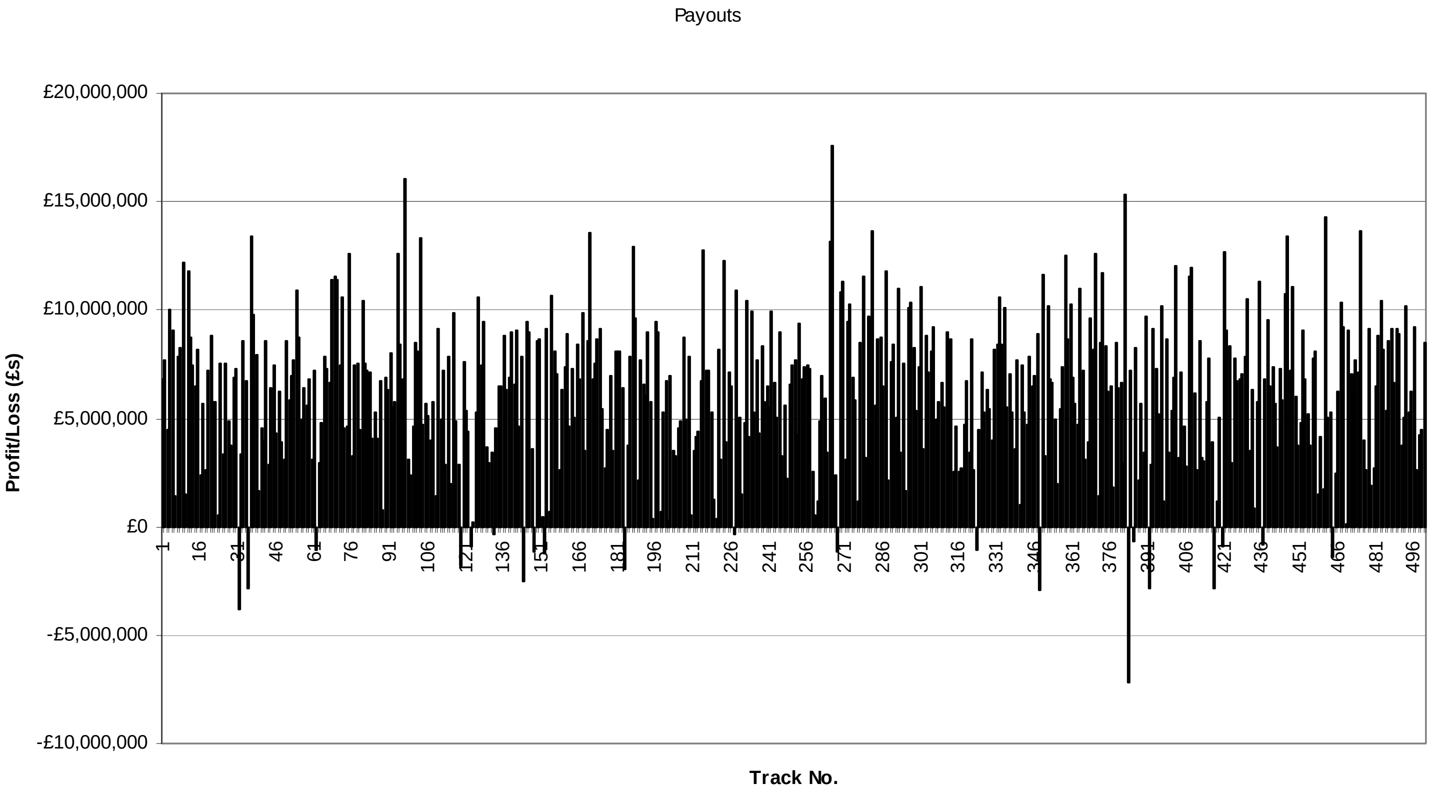


Exhibit 2. Monte Carlo simulated maturity pay outs over 500 antithetic tracks

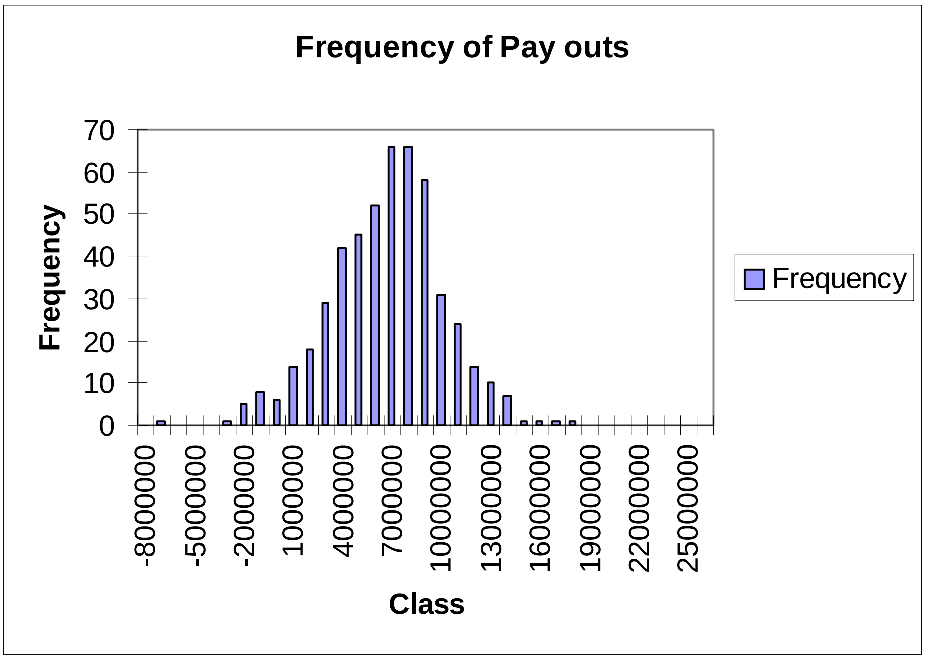


Exhibit 3. Frequency of pay outs at maturity

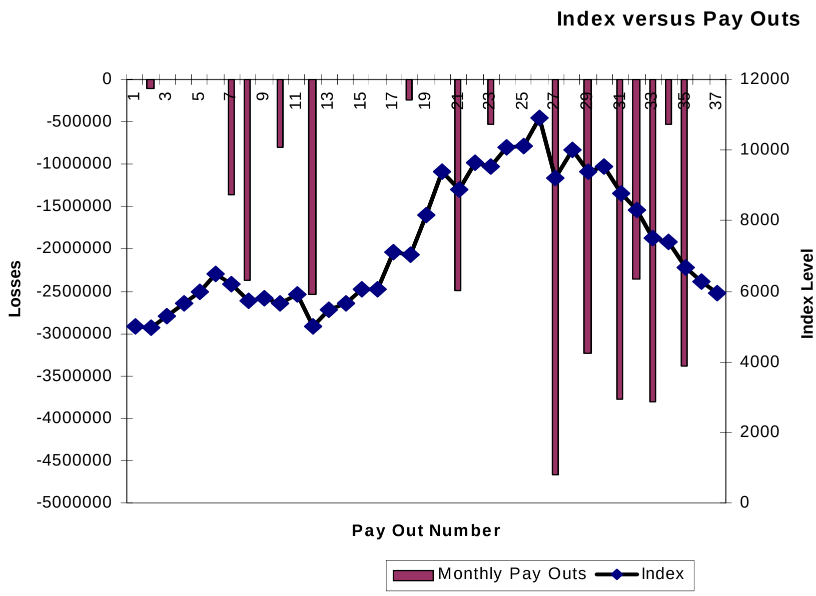
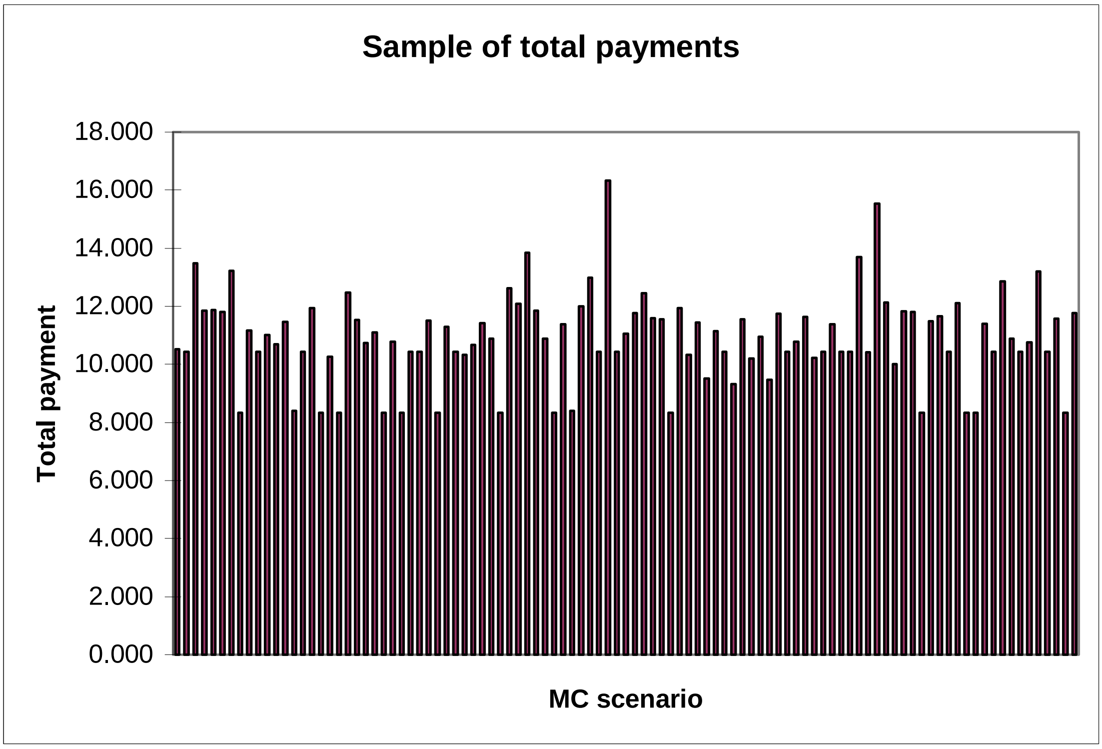


Exhibit 4. Monthly pay out versus Index track

Exhibit 4



Monte Carlo simulations when the coupon rate is x=3% p.a. There are two cases where the total payment is higher than 13.956.