The Evaluation of American Options with the Method of Lines

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Abstract: The valuation of American Options can often be reduced to the study of a free boundary value problem for a partial differential equation. This paper will discuss how the Method of Lines (MOL) can be used to study the numerical valuation of American Options and to determine the early exercise frontier. This method is flexible, accurate and computationally fast.

§1. Introduction

As no analytic solution exists for the free boundary problem describing the pricing of American puts and calls on (dividend paying) stock, various numerical procedures have been proposed in the literature. For a recent discussion and comparison of some of these methods for the Black-Scholes options model see [1], [2].

As observed in these papers, the challenge is to find a method that is both accurate and fast. Both papers note substantial differences in accuracy and computing efficiency between the various methods. A numerical method of long standing for general parabolic free boundary problems is the discrete time method of lines (MOL) approximation and the resulting sequence of free boundary problems for ordinary differential equations with the method of invariant imbedding. This method has been applied over the years to a number of one- and multi-dimensional free boundary problems arising in science and engineering. Its applicability to optimal stopping problems was suggested by one of the authors in [3]. Its advantages for American Puts and Calls were demonstrated in [2] where it is shown that this approach is quite accurate and that it can be implemented to execute on workstations and mainframes as fast as alternative methods with which it was compared.

A similar observation is made in [1] for the analytic method of lines (AMOL) where the solution of time discrete free boundary problems obtained from the MOL is carried out essentially analytically by exploiting the specific Cauchy-Euler structure of equations that arise when the underlying stock price follows the Black-Scholes model.

The present paper does not compare numerical methods. We believe that the results of [1] and [2] show that the two MOL approximation provide competitive efficient numerical methods for American option problems. However, as a fully numerical method, the invariant imbedding approach is not tied to the specific boundary problem that arises in the context of the Black-Scholes model for stock prices, but can be used for equations with time and solution dependent coefficients and boundary conditions as are common in physical science models ([3]). Moreover, trivial modifications make the method formally second order in time. On the other hand, due to the singularity of the solution and the infinite interval there are some technical complications in applying this numerical method. The extensive calculations of [2], primarily for calls but also some puts, are not accompanied by much detail of the actual choice of computational parameters and on how to treat the difficulties due to vanishing coefficients or infinite intervals. Therefore, our discussion will again examine the MOL algorithm with invariant imbedding but will concentrate on the computational issues which need to be resolved.

We shall deal here primarily with American Puts which we consider to be the more challenging problem for an efficient numerical solution because the interval of computation is infinite. The efficiency of any numerical method will depend greatly on how this singularity is treated.

§2. The Model

We will assume the standard model of perfect securities markets (no taxes, restrictions on short sales, transaction costs ...) as described for example in [4], which we repeat here for completeness. The financial market consists of three assets defined on a complete, filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, Q)$ and on a finite time set $[0, T]$. The probability space supports Brownian motion and the filtration will be taken to be the canonical one augmented with the Q-null sets of $\mathcal{F}$.

The savings account, or bond, has its price process subject to variable, but certain, growth on the interval $[0, T]$:

$$dB(t) = r(t)B(t)dt \tag{2.1}$$

where $r(t) \geq 0$ is the interest rate at time t. We restrict ourselves to bounded interest rates and choose the initialization $B(0) = 1$.

Define continuous functions $\mu : I\!R^+ \times [0, T] \to I\!R$, $\rho : [0, T] \to I\!R^+ \cup \{0\}$, $\Sigma : I\!R^+ \times [0, T] \to I\!R^+$. The price process of the stock, or risky asset, is modelled as the diffusion process:

$$dS(t) = [\mu(S(t), t) - \rho(t)]S(t)dt + \Sigma(S(t), t)dW(t) \tag{2.2}$$

with appreciation rate μ; dividend rate ρ; volatility Σ; and where W is a standard Brownian motion on the probability space.

For the third security, we define a reward function ψ to be a continuous, non-negative function on $I\!R^+ \times [0,T]$. An American option on the stock with reward ψ is a financial asset which pays the stochastic reward $\psi(S(t),t)$ when exercised at time $t \in [0,T]$. The American Put has $\psi(s,t) = (\kappa - s)^+ =$ maximum $(\kappa - s, 0)$, and the American Call has $\psi(s,t) = (s-\kappa)^+$, where $\kappa \in I\!R^+$ is referred to as the strike price of the option. This paper will concentrate on these examples so that comparisons can be made with other works. However it may be noted that our method adapts easily to the case where the strike price and the interest rate are functions of time.

The valuation of an American option on the stock with reward ψ and expiration time T can be expressed as $V(S,0)$ where $S(0) = S$ and $V(s,t)$ is the solution of the following free boundary problem, in the case of the American put ([7],[8],[9]): For all $t > 0$, there exists a unique $s^*(t)$ so that

$$\mathcal{L}V(s,t) = 0 \ \text{ if } \ s > s^*(t), \ \ t \in [0,T) \tag{2.3}$$

$$V(s,t) \to (\kappa - s)^+ \ \text{ as } \ t \to T \tag{2.4}$$

$$V(s,t) \to 0 \ \text{ as } \ s \to \infty, \ \ t \in [0,T) \tag{2.5}$$

$$V(s,t) > (\kappa - s)^+ \ \text{ if } \ s > s^*(t), \ \ t \in [0,T) \tag{2.6}$$

$$V(s,t) \to (\kappa - s^*(t)) \ \text{ as } \ s \to s^*(t)+, \ \ t \in [0,T) \tag{2.7}$$

$$\frac{\partial V}{\partial s}(s,t) \to -1 \ \text{ as } \ s \to s^*(t)+, \ \ t \in [0,T) \tag{2.8}$$

where

$$\mathcal{L}V(s,t) \stackrel{\text{def}}{=} \frac{1}{2}\,\Sigma(s,t)^2\,\frac{\partial^2 V}{\partial s^2}(s,t) + [r(t) - \rho(t)]s\,\frac{\partial V(s,t)}{\partial s} - r(t)V(s,t) + \frac{\partial V(s,t)}{\partial t}\,. \tag{2.9}$$

The curve $s = s^*(t)$, the free boundary, is the early exercise frontier which must be determined simultaneously with $V(s,t)$.

As in [1] we shall study the normalized American put obtained by setting

$$u = V/\kappa, \qquad x = s/\kappa. \tag{2.10}$$

In addition, it will be convenient to introduce the new time variable

$$\tau = T - t.$$

Then we get the free boundary problem

$$\sigma(x,\tau)u_{xx} + b(x,\tau)u_x - r(x,\tau)u - u_\tau = 0 \tag{2.11}$$

when $x > s(\tau)$ and $0 < \tau \le T$, together with

$$u(s(\tau)+,\tau) = 1 - s(\tau), \quad 0 < \tau \le T \tag{2.12}$$

$$u_x(s(\tau)+,\tau) = -1, \quad 0 < \tau \le T \tag{2.13}$$

$$u(x,\tau) \to 0 \quad \text{as} \quad x \to \infty, \;\; 0 < \tau \le T \tag{2.14}$$

$$u(x,0) = 0 \quad \text{for} \quad x \ge 1 = s(0) \tag{2.15}$$

The Black-Scholes model is obtained when $\sigma(x,\tau) = \frac{1}{2}\sigma^2 x^2$, $b(x,\tau) = bx = (r-\rho)x$, $r(x,\tau) = r$ where σ, r and ρ are constant. (2.11) becomes

$$\frac{1}{2}\sigma^2 x^2 u_{xx} + (r-\rho)x u_x - ru - u_\tau = 0 \tag{2.16}$$

Its solution will be denoted by $u(x,\tau,r,\rho)$. The specific form of the coefficients of (2.16) allows its transformation to the standard heat equation which has been studied at great length (see for example [19]). In particular, it is known that the solution of (2.16) satisfies the integral equation

$$u(x,\tau) = e(x,\tau) + \int_0^\tau [re^{-rz}\phi(h_2(\tau,z)) - (r-b)e^{-(r-b)z} x\phi(h_1(\tau,z))]dz \tag{2.17}$$

where

$$h_1(\tau,z) = \left[\ln \frac{x}{s(\tau-z)} + \left(b + \frac{\sigma^2}{2}\right) z\right] \Big/ \sigma\sqrt{z}$$

and

$$h_2(\tau,z) = h_1(\tau,z) - \sigma\sqrt{z}\ .$$

Here $e(x,\tau)$ is the value of the corresponding European Put option which is given explicitly by the formula

$$e(x,\tau) = e^{-r\tau}\phi(d - \sigma\sqrt{\tau}) - xe^{-(r-b)\tau}\phi(d)$$

where

$$d = \left[\ln x + \left(b + \frac{\sigma^2}{2}\right)\tau\right] \Big/ \sigma\sqrt{\tau}$$

and

$$\phi(z) = \frac{1}{\sqrt{2\pi}} \int_z^\infty e^{-y^2/2}\, dy.$$

If the integral equation (2.17) is evaluated at $x = s(\tau)$, where $u(s(\tau),\tau) = 1 - s(\tau)$, then a convolution integral equation for the free boundary $s(\tau)$ results. Conversely, if the free boundary $s(\tau)$ is known then the value of the option for any price is readily computed. The initial condition shows that u_x is discontinuous at $x = 1$ and $\tau = 0$. It follows that the solution has a singularity there. We remark that it is shown in [11],[12] that this singularity leads to an asymptotic movement of the free boundary given by

$$S_B(\Delta\tau) \cong 1 - \sqrt{\Delta\tau |\ln \Delta\tau|} \quad \text{as} \quad \Delta\tau \to 0$$

while in [1] the formula

$$S_C(\Delta\tau) \cong \left(\frac{r\sqrt{\Delta\tau}}{\sigma\sqrt{2}}\right)^{\sigma\sqrt{\Delta\tau/2}} \quad \text{as} \quad \Delta\tau \to 0$$

is derived for the AMOL approximation at the first time step. These two expressions are not equivalent since

$$\lim_{\Delta\tau\to 0} \frac{1 - S_C(\Delta\tau)}{1 - S_B(\Delta\tau)} = \infty$$

Finally we note that the steady state solution of (2.11)-(2.16) is readily computed to be

$$u_\infty(x) = (1 - s_\infty)\left(\frac{x}{s_\infty}\right)^\gamma \tag{2.18}$$

where $s_\infty = \frac{\gamma}{\gamma-1}$ and

$$\gamma = \frac{-(b - \sigma^2/2) - \sqrt{(b - \sigma^2/2)^2 + 2r\sigma^2}}{\sigma^2} \tag{2.19}$$

The numerical method should reproduce this solution as $\tau \to \infty$.

§3. Solution Algorithm

One common method of eliminating an infinite interval is to map it to a finite interval. For example, it is known (see [1]) that the value $u(x, \tau, r, \rho)$ of the American Put described by (2.16) is related to the solution $v(x, \tau, \rho, r)$ of the American Call through the transformation

$$v(x, \tau, \rho, r) = xu(1/x, \tau, r, \rho)$$

Thus the solution (2.16) can be obtained by solving a similar free boundary problem on the finite interval $[0, \bar{s}(\tau)]$ for $v(x, \tau, \rho, r)$ subject to

$$\begin{aligned} &v(0, \tau) = 0 \\ &v(\bar{s}(\tau)-, \tau) = \bar{s}(\tau) - 1 \\ &v_x(\bar{s}(\tau)-, \tau) = 1 \\ &v(x, 0) = 0 \quad \text{on} \quad [0, \bar{s}(0) = 1] \end{aligned}$$

and $\bar{s}(\tau) = 1/s(\tau)$. As shown in [2] and briefly discussed below, the MOL algorithm advocated here is well suited for such problems.

In general, however, it is simpler to work with the original problem on a finite interval $[s(\tau), X]$. This eliminates the need for actually transforming the equations which, for the general model (2.11) would fail to have a transparent interpretation. Thus, for the time being let us suppose that the infinite interval $[s(\tau), \infty)$ has been truncated to $[s(\tau), X]$ and that the boundary condition (2.14) is replaced by the numerical boundary condition

$$u(X, \tau) = h(\tau) \tag{3.1}$$

The choice of X and $h(\tau)$ remains to be discussed. Then we have a standard free boundary problem to which the algorithm of [3], [2] applies. In short, the time derivative in (2.11) at time level τ_n is replaced by a difference quotient. If a first order approximation is chosen then the partial differential equation becomes

$$\sigma(x,\tau_n)u'' + b(x,\tau_n)u' - r(x,\tau_n)u - \frac{u - u_{n-1}}{\Delta\tau} = 0 \tag{3.2}$$

where $u \equiv u_n$ is the approximation to $u(x,\tau_n)$. If a second order backward approximation is used then we have

$$\sigma(x,\tau_n)u'' + b(x,\tau_n)u' - r(x,\tau_n)u - \frac{3}{2\Delta\tau}\,(u - u_{n-1}) + \frac{1}{2\Delta\tau}\,(u_{n-1} - u_{n-2}) = 0 \tag{3.3}$$

Either equation is a method of lines approximation to (2.11) which replaces the time dependent problem by a sequence of free boundary problems for ordinary differential equations. Since $u_0 = 0$ on $[s(0), X]$, (3.2) is self-starting. For (3.3) the solution on two previous time levels is required. For ease of notation we shall write (3.2,3) generically as

$$u'' + d(x,\tau_n)u' - c(x,\tau_n)u = g(x,\tau_n) \tag{3.4}$$

where c, d, g are obtained by comparing (3.4) with (3.2) or (3.3). As an aside we remark here that the "method of lines" terminology is ambiguous. In fact, for time dependent differential equations the method of lines approximation frequently describes an approximation of the problem by an initial value problem for time-continuous ordinary differential equations. Continuous time approximations are possible but not common for free boundary problems, see for example [13]. To distinguish the two approaches our discrete time method is sometimes called Rothe's method after its use in [14].

The second order equation (3.4) is written as a first order system

$$u' = v \tag{3.5a}$$

$$v' = c(x,\tau_n)u - d(x,\tau_n)v + g(x,\tau_n) \tag{3.5b}$$

It follows from the general theory of invariant imbedding [3], or by straightforward differentiation, that the solution of this system is related through the so-called Riccati transformation

$$u(x) = R(x)v(x) + w(x) \tag{3.6}$$

where R and w are the solutions of well defined initial value problems

$$R' = 1 + d(x,\tau_n)R - c(x,\tau_n)R^2, \quad R(X) = 0 \tag{3.7}$$

$$w' = -c(x,\tau_n)R(x)w - R(x)g(x,\tau_n), \quad w(X) = h(\tau_n) \tag{3.8}$$

We observe that the Riccati equation has a uniformly bounded non-positive solution on $[s_\infty, X]$ while the linear equation (3.8) has an exponentially decaying fundamental solution. Hence both equations are solvable numerically to any degree of accuracy. Once R and w are known on $[s_\infty, X]$ then it follows from (3.6) and the boundary condition (2.12), (2.13) that the free boundary $s \equiv s_n$ at time level τ_n must be chosen so that

$$1 - s = R(s)(-1) + w(s)$$

In other words, the free boundary s at time τ_n is necessarily a zero of the function

$$\phi(x) = R(x) - w(x) + (1 - x) \tag{3.9}$$

Any point s which satisfies $\phi(s) = 0$ is then an admissible free boundary at this time level. In general s will be the first root to the left of s_{n-1}. Once s has been determined the solution is completed by integrating (numerically)

$$v' = c(x, \tau_n)(R(x)v + w(x)) - d(x, \tau_n)v + g(x, \tau_n) \tag{3.10}$$

with $v(s) = -1$ and substituting the result into (3.6) to obtain $u(x)$ at time level τ_n. As in [1] the computed solution is extended over $[s_\infty, s_n]$ as a linear function. This is the complete algorithm for advancing the solution one time level to the next.

Let us now comment on the choice of the far boundary X. Clearly, X is our approximation to infinity. If X is large then the initial value problems (3.7,8) must be solved over large intervals. On the other hand, since the solution in general changes little outside a narrow region just to the right of the free boundary a fairly coarse integration can be carried out over much of the interval. In light of numerical experiments with typical financial parameters we shall set arbitrarily

$$X = 10.$$

For the Black-Scholes model (2.16) one can estimate *a priori* whether this X is a reasonable approximation to infinity. Since $u(x, 0) = 0$ for $x > 1$, it follows that $u(X, \tau) \cong 0$ for small τ. In fact, it is shown in the appendix that $u(X, \tau) < U(\tau)$ where

$$U(\tau) = X^\alpha u_\infty(1) erfc(\ln X/\sigma\sqrt{2\tau}), \tag{3.11}$$

$\alpha = -(b - \sigma^2/2)/\sigma^2$ and

$$erfc(z) \stackrel{\text{def}}{=} \frac{2}{\sqrt{\pi}} \int_z^\infty \exp(-\eta^2) d\eta$$

As long as $U(\tau) < 10^{-4}$ we set $h(\tau) = 0$ in (3.1). For example, for $r = 0.08$, $\rho = 0.02$ and the high volatility of $\sigma = 0.60$ we find $U(t) < 10^{-4}$ for $\tau < 0.99$, while for $\sigma = 0.4$ we find $\tau < 2.51$. For larger τ the size of X must be increased which requires some numerical experimentation.

Again, for (2.16) an alternative exists which exploits the integral representation (2.17). We can set, for example, $X = 2$. Once $U(\tau) > 10^{-4}$ then one can extrapolate $s(\tau)$ over $[\tau_{n-1}, \tau_n]$ from computed values and determine $u(X, \tau) = h(\tau)$ from (2.17). Since $s(\tau)$ is assumed known the quadrature is straightforward. With $h(\tau)$ given we compute a new free boundary s_n with the above sweep method. This new free boundary will in general differ from the extrapolated value and one can, in principle, now interpolate $s(\tau)$ over $[0, \tau_n]$ and recompute $h(\tau)$ from (2.17). In fact one can iterate until s no longer changes appreciably. However, numerical results show that $h(\tau)$ is quite insensitive to changes in s. Hence only one integration of the convolution integral (2.17) needs to be carried out. Even so, the integration of (2.17) at each time step can be quite time consuming because the probability distribution function has to be repeatedly evaluated.

Yet another alternative for the Black-Scholes model is a semi-analytic method of lines which combines analytic solutions on $[1, \infty)$ with the numerical method of lines on $(s_\infty, 1)$. For definiteness let us consider approximation (3.2) although a similar development applies to (3.3). It is straightforward to verify that the solution $u(x)$ at time level t_n over $[1, \infty)$ can be written as

$$u(x) = x^{\gamma_1} \sum_{i=0}^{n-1} \delta_i^n (\ln x)^i \tag{3.12}$$

where γ_1 is given by (2.19) after the substitution $r \leftarrow \left(r + \frac{1}{\Delta\tau}\right)$ and where the coefficients $\{\delta_i^n\}$ are found from the recursion formula

$$\delta_i^n = \frac{-\gamma_1 \left[\delta_{i-1}^{n-1} + \Delta\tau i(i+1)\frac{\sigma^2}{2}\delta_{i+1}^n\right]}{i[r\Delta\tau + 1 + \gamma_1^2\sigma^2\Delta\tau/2]} \tag{3.13}$$

with

$$\begin{aligned} &i = n-1, n-2, \ldots, 1 \\ &\delta_n^n = 0. \end{aligned}$$

We note that (3.13) does not determine δ_0^n. It follows from (3.12) that

$$\begin{aligned} u(1) &= \delta_0^n \\ u'(1) &= \gamma_1\delta_0^n + \delta_1^n \end{aligned}$$

so that

$$u'(1) \equiv v(1) = \gamma_1 u(1) + \delta_1^n. \tag{3.14}$$

A comparison with (3.6) shows that we account for (3.14) if we integrate (3.7) and (3.8) over $(s_\infty, 1]$ subject to the initial conditions

$$\begin{aligned} R(1) &= 1/\gamma_1 \\ w(1) &= -\delta_1^n/\gamma_1 \end{aligned} \tag{3.15}$$

The coefficient δ_0^n is found at the conclusion of the calculation at time level t_n from

$$\delta_0^n = u(1) = R(1)v(1) + w(1).$$

Clearly, this modification reduces the amount of computation considerably since there are no error functions to be evaluated and since the interval of integration is short. On the down-side is the observation that for small $\Delta\tau$ the exponents γ and γ_1 in (2.19) and (3.12) satisfy

$$\gamma_1 \ll \gamma < 0.$$

Since u in (3.12) must converge to u_∞ in (2.18) it follows that the coefficients δ_i^n must become quite large as $n \to \infty$ (see Table 1 below). This limits the applicability of (3.12) for long-term solutions and requires double-precision even for moderate times to maturity. We have not discovered a scaling of the Black-Scholes model which eliminates this problem.

For the call option the singularity in the problem is due to the vanishing coefficients in (2.16). If the boundary point $x = 0$ is perturbed to $x = \varepsilon$, then it follows from the maximum principle that u is monotonically increasing on $[\varepsilon, \bar{s}(t)]$. Hence the solution u and its first derivative are uniformly bounded with respect to ε and it is safe to replace $u(0,\tau) = 0$ with $u(\varepsilon, \tau) = 0$. Thus the call option appears somewhat simpler to compute than the put option. However, for long-term options the free boundary grows without bounds so that the numerical problems are much the same as in the put.

Finally we observe that the Riccati equation corresponding to the Black-Scholes model has the closed form solution

$$R(x) = \mu x \left[1 + \frac{1}{A - (x/X)^{\alpha}(1+A)} \right] \tag{3.16}$$

where α, μ, and A are found by substituting this expression into (3.7).

§4. Implementation and Numerical Results

Numerical results are presented first for two Black-Scholes problems chosen randomly from the many sample problems described in the literature. The following algorithms were implemented.

Method 1): First and second order method of lines with invariant imbedding on $[s_\infty, 10]$. $h(\tau) = 0$ for $U(\tau) < 10^{-4}$, $h(\tau)$ computed from (2.17) when $U(\tau) \geq 10^{-4}$.

For comparison we also list the value of the put obtained by evaluating (2.17) numerically after the free boundary is known on $[0, T]$.

Method 2): First order semi-analytic method of lines with invariant imbedding on $[s_\infty, 1]$ and initial conditions (3.15).

For both methods we choose a constant time step $\Delta\tau = T/N$ for some integer N. Next we define a fixed mesh $s_\infty = x_0 < x_1 < \ldots < x_j < \ldots < x_M = X$. This mesh need not be uniform. For time independent coefficients in (2.11) the Riccati equation (3.7) is integrated once with the trapezoidal rule from x_{j+1} to x_j, $j = M-1, \ldots, 1$. The results are stored. The algebraic equation resulting from the trapezoidal rule is quadratic in the unknown R_j and can be solved by formula.

We observe next that the linear equation (3.8) always depends on time through the source term and possibly, the initial condition. Hence it must be integrated at each time step. Its solution is also found with the trapezoidal rule which leads to a linear algebraic equation for w_j. As the values for w become available the algebraic sign of $\phi(x)$ defined by (3.9) is monitored at each mesh point. If $\phi(x)$ changes sign between mesh points x_j and x_{j+1} then the free boundary s at time level τ_n is found from the zero of the cubic interpolant through $\{\phi(x_{j-1}), \phi(x_j), \phi(x_{j+1}), \phi(x_{j+2})\}$. $\phi(x)$ was observed to have only one root on $[s_\infty, X]$ in our examples. The equation (3.10) is integrated with the trapezoidal rule, first from s_n to x_{j+1} and then over the fixed mesh to X. The right-hand side of (3.10) required for the first integration step at s is also found by interpolation. Finally, we find $u_n(x)$ from (3.6) and go to the next time step.

For the second order method we generally use the initial condition and the solution from the first order method at $\Delta\tau$ to get started. For the Black-Scholes model one could use the analytic solution at $\Delta\tau$ instead [1].

The numerical results shown here were obtained with a Fortran code. The program was not particularly fine-tuned. Optimization was provided by the compiler. Runs were executed on a workstation. For all runs with Method 1 we chose $\Delta x = (1 - s_\infty)/200$ on $[s_\infty, 1]$, $\Delta x = 1/100$ on $[1, 2]$ and $\Delta x = 8/500$ on $[2, 10]$. For Method 2 we took a uniform mesh of $\Delta x = (1 - s_\infty)/200$. Numerical answers changed little as this mesh size was further reduced.

The first example is chosen from the discussion of the analytic method of lines [1] where equation (3.2) is solved analytically over (s_k, s_{k-1}), $k = 1, \ldots, n$ and $(1, \infty)$ in terms of fundamental solutions and particular integrals like (3.12). Our numerical results are summarized in Table 1.

The analytic method of lines becomes quite involved if many time steps are to be executed because solutions over adjacent subintervals must be linked up smoothly. To reduce the amount of calculations a high order Richardson extrapolation based on the theory of [15] is employed in [1]. However, the extrapolation formulas are developed in [15] for linear boundary value problems with smooth solutions. The free boundary problem, on the other hand, is inherently nonlinear. In addition we know that the solution is not smooth in a neighbourhood at $t = 0$. The ad-hoc “fine-tuning” of the extrapolation formulas required in [1] reflect some of these difficulties. In fact, if we assume a local error bound for the numerical solution of (2.16) of the form

$$u_T(100, 1) - u_{\Delta\tau}(100) \cong K(\Delta\tau)^\alpha$$

where u_T is the analytic solution and K is independent of $\Delta\tau$, then any two numerical solutions $u_{\Delta\tau_1}, u_{\Delta\tau_2}$ and the estimate for the analytic solution can be used to compute α from the formula

$$\alpha \ln\left(\frac{\Delta\tau_1}{\Delta\tau_2}\right) = \frac{u_T - u_{\Delta\tau_1}}{u_T - u_{\Delta\tau_2}} \tag{4.1}$$

Pairwise data of Table 1 obtained with the first order approximation (3.2) quite consistently yield

$$\alpha \cong 0.9$$

which is far less than is required for the theory of [15]. On the other hand, a two-level extrapolation of the above data for, e.g., $\Delta t = 1/100$ and $\Delta t = 1/200$ according to (4.1) yields the improved answer

$$u_T(100, 1) = 8.33820.$$

However, the above data obtained with the second order approximation show no such consistency making extrapolation doubtful. Yet they appear much more accurate. Hence we prefer mesh refinements over extrapolation for checking convergence of the numerical solution.

The second example duplicates a calculation of [16] where an approximate solution of (2.16) is found by assuming a specific functional relationship for $u(x, t)$. This example will show the effect of long times to maturity and the importance of the numerical boundary condition at X. Numerical results are shown in Table 2.

Table 2: Free boundary $s(0)$ (before scaling)
$r = 0.08$, $\rho = 0$, $\sigma = 0.3$

$T = 5$. A numerical approximation to the free boundary at maturity is given in [21] as $s(0) = 66.6219$						
Δt	1st order MOL	CPU sec	2nd order MOL	CPU sec	Semi-anal. MOL	CPU sec
5/100	66.6431	0	66.6133	0	66.6428	0
5/200	66.6280	0	66.6128	0	66.6274	0
5/500	66.6183	2	66.6121	2	66.6180	0
5/1000	66.6150	5	66.6104	6	66.6146	1
$T = 10$. The steady state free boundary is $s_\infty = 64.00$						
10/100	64.9604	0	64.9399	0	64.9600	0
10/200	64.9499	1	64.9397	0	64.9496	0
10/500	64.9437	3	64.9394	4	64.9433	0
10/1000	64.9414	12	64.9393	12	64.9411	1
10/5000	64.9379	246	64.9381	249	64.9392	24

The enormous increase in computing time is due to the calculation of $h(\tau)$ from (2.17) which was carried out during the last 2810 steps where $U(t) > 10^{-4}$. Clearly, the naive integration of (2.17) with a trapezoidal rule should be replaced by a more sophisticated method based on an interpolant of the integrand involving only few function evaluations. Since any such algorithm would be specific to the Black-Scholes model we did not pursue this improvement.

The final example is chosen solely to illustrate the changes in the code required when we no longer have a Black-Scholes model. As our equation we shall take a financial model with residual volatility and an absorbing boundary condition which combines some of the features discussed in [6]. We shall assume that after scaling according to (2.10) the equation (2.11) is

$$\frac{1}{2}\sigma^2(1+x^2)u_{xx} + bxu_x - ru - u_\tau = 0 \tag{4.2}$$

$$u(X,\tau) = 0 \tag{4.3}$$

$$u(x,0) = 0, \qquad x > s(0) = 1 \tag{4.4}$$

The following absorbing free boundary condition is imposed

$$u(s,(\tau),\tau) = 1 - s(\tau) \qquad \text{and} \tag{4.5}$$

$$u(s(\tau),\tau) = -1 \qquad \text{if } s(\tau) > 0.5 \tag{4.6}$$

$$u(.5,\tau) = .5 \qquad \text{otherwise} \tag{4.7}$$

Equation (3.4) obtained by discretizing u_τ is straightforward to derive. Only its coefficients change in comparison to the Black-Scholes formulation. The integration of (3.7) and (3.8) with $h(\tau) = 0$ is carried out over $[0.5, X]$. The function $\phi(x)$ described by (3.9) remains unchanged. If it has a zero on $[0.5, 1]$ then that point is the free boundary at time level t_n. If ϕ does not have a zero on $[0.5, 1]$ then $s_n = 0.5$ and the derivative condition (4.6) is ignored. From (4.7) and (3.6) follows that

$$v(0.5) = \frac{(.5 - w(.5))}{R(.5)}$$

is the correct initial condition for equation (3.10).

Table 3 lists some representative results obtained with the second order method of lines. For $T = 1$ we use $X = 10$, $u(X, t) = 0$ and the same spacial mesh as in Table 1. Representative results are listed in Table 3. For the long-term problem $T = 10$ the boundary point X must be chosen out far enough so that its location has no influence on $u(x, \tau)$. The second part of Table 3 illustrates the numerical experiments one needs to carry out to determine an acceptable X when no numerical boundary value $h(\tau)$ is known at a fixed X so that X must approximate infinity.

Table 3: American put with residual volatility and absorbing boundary

$r = 0.08$, $\rho = 0.02$, $\sigma = .6$

$T = 5$		
Δt	$u(100, 0)$	$T - t = \tau$ to reach absorbing boundary
1/100	21.9546	0.10500
1/200	21.9559	0.10250
1/500	21.9565	0.10100
1/1000	21.9567	0.10100
1/2000[1)]	21.9568	0.10075
$T = 10$		
Δt	X[2)]	$u(100, 0)$
10/5000	10	40.5495
10/5000	20	40.8137
10/5000	30	40.8388
10/5000	40	40.8437[3)]

[1)] 11 CPU sec for a double precision run.

[2)] $\Delta x = 0.016$ on $[2, X]$.

[3)] 89 CPU sec for a double precision run.

The numerical experiments described above allow the following conclusions.

1) The second order method of lines approximation is the numerical method of choice. The solution at $\tau = \Delta\tau$ can be obtained from the first order method or, when applicable, from an analytic solution of the method of lines approximation at $\Delta\tau$.

2) Because the free boundary initially moves with infinite speed but slows down very quickly any commercial implementation of the method should accommodate variable time steps. Note that the Riccati equation must be resolved when $\Delta\tau$ changes.

3) For the Black-Scholes model and short times one can use the semi-analytic method in order to avoid error function evaluations. However, in view of the performance of the first and second order numerical methods a second order semi-analytic method based on (3.4) would appear to be preferable.

§5. Extensions

As indicated earlier our method is not tied to the special structure (2.16), and our method should be able to handle American options with more general models for stock prices like those presented in [5] and [6] where some numerical results are presented based on the recursive implementation method of [17]. As the MOL is not limited in use to one-dimensional free boundary value problems, [18], a further task will be to consider its application to pricing American options on underlying stock described by a multi-factor model (see for example [20], [21]).

§6. Appendix: An a priori bound for $u(X, \tau)$

Consider the Black-Scholes model (2.16) subject to (2.12-15). We take for granted the existence of a classical solution $\{u(x,\tau), s(\tau)\}$ with $s'(\tau) \leq 0$. Standard maximum principle arguments assure that

$$\max\{1-x, 0\} < u(x,\tau) < u_\infty(x)$$

where the steady state solution $u_\infty(x)$ is given by (2.18). Likewise, it follows from the maximum principle that for $1 < x < \infty$ and $\tau > 0$ the inequality

$$u(x,\tau) < W(x,\tau)$$

holds where $W(x,t)$ is the solution of (2.16) subject to

$$W(1,\tau) = u_\infty(1), \quad W(X,\tau) = 0, \quad W(x,0) = 0, \quad 1 < x < X.$$

We shall find a bound on $W(x,\tau)$. If we set

$$W(x,t) = e^{\alpha y + \beta\tau} V(y,t)$$

where

$$y = \ln x$$
$$t = \frac{1}{2}\sigma^2\tau$$

and

$$\alpha = 0.5 - \frac{b}{\sigma^2}$$

$$\beta = -r - \frac{\alpha^2\sigma^2}{2}$$

then one can verify that V is the solution of the problem

$$LV \equiv V_{yy}(y,t) - V_t(t,y) = 0$$

$$V(0,t) = u_\infty(1)e^{-(2\beta/\sigma^2)t}, \quad V(\ln X, t) = 0 \quad V(y,0) = 0, \qquad 0 < y < \ln X.$$

Let $\phi(y,t)$ be the solution of

$$L\phi \equiv \phi_{yy}(y,t) - \phi_t(y,t) = 0$$

$$\begin{aligned} &\phi(0,t) = 1 \\ &\phi(y,0) = 0, \qquad y > 0 \end{aligned}$$

and define

$$\Phi(y,t) = u_\infty(1)e^{-(2\beta/\sigma^2)t}\phi(y,t)$$

then we see from

$$L(\Phi - V) = \frac{2\beta}{\sigma^2}\,\Phi(y,t) \le 0$$

and the maximum principle that

$$V(y,t) \le \Phi(y,t)$$

The solution $\phi(y,t)$ is known to be

$$\phi(y,t) = erfc\left(\frac{y}{\sqrt{4t}}\right)$$

Reversing the transformations we find that

$$u(X,t) < X^\alpha u_\infty(1)erfc\left(\frac{\ln X}{\sigma\sqrt{2\tau}}\right).$$

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