Pricing and Hedging Equity Indexed Annuities with Variance-Gamma Deviates

Sebastian Jaimungal [a]

[a] *Department of Statistics, University of Toronto,*
100 St. George Street, Toronto, Ontario, Canada M5S 3G3

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Abstract

The author analyzes the pricing and hedging problem for Equity Indexed Annuities (EIAs) with underlying risky assets following a geometric Variance-Gamma process. This model allows accurate and parsimonious replication of the implied volatility smiles observed in the financial market. I argue that this model produces consistency in pricing and hedging between the financial and insurance markets. Closed form expressions for prices of Point-to-Point and Cliquet instruments are developed and used to investigate the break-even participation rates. Furthermore, I derive the hedging parameters - Delta, Gamma and Vega - for the Cliquet design. Mortality risk is incorporated through the Actuarial present value principal and I use numerical experiments to investigate the effects of the model parameters.

Key words: , Equity Indexed Annuities, Guaranteed Investments, Jump Processes, Variance Gamma Model
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Email address: `sjaimung@utstat.utoronto.ca` (Sebastian Jaimungal).

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1 Introduction

Equity-indexed annuities (EIAs) are considered to be one of the most innovative products introduced into the insurance market in years. They have attained extremely high popularity in recent times, reaching sales of over US $6 billion per annum. EIAs provide the annuitant with a guaranteed return combined with the ability to earn a percentage of equity returns - borrowing characteristics from both fixed-rate and variable-rate annuities. These insurance products have recently come under intense focus; however, the analysis has been restricted to the case where the underlying index price is modeled by a geometric Brownian motion (GBM). Tiong (2000) successfully obtained closed form pricing formulae for various instruments, through Esscher transformation methods, under the assumptions of GBM. Lin and Tan (2003) investigate the effects of stochastic interest rates by postulating a Vasiček model for the short rate process which is correlated to the GBM of the risky asset. They use simulation techniques to examine the role that the various model parameters play. Their results demonstrate that stochastic interest rates can affect the premiums and the break-even participation rates of the EIA products; however, for small to medium correlation coefficients the magnitude of these corrections are fairly small, and the volatility of the index appears to be more important. This is to be expected, because when the short rate and asset price process are uncorrelated, the interest rate dynamics can be factored from the equity dynamics, and the pricing formulae which arise are those of Tiong (2000) with r replaced by the average interest rate over the term of the option, $r_{avg} = \frac{1}{T-t} \ln \mathbb{E}^Q \left[\exp\{-\int_t^T r_s\, ds\}\right]$. In a market calibrated model, this average corresponds to the prevailing spot interest rate of the appropriate maturity. Furthermore, although EIA products typically have long terms (around 10 years) the embedded reset features are typically applied over $\frac{1}{2} \sim 1$ year periods. This implies that forward prices spanning a single year or less are the most relevant, and such interest rates have significantly lower volatility than, for example, a 10-year spot-rate.

In this paper, I extend the pricing formulae of Tiong (2000) to account for heavy tails and skewness in the distribution of asset returns. The rationale being that most EIAs are written on underlying indices for which there are heavily traded plain-vanilla put and call options. Furthermore, the market prices for such options are inconsistent with the assumption of GBM, with the inconsistency reflected in the so called implied volatility smile and volatility term structure, as well as in the statistical return distribution of asset prices. There are three main classes of models which are capable of correctly matching market implied volatilities: (i) state-dependent volatility models, (ii) stochastic volatility models, and (iii) jump models. State-dependent volatility models attempt to capture the correlation between asset prices and volatility. Non-parametric specifications have been investigated by Derman and Kani (1998) and Duprie (1994); while parametric specifications have been invested by Cox and Ross (1976) among many others. Stochastic volatility models assume that the volatility is driven by a separate stochastic process, which may be correlated to the asset price process. Hull and White (1987) model the variance process as a diffusion process, while Duan (1996) utilizes a GARCH model to model the two-dimensional dynamics. Jump models postulate that the asset price follows a jump process, such as in the Poisson jump-diffusion model of Merton (1976), and in the Variance-Gamma (VG) model introduced by Madan and Seneta (1990) and extended in Madan, Carr, and Chang (1998).

I propose to use the geometric variance Gamma (VG) process to obtain EIA premiums which are consistent with the market prices of puts and calls. The VG model is characterized by a time-changed Brownian motion, where the time-change is driven by a Gamma process. Log-returns of the underlying are assumed to be normally distributed in *financial time* rather than in real time. The financial clock ticks in tandem with business activity; that is, during periods of high activity the financial clock runs faster than the real clock, while during periods of low activity the financial clock runs slower than the real clock. This model parsimoniously captures implied volatility smile effects. Basic properties of the geometric VG process are reviewed in section 2 where I also introduce the main pricing results which are used throughout the remainder of the paper.

In section 3, I obtain the premiums for EIAs with Point-to-Point and Cliquet designs and explore how the model parameters affect this premium. In addition to consistently pricing the EIA products, in section 4 I derive the dynamic hedging parameters Delta, Gamma, and Vega implied by the VG model and illustrate how they differ from those implied by a GBM asset model. I find that the differences can be dramatic and argue that heavy tails and skewness must be taken into account to hedge EIAs properly. The financial risks in the EIA products can be managed and priced through option theoretic methods; however, mortality risk must also be accounted, and I achieve this through through standard Actuarial principles. In section 5 I show how the Actuarial present value principal can be used to price these products, and also obtain the mortality adjusted hedging strategies.

2 The Variance-Gamma Model

The Variance-Gamma (VG) model has been chosen as a representative of the class of jump models as it parsimoniously characterizes volatility smiles with three parameters: overall level (σ), skewness (θ), and excess kurtosis (ν). Monroe (1978) demonstrated that any semimartingale has a representation in terms of a time-changed Brownian motion. The VG model specifies the time-change to be driven by a Gamma process. Monroe's result implies that log-returns of the underlying asset can be assumed to be normally distributed in *financial time* rather than in calendar time. The ticking of the financial clock is correlated with business activity, so that during periods of high activity the financial clock runs more quickly than the calendar clock, while during periods of low activity the financial clock runs more slowly than the calendar clock. The Gamma process is chosen to drive the financial clock for two reasons: (i) it is the simplest continuous state-space analog of the Poisson process and (ii) it leads to a parsimonious characterization of the two most important inherent biases in the Black-Scholes formula: skewness and excess kurtosis.

Let $\{X_t | t \geq 0\}$ denote a Brownian process with drift θ and volatility σ; further, let $\{\Gamma_t | t \geq 0\}$

denote a Gamma process with variance rate ν and mean rate unity [2]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space with filtration $\mathcal{F} = \{\mathcal{F}_t | t \geq 0\}$ where $\mathcal{F}_t$, for every $t \geq 0$, is the product sigma field $\mathcal{F}_t^X \otimes \mathcal{F}_t^\Gamma$, $\mathcal{F}^X = \{F_t^X | t \geq 0\}$ is the natural filtration generated by X_t, and $\mathcal{F}^\Gamma = \{F_t^\Gamma | t \geq 0\}$ is the natural filtration generated by Γ_t. The VG process is defined as a Brownian motion subordinated by a Gamma process:

$$Y_t \equiv X_{\Gamma_t} \; . \tag{1}$$

Remarkably, this process is in fact a pure jump Lévy process and can be represented in terms of its random jump measure $\mu(dy, dt)$, which counts the number of jumps arriving in the interval $[t, t+dt)$ of magnitude $[y, y+dy)$ (see Bertoin, 1996; Sato, 1999). In terms of this jump measure the VG process can be written as

$$Y_t = \int_0^t \int_{-\infty}^{\infty} y \; \mu(dy, dt) \; . \tag{2}$$

The predictable compensator of the jump process $\nu(dy)$ renders the compensated process $J_t - t \int_{-\infty}^{\infty} \nu(dy)$ a Martingale (see Bertoin, 1996; Sato, 1999). The compensator $\nu(dy)$ is known as the Lévy measure (or density), and is independent of t because of the stationarity of Lévy processes. For the VG model the Lévy density can be written as

$$\nu(dy) = \frac{1}{\nu |y|} \exp\left(by - a|y|\right) dy \; , \tag{3}$$

where,

$$a = \left[\frac{2}{\nu\sigma^2} + \left(\frac{\theta}{\sigma^2} \right)^2 \right]^{1/2} \qquad \text{and} \qquad b = \frac{\theta}{\sigma^2} \; . \tag{4}$$

The characteristic function of Y_t is supplied by the Lévy-Khintchine formula

$$\Phi(z) = \mathbb{E}^{\mathbb{Q}} \left[\mathrm{e}^{izX_{\Gamma_t}} \right] = \exp\left\{ t\psi(z) \right\} \; , \tag{5}$$

[2] A pure jump process and pure diffusion process contain orthogonal stochastic components as a result of the Lévy representation theorem, and are therefore immediately independent.

where,

$$\psi(z) = \int_{-\infty}^{\infty} \left(e^{izx} - 1\right) \nu(dx) = -\frac{1}{\nu} \left\{ \ln\left(1 - i(a-b)z\right) + \ln\left(1 - i(a+b)z\right) \right\} \ . \tag{6}$$

From the above expression, the VG process can be viewed as the sum of two Gamma processes, one with strictly positive and the other with strictly negative jump sizes. This proves the earlier assertion that the VG process is indeed a pure jump model even though it is built as a time-changed Brownian process.

The moments of the VG process can be computed from the cumulant $\psi(x)$. The first four moments are found to be:

$$\mathbb{E}[X_{\Gamma_t}] = \theta\, t \ , \tag{7}$$

$$\mathbb{E}[(X_{\Gamma_t} - \mathbb{E}[X_{\Gamma_t}])^2] = (\theta^2\nu + \sigma^2)\, t \ , \tag{8}$$

$$\mathbb{E}[(X_{\Gamma_t} - \mathbb{E}[X_{\Gamma_t}])^3] = (2\theta^2\nu + 3\sigma^2)\, \nu\, \theta\, t \ , \tag{9}$$

$$\mathbb{E}[(X_{\Gamma_t} - \mathbb{E}[X_{\Gamma_t}])^4] = (3\sigma^4 + 12\sigma^2\nu + 6\theta^4\nu^2)\, \nu\, t + (3\sigma^4 + 6\sigma^2\theta^2\nu + 3\theta^4\nu^2)\, t^2 \ . \tag{10}$$

Notice that as θ tends to 0 the third moment vanishes; furthermore, the sign of θ determines the sign of the skewness - positive skewness corresponds to positive θ, while negative skewness corresponds to negative θ. Finally, in this limit, the fourth moment divided by the square of the second is $3(1+\nu)$; consequently, ν characterizes the excess kurtosis in the model.

In this paper, the asset price process S_t is postulated to be a geometric VG process with a drift adjustment directly under a given risk-neutral probability measure $\mathbb{Q}$. Explicitly,

$$S_t = S_0 \exp\left\{ \omega\, t + X_{\Gamma_t} \right\} \ . \tag{11}$$

The asset is tradable only in real time, after the stochastic time change, and not in business activity time (so that Γ_t is not an observable). The drift parameter ω is fixed by forcing the dividend compensated asset price relative to the money-market account $M_t = \exp(\int_0^t r_s\, ds)$ to be a martingale, as follows:

$$\mathbb{E}^{\mathbb{Q}}\left[\frac{e^{\delta t}S_t}{M_t}\middle|\mathcal{F}_s\right]=\frac{e^{\delta s}S_s}{M_s}\ \Rightarrow\ \omega=r-\delta+\frac{1}{\nu}\ln\left(1-\left(\theta+\frac{1}{2}\sigma^2\right)\nu\right)\ . \tag{12}$$

Here δ denotes the (constant) dividend yield rate of the asset and I assume that the short-rate process is non-stochastic for simplicity. In particular r_s is equal to r which is assumed to be positive. The rate r is known as the force of interest by Actuaries.

The price of a European option written on a risky asset following a geometric VG process can be obtained in three ways. Firstly, by utilizing Ito's lemma for Lévy processes to derive a partial-integral-differential-equation (PIDE) that the price function solves; secondly, by utilizing the characteristic function (5) to write the price in terms of the Fourier transformation of the pay-off and the density; and thirdly, by utilizing the representation as a time changed Brownian process. I find that the most useful representation for pricing EIAs is the third and the main tool for determining their price is the following theorem.

Theorem 2.1 *Given a European contingent claim which pays $\varphi(\ln(S_T))$ at time T, and assume that the asset price process follows (11) under a given risk-neutral measure $\mathbb{Q}$. The arbitrage-free price at time t of such a claim, denoted $P_t(s,T)$ where $s=\ln(S_t)$, equals the time weighted average of shifted Black-Scholes prices:*

$$P_t(s,T)=\int_0^\infty \exp\left\{\left(\theta+\tfrac{1}{2}\sigma^2\right)g-r\tau\right\}\ P_{BS}\left(s+\omega\tau,\sigma,\theta+\frac{\sigma^2}{2},g\right)\ f_{\Gamma_\tau}(g)\ dg\ , \tag{13}$$

where $\tau\equiv T-t$, $f_{\Gamma_\tau}(g)$ denotes the probability density function (pdf) of a Gamma random variable,

$$f_{\Gamma_\tau}(g)=\frac{g^{\frac{\tau}{\nu}-1}\,\mathrm{e}^{-\frac{g}{\nu}}}{\Gamma(\frac{\tau}{\nu})\,\nu^{\frac{\tau}{\nu}}}\ , \tag{14}$$

and the price of the option in the Black-Scholes model is,

$$P_{BS}(s,\sigma,r,u)=\frac{e^{-ru}}{\sqrt{2\pi\sigma^2 u}}\int_{-\infty}^{\infty}\varphi(s+x)\ \exp\left\{-\frac{\left(x-(r-\frac{1}{2}\sigma^2)u\right)^2}{2\sigma^2 u}\right\}\ dx\ . \tag{15}$$

Madan, Carr, and Chang (1998) use a version of the above theorem to demonstrate that puts and calls can be written in terms of degenerate Hypergeometric functions and Bessel functions. The pricing formula they obtain are similar to the Black-Scholes case with the density function for the standard normal replaced by the special function

$$\Psi(a,b,c) \equiv \int_0^\infty \Phi\left(\frac{a}{\sqrt{g}} + b\sqrt{g}\right) \frac{g^{c-1}e^{-g}}{\Gamma(c)}\, dg \ , \tag{16}$$

where $\Phi(x)$ denotes the cumulative density function (cdf) of a standard normal random variable. The Ψ function is a weighted average of the normal cdf which arises in Black-Scholes pricing equations - Ψ accounts for the effects of the stochastic time change. In the appendix, the Ψ function is written in terms of Hypergeometric and Bessel functions. The price of European call and put options can be reduced to combinations of the Ψ function using proposition 2.1 resulting in the following pricing equation.

Theorem 2.2 Call Price Function. *The value at time $t < T$ of a call option with strike K and maturity T written on an asset following a geometric VG process (11) is*

$$C_t(T,K) = S_t\, e^{-\delta\,\tau}\Psi\left(\varsigma\sqrt{\frac{\vartheta}{\nu}}, \frac{\theta+\sigma^2}{\sigma}\sqrt{\frac{\nu}{\vartheta}}, \frac{\tau}{\nu}\right) - K\, e^{-r\,\tau}\, \Psi\left(\frac{\varsigma}{\sqrt{\nu}}, \frac{\theta}{\sigma}\sqrt{\nu}, \frac{\tau}{\nu}\right) \ , \tag{17}$$

where,

$$\varsigma = \frac{\ln(S_t/K) + \omega\tau}{\sigma} \ , \tag{18}$$

$$\vartheta = 1 - (\theta + \tfrac{1}{2}\sigma^2)\, \nu \ , \tag{19}$$

and $\tau \equiv T - t$.

The VG model is able to capture a variety of features in the implied volatility smiles. Recall that implied volatility σ_{imp} is the volatility in the Black-Scholes model which matches quoted prices and is defined as the solution to the non-linear equation

$$\mathrm{Call}_{BS}(S,K,\sigma_{imp},\tau) = \mathrm{Call}(S,K,\tau) \ , \tag{20}$$

where $\mathrm{Call}(S, K, \tau)$ denotes the quoted price. Figure C.1 depicts several implied volatility smiles at maturities of $\frac{1}{4}$, $\frac{1}{2}$ and 1 year using the VG model prices as a proxy for market prices. The risk-free rate has little effect on the general shape of the curves and is fixed at 5% per annum for all maturities. Notice that the smiles are negatively skewed and steep in the shorter term, while they flatten and become more symmetric in the long term. Such behavior is typically observed in the financial markets across all sectors. The VG kurtosis parameter ν controls the steepness while θ controls the asymmetry of the smiles. Negative θ is typically observed in options on index futures as well as individual stock options. This implies that the market values out-of-the money puts more highly than similar out-of-the-money calls.

3 Pricing Formulae

3.1 The Point-To-Point Design

A Point-to-Point EIA supplies the annuitant with a guaranteed minimum (continuously compounded) return of γ on a percentage β of the notional amount from the time of entering the contract to maturity T. The equity-linked return, which the minimum is compared with, is scaled by the participation rate which I denote by α. The value of this contract at expiry is written as follows:

$$\varphi(S_0, S_T) = \max\left(\beta\, e^{\gamma(T-t)}, \left(\frac{S_T}{S_0}\right)^{\alpha}\right) . \tag{21}$$

The value of the contract at an arbitrary time t, such that $0 \leq t \leq T$, is obtained by conditioning on the Gamma time change and using (16). I record the pricing result in the following theorem and delegate the proof to the appendix (in the remainder of this paper, all proofs are deferred to the appendix).

Theorem 3.1 *The value at time t, such that $0 \leq t \leq T$, of a Point-to-Point EIA contract* *written on an asset following a geometric VG process (11), is*

$$P_t^{pp}(S_0, S_t, \alpha, \beta, \gamma, T)$$
$$= \beta\, e^{-r\,\tau + \gamma T}\, \Psi\left(\frac{\varsigma}{\sqrt{\nu}}, -\frac{\theta}{\sigma}\sqrt{\nu}, \frac{\tau}{\nu}\right) + \left(\frac{S_t}{S_0}\right)^{\alpha} \frac{e^{-(r-\alpha\omega)\,\tau}}{\vartheta^{\tau/\nu}}\, \Psi\left(-\varsigma\sqrt{\frac{\vartheta}{\nu}}, \frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}}, \frac{\tau}{\nu}\right) \; , \quad (22)$$

where $\tau \equiv T - t$ represents the time to maturity,

$$\varsigma = \frac{\gamma\, T - \alpha(\omega\,\tau + \ln(S_t/S_0)) + \ln(\beta)}{\alpha\sigma} \; , \quad (23)$$

$$\overline{\theta} = \theta + \alpha\sigma^2 \; , \quad (24)$$

and

$$\vartheta = 1 - \alpha\,\nu\left(\theta + \frac{1}{2}\alpha\sigma^2\right) \; . \quad (25)$$

Figure C.2 shows the dependence of the initial premium of a Point-To-Point EIA as a function of the time to maturity and the excess kurtosis parameter ν. For α less than one the behavior of the premium is predominantly determined by the time to maturity while the excess kurtosis has little affect. This is reassuring, since most products are being sold with participation rates considerably less than unity, while most risk management systems are built around the assumptions of Black-Scholes. However, when α is greater than one, the premiums are very sensitive to the excess kurtosis parameter ν, and it becomes important to account for departures from Black-Scholes.

The break-even participation rate, defined as the participation rate which makes the premium of the contract exactly one, provides a benchmark for setting α. Tiong (2000) finds that the theoretical Black-Scholes break-even points are considerably higher than those currently being offered in the market. In Table C.1 the break-even participation rates for the Point-to-Point instrument under several market conditions are shown. In this table, the VG volatility parameter σ is chosen such that the at-the-money implied volatility of a call option is equal to either 20% or 30%. This allows for meaningful comparisons across different terms and interest rate levels.

On comparing Table C.1 with the results in Tiong (2000) one see that VG model implies *lower* break-even participation rates than those implied by GBM. This agrees with intuition, since accounting for jump risk induces an increase in the price of the contract for the same level of participation; consequently, a lower participation is required to break-even. Even though the VG-model implies lower participation rates, they are not sufficiently low to fully account for current market practices.

3.2 The Cliquet Design

The Cliquet (or ratchet) design EIA provides several guarantees over the lifetime of the product. At the end of each time period of length Δt this EIA locks in a percentage α of the return on the index; however, if the scaled return is lower the guaranteed minimum return of γ, then the investor receives a return of γ for that period. The pay-off from this investment scheme is paid at the maturity date, and can be expressed as,

$$\varphi(\{S_0, S_{\Delta t}, \ldots, S_{n\Delta t}\}) = \prod_{i=1}^{n} \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right) . \tag{26}$$

The value of such a contract at t less than Δt is therefore

$$\begin{aligned}
&P_t^c\left(S_0, S_t, \gamma, \alpha, \Delta t, n\right) && (27)\\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\,(n\Delta t - t)} \prod_{i=1}^{n} \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\middle|\,\mathcal{F}_t\right]\\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\,(n\Delta t - t)} \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{\Delta t}}{S_0}\right)^{\alpha}\right) \prod_{i=2}^{n} \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\middle|\,\mathcal{F}_t\right]\\
&\quad = e^{-r\,(\Delta t - t)}\mathbb{E}^{\mathbb{Q}}\left[\max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{\Delta t}}{S_0}\right)^{\alpha}\right)\middle|\,\mathcal{F}_t\right]\\
&\qquad\qquad \times \prod_{i=2}^{n} e^{-r\,\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\middle|\,\mathcal{F}_{(i-1)\Delta t}\right]\\
&\quad = P_t^{pp}(S_0, S_t, \alpha, 1, \gamma, \Delta t)\ \left(P_0^{pp}(1, 1, \alpha, 1, \gamma, \Delta t)\right)^{n-1} . && (28)
\end{aligned}$$

The third equality follows from the independent and stationary increment properties of the Brownian motion and the subordinating Gamma process which generate the VG dynamics. Notice that the premium, at $t = 0$, with n resets is equal to the premium of a one period EIA raised to the power of n. The break-even points for these instruments are therefore equal to those in a Point-to-Point design with a term of Δt and a β of 100%. If the underlying process is not stationary, then the pricing problem would be much less tractable. This is one major advantage of the VG model - it is able to capture heavy tails and skewness while maintaining analytical tractability.

3.3 The Cliquet with a Cap Design

The Cliquet design is also offered with a cap κ. The cap feature reduces the price of the guarantee, and therefore admits a higher break-even participation rate than the same instrument without a cap. The embedded protection provided by the floor, along with the higher participation rate, makes this product much more attractive than the Cliquet without a cap. The pay-off at maturity is

$$\varphi(\{S_0, S_{\Delta t}, \ldots, S_{n\Delta t}\}) = \prod_{i=1}^{n} \min\left(e^{\kappa\,\Delta t},\ \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\right) . \tag{29}$$

In the limit as κ tends to ∞ the capped Cliquet design reduces to the Cliquet design without a cap. The general pricing problem can be treated by conditioning on the stochastic time change and integrating over the remaining Brownian degree of freedom.

Theorem 3.2 *The value at time t, such that $0 \leq t < \Delta t$, of a capped Cliquet with n remaining protection periods, written on an asset following a geometric VG process (11) is*

$$P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) = P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)\ \left(P_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t)\right)^{n-1} , \tag{30}$$

where,

$$
\begin{aligned}
&P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t) \\
&\quad = e^{-r\tau}\Bigg\{ e^{\kappa \Delta t}\Psi\left(-\frac{\varsigma(\kappa)}{\sqrt{\nu}}, \frac{\theta}{\sigma}\sqrt{\nu}, \frac{\tau}{\nu}\right) + e^{\gamma\Delta t}\Psi\left(\frac{\varsigma(\gamma)}{\sqrt{\nu}}, -\frac{\theta}{\sigma}\sqrt{\nu}, \frac{\Delta t}{\nu}\right) \\
&\qquad\qquad + \left(\frac{S_t}{S_0}\right)^{\alpha} \frac{\mathrm{e}^{\alpha\omega\tau}}{\vartheta^{\tau/\nu}}\left[\Psi\left(\varsigma(\kappa)\sqrt{\frac{\vartheta}{\nu}}, -\frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}}, \frac{\tau}{\nu}\right) - \Psi\left(\varsigma(\gamma)\sqrt{\frac{\vartheta}{\nu}}, -\frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}}, \frac{\tau}{\nu}\right)\right]\Bigg\}, \quad (31)
\end{aligned}
$$

and

$$
\varsigma(x) = \frac{1}{\sigma}\left\{\frac{x\Delta t}{\alpha} - (\omega\tau + \ln(S_t/S_0))\right\} . \quad (32)
$$

Figure C.3 illustrates how the premium varies as a function of the participation rate and excess kurtosis. As in the Point-to-Point case, the Black-Scholes premium forms a lower boundary for the premiums at all levels of participation. The heavy tails and negative skewness place a higher burden on the embedded guarantee resulting in higher premiums than the case without such heavy tails or negative skewness. The general trend found in the GBM case is maintained; however, the values themselves and curvatures are clearly larger in the VG model. Table C.2 contains the break-even participation rates for several model and contract parameters. The excess kurtosis in the VG model induces at least a 10% reduction in the break-even participation rates compared with the results reported in Tiong (2000).

4 Dynamic Hedging

In this section, I obtained the Delta, Gamma and Vega of a capped Cliquet (referred to as a Cliquet in the sequel). These parameters can be used to dynamically hedge the market risk component of the EIAs. Within an incomplete market model, such as the VG model, dynamic hedging cannot be used to eliminate all risks; however, it will offer the risk manager the ability to partially hedge them. The hedging parameters developed in this section will certainly outperform those implied by the Black-Scholes model, as heavy tails and skewness are explicitly incorporated into the pricing problem.

The Delta $\Delta_t(S_t)$ of an option is defined as the relative change of the price of the option with respect to small changes in the spot price. Let $P_t(S_t)$ denote the price of an option with a spot of S_t, then

$$\Delta_t(S_t) \equiv \frac{\partial}{\partial S_t} P_t(S_t) \ . \tag{33}$$

The Delta prescribes the number of units in the underlying asset that the risk manager can hold to offset changes in the option price due to changes in the spot price. However, it is well known that due to the curvature of option prices, as a function of the spot-level, Delta hedging is insufficient for risk management purposes. This is especially true for options in which the convexity changes sign, such as those embedded in EIA products. To properly hedge such risks the options Gamma $\Gamma_t(S_t)$ must be utilized, and offsetting positions in put and call options must be used in conjunction with positions in the underlying asset to properly hedge the position. The Gamma is defined as the sensitivity of the Delta as the spot changes and is explicitly given by:

$$\Gamma_t(S_t) \equiv \frac{\partial}{\partial S_t} \Delta_t(S_t) = \frac{\partial^2}{\partial S_t^2} P_t(S_t) \ . \tag{34}$$

Finally, although spot price changes constitute the largest affect on the hedging strategy, the market volatility may also change through time. To manage such risks the options Vega $\mathcal{V}_t(S_t)$, measured by the sensitivity of price to volatility level, is used. Explicitly,

$$\mathcal{V}_t(S_t) \equiv \frac{\partial}{\partial \sigma} P_t(S_t) \ . \tag{35}$$

In the VG-model, the parameter σ is not an absolute measure of the volatility level (it also affects kurtosis and skewness, see equations (7-10)); however, it predominantly sets the vertical position of the implied volatility smiles. Consequently, it is a reasonable proxy for the constant volatility parameter in the Black-Scholes model and I propose that it be used to hedge against changes in volatility level.

I now record the main results of this section, while delegating the proofs to the appendix.

Theorem 4.1 *The Delta of the capped Cliquet EIA is,*

$$\Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) = \Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)\ \left(P_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t)\right)^{n-1} \tag{36}$$

where,

$$\begin{aligned}
&\Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t) \\
&\quad = \frac{\alpha}{\vartheta^{\tau/\nu}} \frac{S_t^{\alpha-1}}{S_0^{\alpha}}\ e^{-(r-\alpha\omega)\tau} \left[\Psi\left(\varsigma(\kappa)\sqrt{\frac{\vartheta}{\nu}},\ -\frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}},\ \frac{\tau}{\nu}\right) - \Psi\left(\varsigma(\gamma)\sqrt{\frac{\vartheta}{\nu}},\ -\frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}},\ \frac{\tau}{\nu}\right)\right] .
\end{aligned} \tag{37}$$

Theorem 4.2 *The Gamma of the capped Cliquet EIA is,*

$$\Gamma_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) = \Gamma_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)\ \left(P_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t)\right)^{n-1} \tag{38}$$

where,

$$\begin{aligned}
&\Gamma_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t) \\
&\quad = \frac{\alpha-1}{S_t}\ \Delta_t^{cc}(s_t, \gamma, \alpha, \kappa, \Delta t) \\
&\qquad + \frac{2\alpha\, e^{-r\tau}}{\sigma\, S_t^2\, \nu^{\frac{\tau}{\nu}} \Gamma(\frac{\tau}{\nu})} \left\{ \left(\frac{\overline{\varsigma}^2(\gamma)}{a}\right)^{\frac{\tau}{2\nu}-\frac{1}{4}} e^{\theta\,\overline{\varsigma}(\gamma)/\sigma + \gamma T}\, K_{\frac{\tau}{\nu}-\frac{1}{2}}\left(2\sqrt{a\,\overline{\varsigma}^2(\gamma)}\right) \right. \\
&\qquad\qquad \left. - \left(\frac{\overline{\varsigma}^2(\kappa)}{a}\right)^{\frac{\tau}{2\nu}-\frac{1}{4}} e^{\theta\,\overline{\varsigma}(\kappa)/\sigma + \kappa T}\, K_{\frac{\tau}{\nu}-\frac{1}{2}}\left(2\sqrt{a\,\overline{\varsigma}^2(\kappa)}\right) \right\} ,
\end{aligned} \tag{39}$$

and,

$$a = \frac{1}{\nu} - \frac{1}{2}\frac{\theta^2}{\sigma^2} . \tag{40}$$

Furthermore, $K_v(x)$ denotes the modified Bessel function of the second kind of order v.

Theorem 4.3 *The Vega of the capped Cliquet EIA is,*

$$\begin{aligned}
&\mathcal{V}_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) \\
&\quad = \mathcal{V}_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)\ \left(P_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t)\right)^{n-1} \\
&\qquad + (n-1)\, P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)\, \mathcal{V}_0^{cc}(1, 1, S_0, \gamma, \alpha, \kappa, \Delta t)\ \left(P_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t)\right)^{n-2} ,
\end{aligned} \tag{41}$$

where,

$$
\begin{aligned}
&\mathcal{V}_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t) \\
&\quad = \alpha \left(\frac{S_t}{S_0}\right)^{\alpha} e^{(\omega\alpha - r)\tau} \Bigg\{ \\
&\qquad \frac{\alpha\sigma\nu}{\vartheta^{\frac{\tau}{\nu}+1}} \left[\Psi\left(\varsigma(\kappa)\sqrt{\frac{\vartheta}{\nu}}, -\frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}}, \frac{\tau}{\nu}+1\right) - \Psi\left(\varsigma(\gamma)\sqrt{\frac{\vartheta}{\nu}}, -\frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}}, \frac{\tau}{\nu}+1\right)\right] \\
&\qquad - \frac{\tau\sigma}{(1-(\theta+\frac{1}{2}\sigma^2)\nu)\vartheta^{\frac{\tau}{\nu}}} \left[\Psi\left(\varsigma(\kappa)\sqrt{\frac{\vartheta}{\nu}}, -\frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}}, \frac{\tau}{\nu}\right) - \Psi\left(\varsigma(\gamma)\sqrt{\frac{\vartheta}{\nu}}, -\frac{\overline{\theta}}{\sigma}\sqrt{\frac{\nu}{\vartheta}}, \frac{\tau}{\nu}\right)\right] \\
&\qquad - \frac{1}{\Gamma(\frac{\tau}{\nu})\nu^{\tau/\nu}}\sqrt{\frac{2}{\pi}} \left[e^{\varsigma(\kappa)\overline{\theta}} \left(\frac{\varsigma^2(\kappa)}{2\frac{\vartheta}{\nu}+\frac{\overline{\theta}^2}{\sigma^2}}\right)^{\frac{\tau}{2\nu}+\frac{1}{4}} K_{\frac{\tau}{\nu}+\frac{1}{2}}\left(\sqrt{\varsigma^2(\kappa)\left(2\frac{\vartheta}{\nu}+\frac{\overline{\theta}^2}{\sigma^2}\right)}\right)\right. \\
&\qquad\qquad \left.\left. - e^{\varsigma(\gamma)\overline{\theta}} \left(\frac{\varsigma^2(\gamma)}{2\frac{\vartheta}{\nu}+\frac{\overline{\theta}^2}{\sigma^2}}\right)^{\frac{\tau}{2\nu}+\frac{1}{4}} K_{\frac{\tau}{\nu}+\frac{1}{2}}\left(\sqrt{\varsigma^2(\gamma)\left(2\frac{\vartheta}{\nu}+\frac{\overline{\theta}^2}{\sigma^2}\right)}\right)\right]\right\}. \qquad (42)
\end{aligned}
$$

Figure C.4 depicts how the Delta, as a function of the spot price, evolves through time. The top panel depicts the results in the VG-model and for comparison the Black-Scholes model results are displayed in the lower panel. Notice that the Delta approaches its asymptotic value, at the maturity date, much more quickly in the VG model than it does in the Black-Scholes model.

Figure C.5 depicts the behavior of the Price, Delta, Gamma and Vega, as a function of the spot price, on the excess kurtosis parameter ν. In the in-the-money region the price increases as ν increases, this results from an increase in the probability that the process jumps deeper into the money. In the out-of-the-money region the price decreases as ν increases, this results from an increase in the probability that the process jumps deeper out of the money.

The Delta also has an interesting behavior. As ν increases, the Delta increases in the region in which the EIA pays the equity return, while it decreases outside this region. Thia is to say, the Delta tends towards its value at maturity as ν increases. Unlike the Black-Scholes Delta, which is essentially symmetric, the VG Delta is highly asymmetric due to the asymmetry in the jump sizes.

The behavior of the Gamma is also consistent with the previous observations. As ν increases the Gamma becomes more highly peaked near the spot prices where the floor and the cap become effective. Also notice that the sign of Gamma changes from positive to negative.

Finally, the absolute size of the Vega of the EIA is smaller than the Black-Scholes case, indicating that the option prices are less sensitive to changes in volatility level. This is to be expected since the presence of tail events has a larger effect on price than small changes in the volatility level.

5 Incorporating Mortality Risk

EIA contracts typically contain mortality clauses such as the following: if the annuitant dies in year m, then the guaranteed return up to the end of that year is paid to the investors beneficiary. If the insurance company is well diversified, in that the number of policies, with the same contract parameters, are sold to a large number of individuals, then mortality risk can be priced into the contracts rather simply using the Actuarial present value principle (see Bowers, Gerber, Hickman, Jones, and Nesbitt, 1997). This assumes that the future lifetime $T(x)$ of the individual (x) is stochastically independent of the asset price process, and is certainly a just assumption. Let ${}_tp_x \equiv \mathbb{P}(T(x) > t)$ denote the probability that an individual (x) will survive past age t given that the individual is aged x. As usual the mortality probability is defined as ${}_tq_x = 1 - {}_tp_x$. The one year survival and mortality probability are defined as $p_x \equiv {}_1p_x$ and $q_x \equiv {}_1q_x$ respectively. Assume that if the annuitant dies during the time interval $[k, k+1)$, then the annuity closes at the end of the year $k+1$. The mortality modified pay-off for the EIA contract, accumulated to the maturity date $T = n\Delta t$, is therefore

$$
\begin{aligned}
&\overline{\varphi}\left(\{S_0, S_{\Delta t}, \ldots, S_{n\Delta t}\}, T(x)\right) \\
&\quad = e^{r\,(n-1)\Delta t}\varphi\left(\{S_0, S_{\Delta t}\}\right)\, \mathbb{I}\left(T(x) \in (t, \Delta t]\right)
\end{aligned}
$$

$$+\sum_{k=2}^{n} e^{r\,(n-k)\Delta t}\,\varphi\left(\{S_0, S_{\Delta t}, \ldots, S_{k\Delta t}\}\right)\mathbb{I}\left(T(x) \in ((k-1)\Delta t, k\Delta t]\right)\ , \tag{43}$$

where $\mathbb{I}(\mathcal{A})$ represents the indicator function for the event $\mathcal{A}$ and $\varphi\left(\{S_0, S_{\Delta t}, \ldots, S_{k\Delta t}\}\right)$ represents the pay-off of a $k\Delta t$-maturity EIA without mortality risk.

The Actuarial fair value of the contract is

$$\bar{P}_t = \mathbb{E}^{\mathbb{Q}}[\overline{\varphi}\left(\{S_0, S_{\Delta t}, \ldots, S_{n\Delta t}\}, T(x)\right)|\overline{\mathcal{F}}_t] = P_t(N)\ {}_N p_x + \sum_{k=1}^{N} P_t(k)\ {}_{k-1}p_x\ q_{x+k-1}\ , \tag{44}$$

where $P_t(k)$ denotes the price, at time t, of the k-maturity EIA without mortality risk and $\overline{\mathcal{F}}_t$ represents the product filtration generated by the asset and the time of death stochastic processes. The break-even participation rate in a well diversified portfolio therefore solves

$$\bar{P}_0 = P(N)_0\ {}_N p_x + \sum_{k=0}^{N-1} P(k)_0\ {}_k p_x\ q_{x+k} = 1\ . \tag{45}$$

If the portfolio is not well diversified, in that the number of issued annuities is small, one can instead use a loaded participation rate, defined as the participation rate which solves

$$\mathbb{E}^{\mathbb{Q}}[\overline{\varphi}|\overline{\mathcal{F}}_0] + \epsilon\sqrt{\mathrm{Var}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[\overline{\varphi}|T(x)]|\overline{\mathcal{F}}_0]} = 1\ . \tag{46}$$

A participation rate satisfying (46) will guarantee that $\epsilon\%$ of the time there are sufficient funds to cover the EIA benefits. Clearly this results in an equal or lower participation rate than that computed by (45). The variance appearing in (46) is easily computed in terms of the prices of the EIA product without mortality risk as follows:

$$\mathrm{Var}^{\mathbb{Q}}[\mathbb{E}^{\mathbb{Q}}[\overline{\varphi}|T(x)]|\mathcal{F}_0] = P_0^2(N)\ {}_N p_x + \sum_{k=0}^{N-1} P_0^2(k)\ {}_k p_x\ q_{x+k}\ - \bar{P}_0^2\ . \tag{47}$$

Interestingly, in an asset model which has stationary and independent increments, the break-even participation rates with and without mortality risk are identical; furthermore, the loaded break-even participation rate is equal to the unloaded rate. This is because in the case without mortality there exists a single participation α, across all maturity levels, that sets the premium equal to one. This results from the power law behavior in for example equation (30). Consequently, if such a rate is used in the presence of mortality risk, then summing over all mortality events, weighted by their respective probabilities, also results in a premium of one. Furthermore, since the premiums for all maturities are equal, the variance of the premiums is identically zero and the loaded participation rate calculation reduces to the unloaded case.

Even though the break-even participation rates are unaffected by mortality risk in a stationary model, the hedging parameters are are affected when mortality is present. In the presence of mortality risk one finds the following hedging parameters:

$$\Delta_t = {}_N p_x\, \Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, N) + \sum_{k=1}^{N} \Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, k)\ {}_{k-1}p_x\ q_{x+k-1} \tag{48}$$

$$\Gamma_t = {}_N p_x\, \Gamma_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, N) + \sum_{k=1}^{N} \Gamma_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, k)\ {}_{k-1}p_x\ q_{x+k-1} \tag{49}$$

$$\mathcal{V}_t = {}_N p_x\, \mathcal{V}_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, N) + \sum_{k=1}^{N} \mathcal{V}_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, k)\ {}_{k-1}p_x\ q_{x+k-1}\ . \tag{50}$$

Figure C.6 depicts how the hedging parameters vary as mortality is incorporated into the pricing problem. I assume that the time of death random variable $T(x)$ is exponentially distributed with hazard rate μ. The diagrams clearly show that the Delta and Vega acquire significant corrections due to the presence of mortality, while Gamma remains essentially unaffected. The diagrams also indicate that the absolute size of the hedging parameters decreases as the hazard rate increases.

6 Conclusions

In this paper I derived closed form formulae for the prices of Point-to-Point and Cliquet designed EIA instruments when the underlying asset price is modeled as geometric variance Gamma process. This allows for consistency between the financial market prices of put and call options and EIA products written on the same underlying asset. The pricing results are more complex than those found in the Black-Scholes model; however, they are tractable and allow for efficient numerical implementation. I explored several features of the pricing results and determined the break-even participation rates under various contract and model specifications. The results indicate that the break-even rates are less than those implied by the Black-Scholes model; however, they are still higher than those used in the market. I further derived closed form equations for the hedging parameters: Delta, Gamma and Vega. These parameters can be used to manage market risk due to spot price changes and changes in volatility level. I then used the actuarial present value principle to determine the mortality adjustments to prices and hedging parameters. The results of this work illustrate that jumps do affect both pricing and hedging significantly, albeit not drastically.

Future directions of research include the addition of stochastic volatility through further stochastic time changes. Such generalizations will allow for consistent pricing between European put and call options, forward start put and call options, and EIA products. It will be interesting to see whether the corrections due to stochastic volatility are of the same size as the corrections found in this work. Incorporating stochastic interest rates is also a fruitful direction to explore. Finally, the treatment of mortality risk can be improved by investigating indifference pricing methodologies as in Young (2003) for GBM and in Jaimungal and Young (2004) for Lévy processes.

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A The Psi Function

The following two lemmas (whose proof can be found in Madan, Carr, and Chang (1998)) allow prices to be written explicitly in terms of special functions.

Lemma A.1 *The two functions,*

$$H_1(x,y,z) = \int_{-1}^{u} e^{-ys}\,(1-s^2)^{z-1}\,ds\ , \tag{A.1}$$

$$H_2(x,y,z) = \int_{-1}^{u} e^{-ys}\,s\,(1-s^2)^{z-1}\,ds\ , \tag{A.2}$$

can be written in terms of the degenerate hypergeometric function of two variables,

$$\Theta(a,b,c;x,y) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_{0}^{1} u^{\alpha-1}\,(1-u)^{c-a-1}\,(1-u\,x)^{-b}\,e^{u\,y}\,du\ , \tag{A.3}$$

as follows,

$$H_1(x,y,z) = \frac{1}{z}\,e^{y}\,(1+x)^{z}2^{z-1}\Theta\left(z, 1-z, 1+z; \frac{1+x}{2}, -y(1+x)\right)\ , \tag{A.4}$$

$$\begin{aligned} H_2(x,y,z) = {} & -H_1(x,y,z) + \frac{1}{1+z}\,e^{y}\,(1+x)^{1+z}2^{z-1} \times \\ & \Theta\left(1+z, 1-z, 2+z; \frac{1+x}{2}, -y(1+x)\right)\ . \end{aligned} \tag{A.5}$$

Lemma A.2 *The function Ψ can be written in terms of H_1 and H_2 as follows,*

$$\begin{aligned} \Psi(a,b,c) = {} & \frac{d^{c+\frac{1}{2}}}{\sqrt{2\pi}\,\Gamma(c)2^{c-1}} \left\{ K_{c+\frac{1}{2}}(d)\,H_1\left(\frac{b}{\sqrt{2+b^2}}, sgn(a)\,d, c\right) \right. \\ & \left. -sgn(a)K_{c-\frac{1}{2}}(d)\,H_2\left(\frac{b}{\sqrt{2+b^2}}, sgn(a)\,d, c\right)\right\} \end{aligned} \tag{A.6}$$

where,

$$d = |a|\sqrt{2+b^2}, \tag{A.7}$$

and $K_v(x)$ is the modified Bessel function of the second kind of order v.

B Pricing Formula Proofs

Proof of Proposition 2.1. The price of a European contingent claim is equal to the discounted expected pay-off in a risk-neutral measure, $\mathbb{Q}$. Therefore,

$$\begin{aligned}
&P_t(s_t, T) \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \varphi\left(\ln S_T\right) \middle| \mathcal{F}_t \right] \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \varphi\left(s_t + \omega(T-t) + X_{\Gamma_T} - X_{\Gamma_t}\right) \middle| \mathcal{F}_t \right] \\
&\quad = \int_0^\infty \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \varphi\left(s_t + \omega(T-t) + X_{\Gamma_{t+g}} - X_{\Gamma_t}\right) \middle| \mathcal{F}_t \right] f_{\Gamma_T - \Gamma_t}(g)\, dg\ .
\end{aligned} \tag{B.1}$$

Conditional on $\mathcal{F}_t$, $X_{\Gamma_{t+g}} - X_{\Gamma_t} \triangleq X_g$ where $\triangleq$ represents equality in law; consequently,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \varphi\left(s_t + \omega(T-t) + X_g\right) \middle| \mathcal{F}_t \right] \\
&\quad = e^{(\theta + \frac{1}{2}\sigma^2)g - r(T-t)} \mathbb{E}^{\mathbb{Q}}\left[e^{-(\theta + \frac{1}{2}\sigma^2)g} \varphi\left(s_t + \omega(T-t) + X_g\right) \middle| \mathcal{F}_t \right] \\
&\quad = e^{(\theta + \frac{1}{2}\sigma^2)g - r(T-t)} P_{BS}\left(s_t + \omega(T-t), \sigma, \theta + \frac{1}{2}\sigma^2, 0, g \right)\ .
\end{aligned} \tag{B.2}$$

Substituting (B.2) into (B.1) leads immediately to the pricing equation (13). $\square$

Proof of Theorem 3.1 The price is,

$$\begin{aligned}
&P_t^{pp}(S_0, S_t, \alpha, \beta, \gamma, T) \\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r(T-t)} \max\left(\beta\, e^{\gamma(T-t)}, \left(\frac{S_T}{S_t}\right)^{\alpha} \right) \middle| \mathcal{F}_t \right] \\
&\quad = \int_0^\infty f_{\Gamma_T - \Gamma_t}(g)\, e^{-r(T-t)}\ \mathbb{E}^{\mathbb{Q}}\left[\max\left(\beta\, e^{\gamma\,(T-t)}, e^{\alpha(\omega(T-t) + X_g)} \right) \middle| \mathcal{F}_t \right]\ dg \\
&\quad = \int_0^\infty f_{\Gamma_T - \Gamma_t}(g)\ \left\{ \beta\, e^{-(r-\gamma)(T-t)} \Phi(d_+) +\ e^{(\alpha\theta + \frac{1}{2}\alpha^2\sigma^2)\tau - (r - \alpha\omega)(T-t)}\ \Phi(d_-) \right\}\ dg\ ,
\end{aligned} \tag{B.3}$$

where,

$$d_+ = \frac{(\gamma - \alpha\omega)\,(T-t) + \ln(\beta)}{\alpha\sigma\,\sqrt{g}} - \frac{\theta}{\sigma}\sqrt{g}, \quad \text{and} \quad d_- = \alpha\sigma\sqrt{g} - d_+\,, \tag{B.4}$$

and the equality in law, $X_{\Gamma_t+g} - X_{\Gamma_t} \triangleq X_g$, has been used. Note that two different time scales, g and $T-t$, appear in (B.3). Furthermore, the discount factors that appear in the analogous formula in Tiong (2000) are absent. Applying the definition of (16) to B.3 leads to (22). □

Proof of Theorem 3.2 The price is,

$$\begin{aligned}
&P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n)\\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\,(n\Delta t - t)} \prod_{i=1}^{n} \min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_t\right]\\
&\quad = \mathbb{E}^{\mathbb{Q}}\left[e^{-r\,(n\Delta t - t)} \min\left(e^{\kappa\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{\Delta t}}{S_0}\right)^{\alpha}\right)\right)\right.\\
&\qquad\qquad \left.\times \prod_{i=2}^{n} \min\left(e^{\kappa\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_t\right]\\
&\quad = e^{-r\,(\Delta t - t)}\,\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\,\frac{S_{\Delta t}}{S_t}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_t\right]\\
&\qquad\qquad \times \prod_{i=2}^{n} e^{-r\,\Delta t}\,\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_{(i-1)\Delta t}\right]\,.
\end{aligned} \tag{B.5}$$

The last equality follows from the independent and stationary increment properties of the VG model. Conditioning on the stochastic time-change reduces the expectation to the following integral:

$$\begin{aligned}
&P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)\\
&\quad \equiv e^{-r(\Delta t - t)}\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\,\frac{S_{\Delta t}}{S_t}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_t\right]\\
&\quad = e^{-r(\Delta t - t)} \int_0^{\infty} f_{\Gamma_T - \Gamma_t}(g)\,\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\right)^{\alpha} e^{\alpha(\omega(T-t) + X_g)}\right)\right)\middle|\,\mathcal{F}_t\right] dg
\end{aligned} \tag{B.6}$$

The expectation under the integral can be computed easily and results in:

$$\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\right)^{\alpha} e^{\alpha(\omega(T-t) + X_g)}\right)\right)\middle|\,\mathcal{F}_t\right]$$

$$= \int_0^\infty f_{\Gamma_T - \Gamma_t}(g) \Big\{ \; e^{\kappa \Delta t}\, \Phi(-d(\kappa)) + e^{\gamma \Delta t}\, \Phi(d(\gamma))$$

$$+ e^{\alpha[\omega \Delta t + (\theta + \frac{1}{2}\alpha\sigma^2)g]} \Big[\Phi\Big(d(\kappa) - \alpha\sigma\sqrt{g}\Big) - \; \Phi\Big(d(\gamma) - \alpha\sigma\sqrt{g}\Big)\Big]\Big\}\, dg \; , \qquad \text{(B.7)}$$

where,

$$d(x) \equiv \frac{\varsigma(x)}{\sqrt{g}} - \frac{\theta}{\sigma}\sqrt{g} \; . \qquad \text{(B.8)}$$

Through (16) the expectation can be written in terms of Ψ and on substituting these results into (B.5) one obtains (31). $\square$

Proof of Theorem 4.1 The Delta is defined as the partial derivative of the price with respect to the spot level,

$$\begin{aligned}
&\Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) \\
&\quad = \frac{\partial}{\partial S_t} P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) \\
&\quad = \frac{\partial}{\partial S_t} \mathbb{E}^{\mathbb{Q}} \left[e^{-r\,(n\Delta t - t)} \prod_{i=1}^{n} \min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{t+i\Delta t}}{S_{t+(i-1)\Delta t}} \right)^{\alpha} \right) \right) \middle| \mathcal{F}_t \right] \\
&\quad = e^{-r\,(\Delta t - t)}\, \mathbb{E}^{\mathbb{Q}} \left[\frac{\partial}{\partial S_t} \min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0} \frac{S_{\Delta t}}{S_t} \right)^{\alpha} \right) \right) \middle| \mathcal{F}_t \right] \\
&\qquad\qquad \times \prod_{i=2}^{n} e^{-r\,\Delta t}\, \mathbb{E}^{\mathbb{Q}} \left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} \right)^{\alpha} \right) \right) \middle| \mathcal{F}_{(i-1)\Delta t} \right] \; . \qquad \text{(B.9)}
\end{aligned}$$

The expectations under the product sign are of same form as in (B.6). Explicitly,

$$e^{-r\,\Delta t}\, \mathbb{E}^{\mathbb{Q}} \left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}} \right)^{\alpha} \right) \right) \middle| \mathcal{F}_{(i-1)\Delta t} \right] = P_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t) \; . \; \text{(B.10)}$$

The computation now reduces to the expectation containing the partial derivative term. Through straightforward calculations one finds that,

$$\begin{aligned}
&\Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) \\
&\quad \equiv e^{-r\tau} \mathbb{E}^{\mathbb{Q}} \left[\frac{\partial}{\partial S_t} \min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0} \frac{S_{\Delta t}}{S_t} \right)^{\alpha} \right) \right) \middle| \mathcal{F}_t \right]
\end{aligned}$$

$$= \frac{\alpha}{S_0}\left(\frac{S_t}{S_0}\right)^{\alpha-1} e^{-(r-\omega\alpha)\tau} \int_0^\infty f_{\Gamma_T-\Gamma_t}(g)\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha X_g}\mathbb{I}\left(a(\gamma,g) < S_t < a(\kappa,g)\right)\Big|\, \mathcal{F}_t\right]\, dg\ , \quad \text{(B.11)}$$

where,

$$a(x,g) = S_0 e^{x\Delta t - \omega\tau + X_g}\ ,$$

and $\tau \equiv \Delta t - t$. Finally, the expectation under the integral evaluates to

$$\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha X_g}\mathbb{I}\left(a(\gamma,g) < S_t < a(\kappa,g)\right)\Big|\, \mathcal{F}_t\right] \\
&= \int_0^\infty f_{\Gamma_T-\Gamma_t}(g) e^{\alpha(\theta+\frac{1}{2}\alpha\sigma^2)g}\left[\Phi\left(\frac{\varsigma(\kappa)}{\sqrt{g}} - \frac{\overline{\theta}}{\sigma}\sqrt{g}\right) - \Phi\left(\frac{\varsigma(\gamma)}{\sqrt{g}} - \frac{\overline{\theta}}{\sigma}\sqrt{g}\right)\right]\, dg\ . \qquad \text{(B.12)}
\end{aligned}$$

The integral can now be expressed in terms of Ψ through (16) and one finds the result (36). $\square$

Proof of Theorem 4.2 The Gamma is defined as the second partial derivative of the price with respect to the spot level

$$\begin{aligned}
&\Gamma_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) \\
&\quad = \frac{\partial^2}{\partial^2 S_t} P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t, n) \\
&\quad = \frac{\partial}{\partial S_t}\mathbb{E}^{\mathbb{Q}}\left[e^{-r\,(n\Delta t - t)}\prod_{i=1}^{n}\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{t+i\Delta t}}{S_{t+(i-1)\Delta t}}\right)^{\alpha}\right)\right)\Big|\, \mathcal{F}_t\right] \\
&\quad = e^{-r\,(\Delta t - t)}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial^2}{\partial^2 S_t}\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\frac{S_{\Delta t}}{S_t}\right)^{\alpha}\right)\right)\Big|\, \mathcal{F}_t\right] \\
&\qquad\quad \times \prod_{i=2}^{n} e^{-r\,\Delta t}\,\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\right)\Big|\, \mathcal{F}_{(i-1)\Delta t}\right]\ . \qquad \text{(B.13)}
\end{aligned}$$

The last equality follows from the independent and stationary increment properties of the VG model. The expectations under the product sign can be written in terms of the price as before. Conditioning on the stochastic time-change reduces the expectation containing the partial derivatives to the following integral:

$$\begin{aligned}
&\Gamma_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t) \\
&\quad \equiv e^{-r\,\tau}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial^2}{\partial^2 S_t}\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\frac{S_{\Delta t}}{S_t}\right)^{\alpha}\right)\right)\Big|\, \mathcal{F}_t\right]
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\alpha-1}{S_t}\Delta_t^{cc}(S_0,S_t,\gamma,\alpha,\kappa,\Delta t) \\
&\quad + \frac{\alpha}{S_0}\left(\frac{S_t}{S_0}\right)^{\alpha-1} e^{-(r-\alpha\omega)\tau} \int_0^\infty f_{\Gamma_T-\Gamma_t}(g)\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha X_g}\left(\delta(S_t-a(\gamma,g))\right.\right. \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad \left.\left.-\delta(S_t-a(\kappa,g))\right)\right|\mathcal{F}_t\Big]\, dg \ ,
\end{aligned} \tag{B.14}
$$

where $\delta(x-a)$ denotes the Dirac delta function at the point a, i.e. for any random variable Y the expectation $\mathbb{E}[g(Y)\delta(Y-z)]$ equals $g(z)f_Y(z)$, where $f_Y(y)$ denotes the pdf of Y. The expectation under the integral evaluates to

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha X_g}\left(\delta(S_t-a(\gamma,g))-\delta(S_t-a(\kappa,g))\right)\Big|\,\mathcal{F}_t\right] \\
&= \frac{S_0^\alpha e^{-\alpha\omega\tau-\alpha\theta g-\frac{1}{2}\frac{\theta^2}{\sigma^2}g}}{\sigma\, S_t^{\alpha+1}\sqrt{g}}\left[e^{\gamma\Delta t-\frac{\varsigma(\gamma)^2}{2g}+\frac{\theta}{\sigma}\varsigma(\gamma)} - e^{\kappa\Delta t-\frac{\varsigma(\kappa)^2}{2g}+\frac{\theta}{\sigma}\varsigma(\kappa)}\right] \ .
\end{aligned} \tag{B.15}
$$

The integral over the time change g can be carried by making use of the integral definition of the modified Bessel function of the second kind

$$
\int_0^\infty g^{c-1}e^{-ag-b/g}dg = 2\left(\frac{b}{a}\right)^{c/2} K_c(2\sqrt{ab}) \ . \tag{B.16}
$$

Putting the results together one arrives at (38). □

Proof of Theorem 4.3 The Vega is defined as the partial derivative of the price with respect to the volatility σ level

$$
\begin{aligned}
&\mathcal{V}_t^{cc}(S_0,S_t,\gamma,\alpha,\kappa,\Delta t,n) \\
&= \frac{\partial}{\partial\sigma}P_t^{cc}(S_0,S_t,\gamma,\alpha,\kappa,\Delta t,n) \\
&= \frac{\partial}{\partial\sigma}\mathbb{E}^{\mathbb{Q}}\left[e^{-r\,(n\Delta t-t)}\prod_{i=1}^n \min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{t+i\Delta t}}{S_{t+(i-1)\Delta t}}\right)^\alpha\right)\right)\Bigg|\,\mathcal{F}_t\right] \\
&= e^{-r\,(\Delta t-t)}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial\sigma}\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\frac{S_{\Delta t}}{S_t}\right)^\alpha\right)\right)\Bigg|\,\mathcal{F}_t\right] \\
&\qquad \times\prod_{i=2}^n e^{-r\,\Delta t}\,\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^\alpha\right)\right)\Bigg|\,\mathcal{F}_{(i-1)\Delta t}\right]
\end{aligned}
$$

$$
\begin{aligned}
&\quad + e^{-r\,(\Delta t - t)}\,\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\,\frac{S_{\Delta t}}{S_t}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_t\right] \\
&\qquad\qquad \times \sum_{k=2}^{n} e^{-r\Delta t}\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial\sigma}\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{k\Delta t}}{S_{(k-1)\Delta t}}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_{(k-1)\Delta t}\right] \\
&\qquad\qquad\quad \times \prod_{i=2, i\neq k}^{n} e^{-r\,\Delta t}\,\mathbb{E}^{\mathbb{Q}}\left[\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_{i\Delta t}}{S_{(i-1)\Delta t}}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_{(i-1)\Delta t}\right] \\
&= \mathcal{V}_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)(P_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t))^{n-1} \\
&\quad + (n-1)P_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)\mathcal{V}_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t)(P_0^{cc}(1, 1, \gamma, \alpha, \kappa, \Delta t))^{n-2}\ ,
\end{aligned}
\tag{B.17}
$$

where,

$$
\begin{aligned}
&\mathcal{V}_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t) \\
&\quad \equiv e^{-r\tau}\,\mathbb{E}^{\mathbb{Q}}\left[\frac{\partial}{\partial\sigma}\min\left(e^{\kappa\,\Delta t}, \max\left(e^{\gamma\,\Delta t}, \left(\frac{S_t}{S_0}\,\frac{S_{\Delta t}}{S_t}\right)^{\alpha}\right)\right)\middle|\,\mathcal{F}_t\right] \\
&\quad = \left(\frac{S_t}{S_0}\right)^{\alpha} e^{-(r-\alpha\omega)\tau}\int_0^{\infty} f_{\Gamma_T-\Gamma_t}(g)\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha X_g}\mathbb{I}(a(\gamma, g) < S_t < a(\kappa, g))\right. \\
&\qquad\qquad\qquad\qquad \left.\times\left(\alpha\sqrt{g}(X_g - \theta g) - \frac{\alpha\tau\sigma}{1-(\theta+\frac{1}{2}\sigma^2)\nu}\right)\middle|\,\mathcal{F}_t\right]\ .
\end{aligned}
\tag{B.18}
$$

The second term in in the integral is proportional to $\Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)$ explicitly,

$$
\begin{aligned}
&-\frac{\alpha\tau\sigma e^{-(r-\alpha\omega)\tau}}{1-(\theta+\frac{1}{2}\sigma^2)\nu}\left(\frac{S_t}{S_0}\right)^{\alpha}\int_0^{\infty} f_{\Gamma_T-\Gamma_t}(g)\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha X_g}\mathbb{I}(a(\gamma, g) < S_t < a(\kappa, g))\middle|\,\mathcal{F}_t\right]\,dg \\
&\qquad\qquad = -\frac{\sigma\tau}{1-(\theta+\frac{1}{2}\sigma^2)\nu}S_t\,\Delta_t^{cc}(S_0, S_t, \gamma, \alpha, \kappa, \Delta t)\ .
\end{aligned}
\tag{B.19}
$$

The expectation of the first term evaluates to

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{Q}}\left[e^{\alpha X_g}\mathbb{I}(a(\gamma, g) < S_t < a(\kappa, g))\alpha\sqrt{g}(X_g - \theta g)\middle|\,\mathcal{F}_t\right] \\
&\quad = e^{\alpha(\theta+\frac{1}{2}\alpha\sigma^2)g}\left[\alpha^2\sigma g\left(\Phi\left(\frac{\varsigma(\kappa)}{\sqrt{g}} - \frac{\overline{\theta}}{\sigma}\sqrt{g}\right) - \Phi\left(\frac{\varsigma(\gamma)}{\sqrt{g}} - \frac{\overline{\theta}}{\sigma}\sqrt{g}\right)\right)\right. \\
&\qquad\qquad \left. -\alpha\sqrt{g}\left(\Phi'\left(\frac{\varsigma(\kappa)}{\sqrt{g}} - \frac{\overline{\theta}}{\sigma}\sqrt{g}\right) - \Phi'\left(\frac{\varsigma(\gamma)}{\sqrt{g}} - \frac{\overline{\theta}}{\sigma}\sqrt{g}\right)\right)\right]\ .
\end{aligned}
\tag{B.20}
$$

Once again applying (16) and making use of (B.16) one arrives at the result in (41). □

C Tables and Figures

		ATM vol = 20%			ATM vol = 30%		
δ	Term	$r = 4\%$	$r = 5\%$	$r = 6\%$	$r = 4\%$	$r = 5\%$	$r = 6\%$
	3	0.7288	0.8079	0.8667	0.5897	0.6648	0.7266
1%	5	0.7700	0.8597	0.9225	0.6326	0.7235	0.7931
	10	0.8474	0.9474	1.0054	0.7218	0.8291	0.8996
	3	0.8111	0.8966	0.9565	0.63395	0.71185	0.77576
2%	5	0.8678	0.9628	1.0267	0.68508	0.78081	0.85300
	10	0.9704	1.0765	1.1331	0.79245	0.90573	0.97807

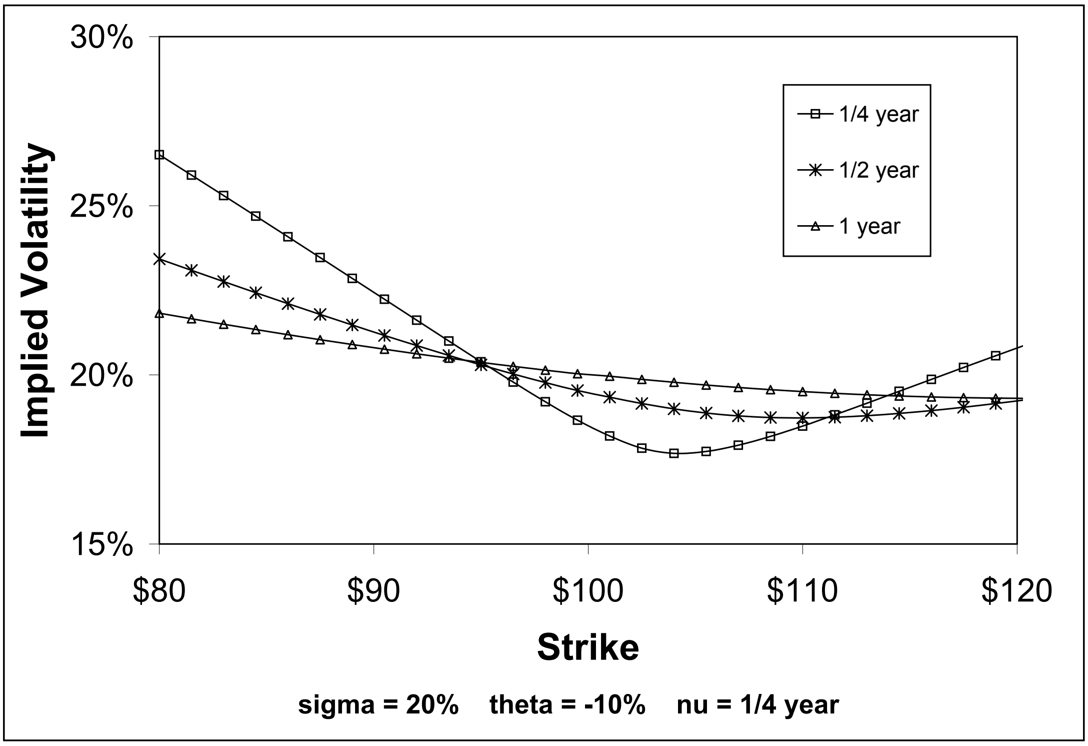
Table C.1

The break even participation rates under several market conditions with $\beta = 90\%$, $\gamma = 3\%$, $\theta = -20\%$ and $\nu = \frac{1}{2}$ year. The volatility parameter σ is chosen so that the at-the-money implied volatility of a call is equal to 20% and 30% in the left and right panels respectively.

δ	κ	$\nu = 1/4$ year			$\nu = 1/2$ year		
		$r = 4\%$	$r = 5\%$	$r = 6\%$	$r = 4\%$	$r = 5\%$	$r = 6\%$
	10%	0.26250	0.42033	0.72806	0.25823	0.39153	0.61977
1%	12%	0.25755	0.38117	0.53736	0.25472	0.36545	0.49186
	14%	0.25603	0.36727	0.48327	0.25362	0.35625	0.45629
	10%	0.27529	0.45029	0.81914	0.27088	0.41718	0.68379
2%	12%	0.26931	0.40346	0.57952	0.26661	0.38601	0.52640
	14%	0.27529	0.45029	0.81914	0.27088	0.41718	0.68379

Table C.2

The break-even participation rates of a capped Cliquet. The remaining model and contract parameters not shown in the table are kept constant at $\sigma = 20\%$, $\theta = -20\%$, $\gamma = 3\%$ and the reset period $\Delta t = 1$ year.



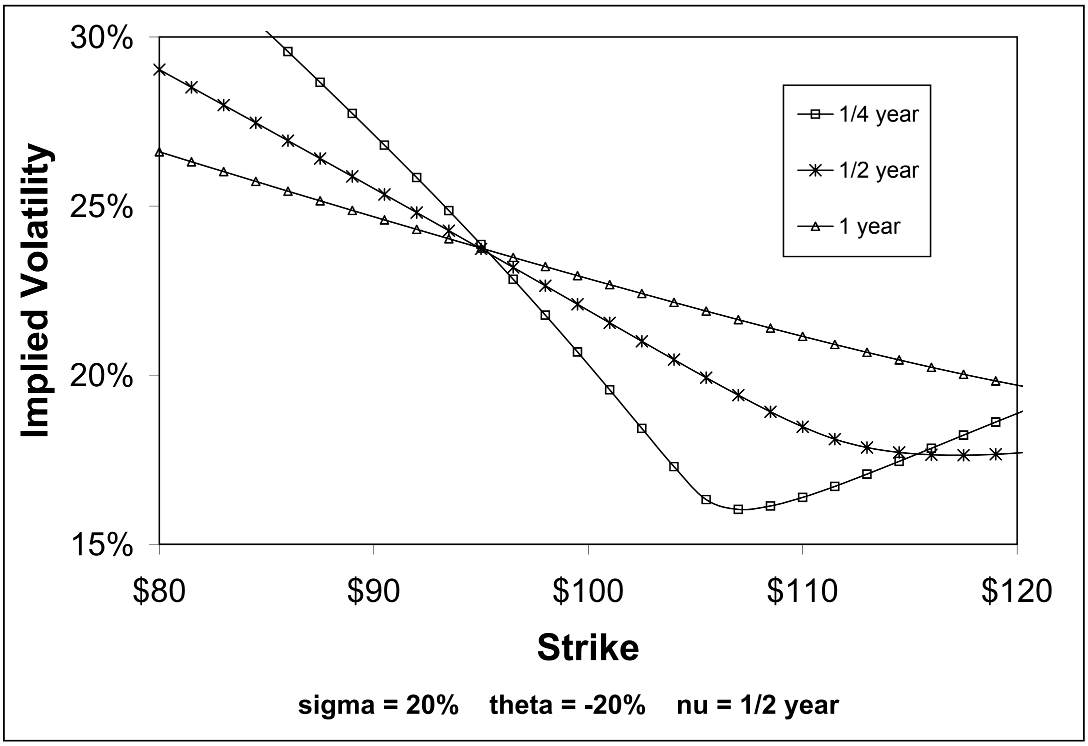
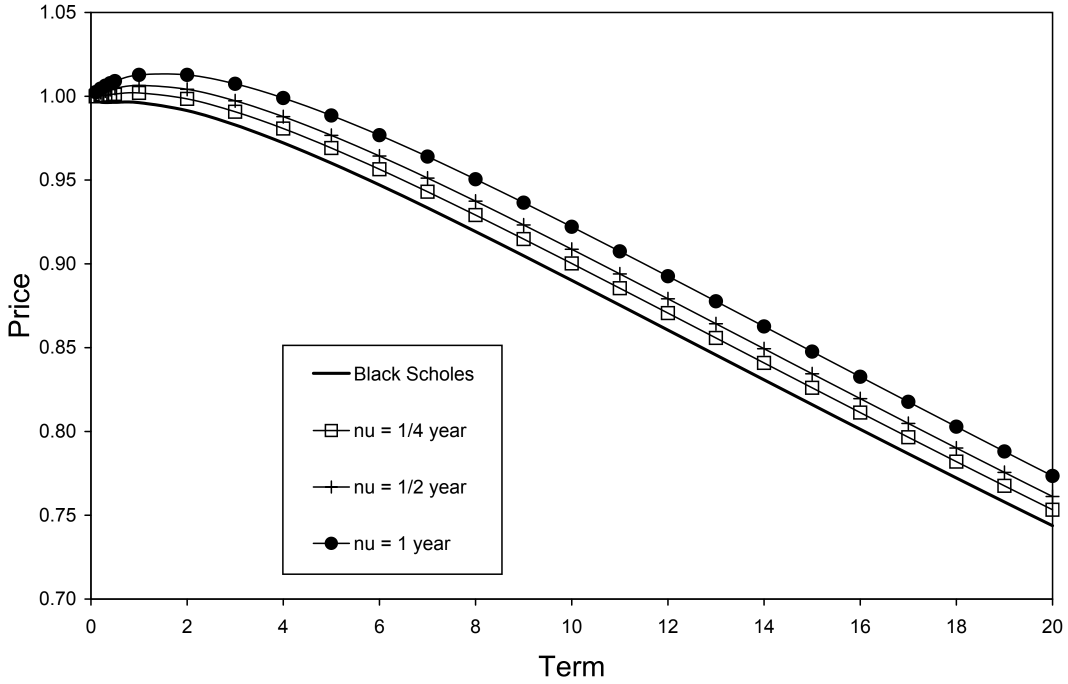


Fig. C.1. These diagrams depict implied volatility smiles using the VG model at maturities of $\frac{1}{4}$, $\frac{1}{2}$ and 1 year. The VG parameters are as shown. The spot price is \$100, the risk free interest rate is 5% and the dividend yield is set to zero for both diagrams.



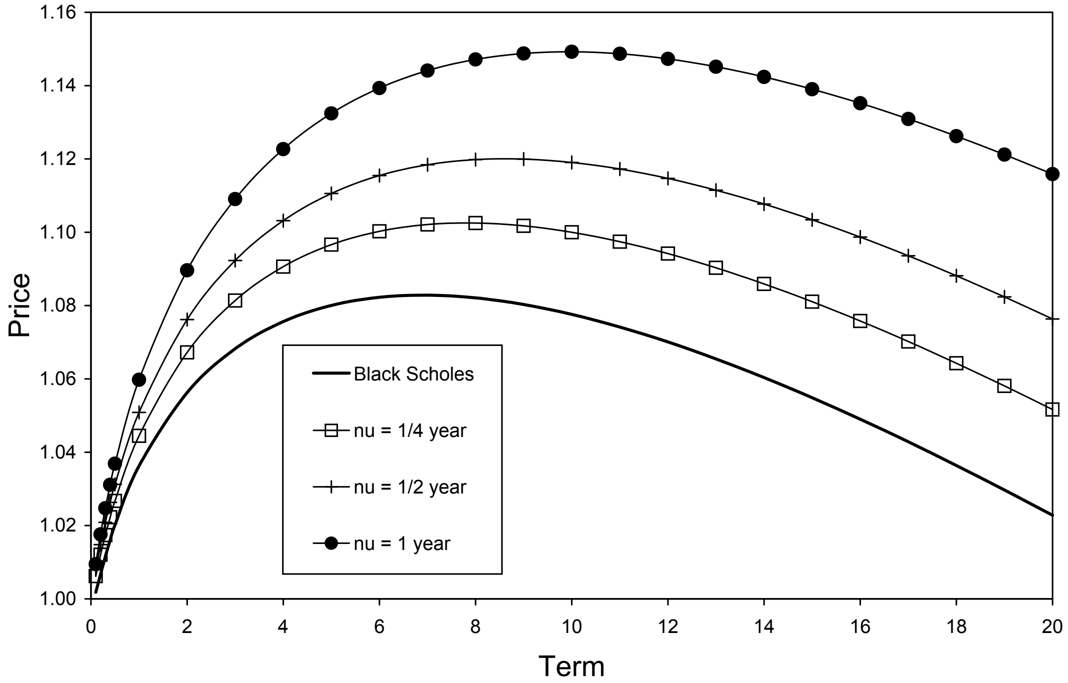


Fig. C.2. Point-to-Point premiums as a function of maturity for several choices of the excess kurtosis parameter. The model and contract parameters are $\sigma = 20\%$, $\theta = -20\%$, $\delta = 2\%$, $\beta = 0.9$, $\gamma = 3\%$ and $r = 5\%$. In the top panel the participation rate $\alpha = 0.8$ while $\alpha = 1.2$ in the bottom panel.

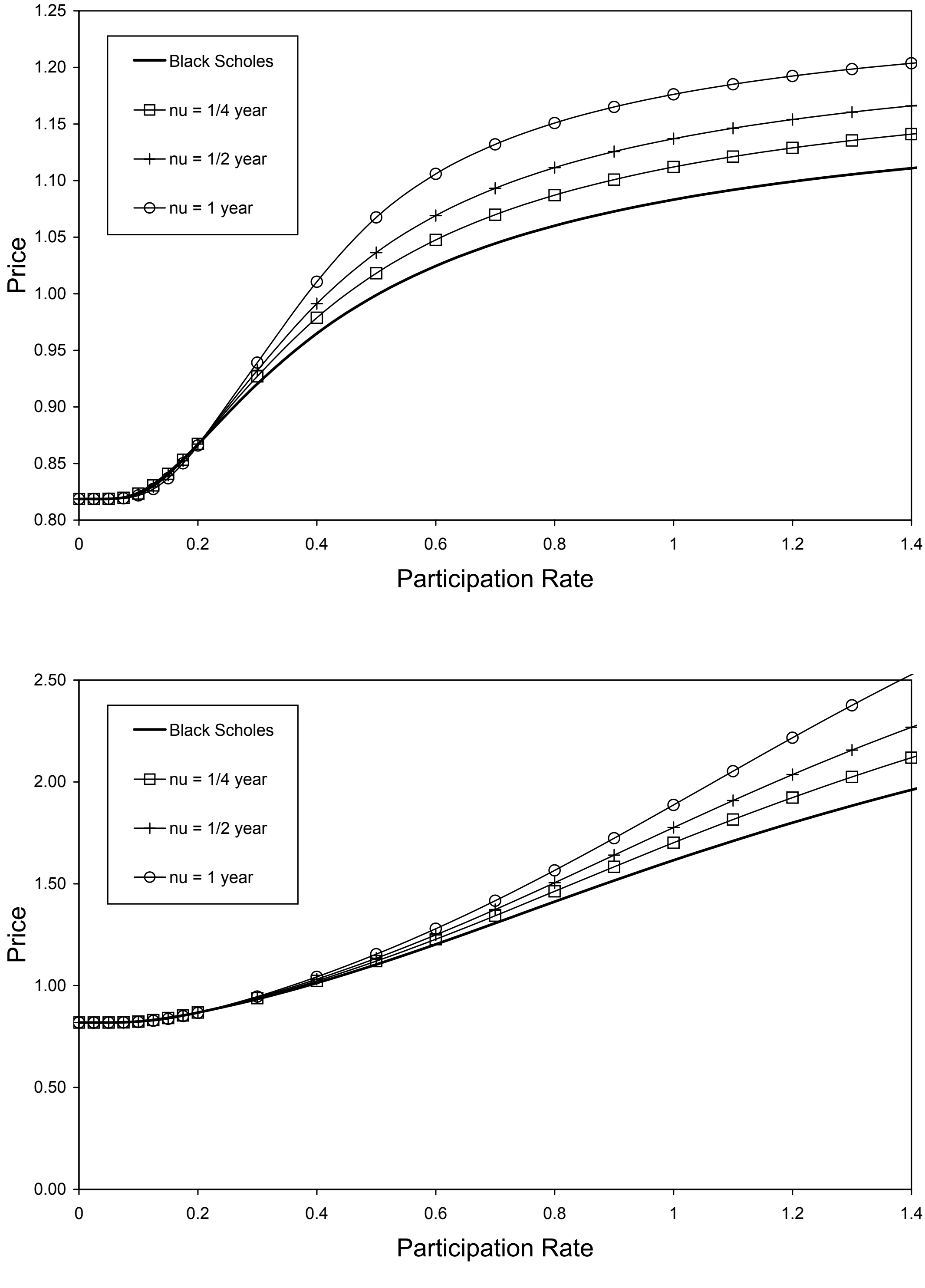
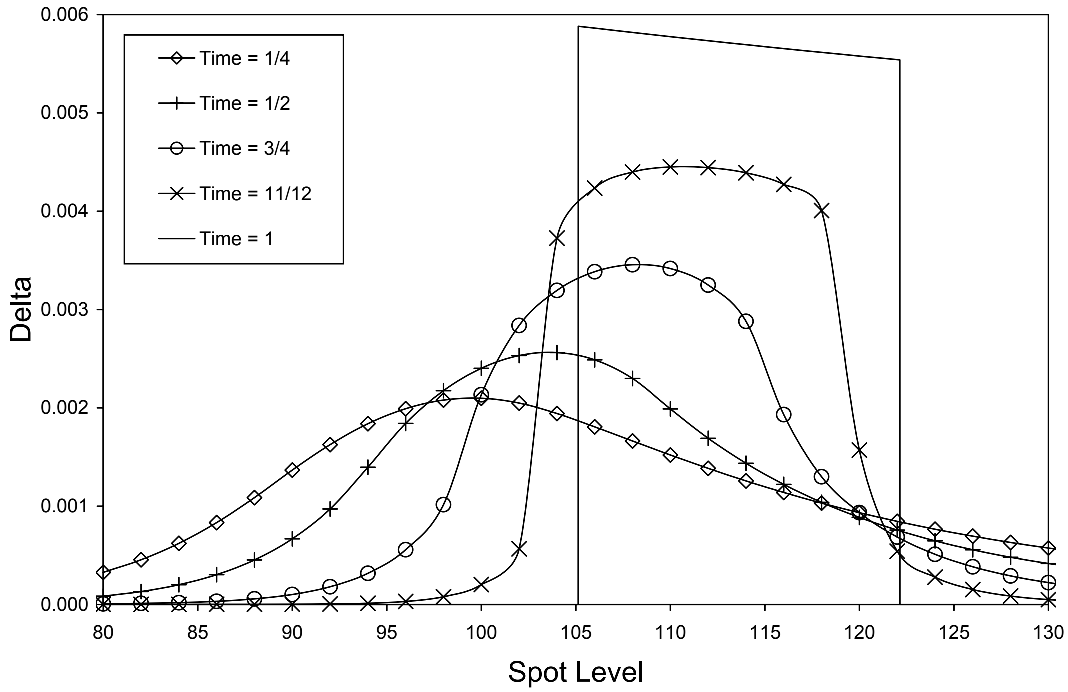


Fig. C.3. Cliquet premiums as a function of participation rate for several choices of the excess kurtosis parameter. The model and contract parameters are $n = 10$, $\Delta t = 1$ year, $\sigma = 20\%$, $\theta = -20\%$, $\delta = 2\%$, $\gamma = 3\%$, and $r = 5\%$. In the top panel $\kappa = 10\%$, in the bottom panel $\kappa = 30\%$.



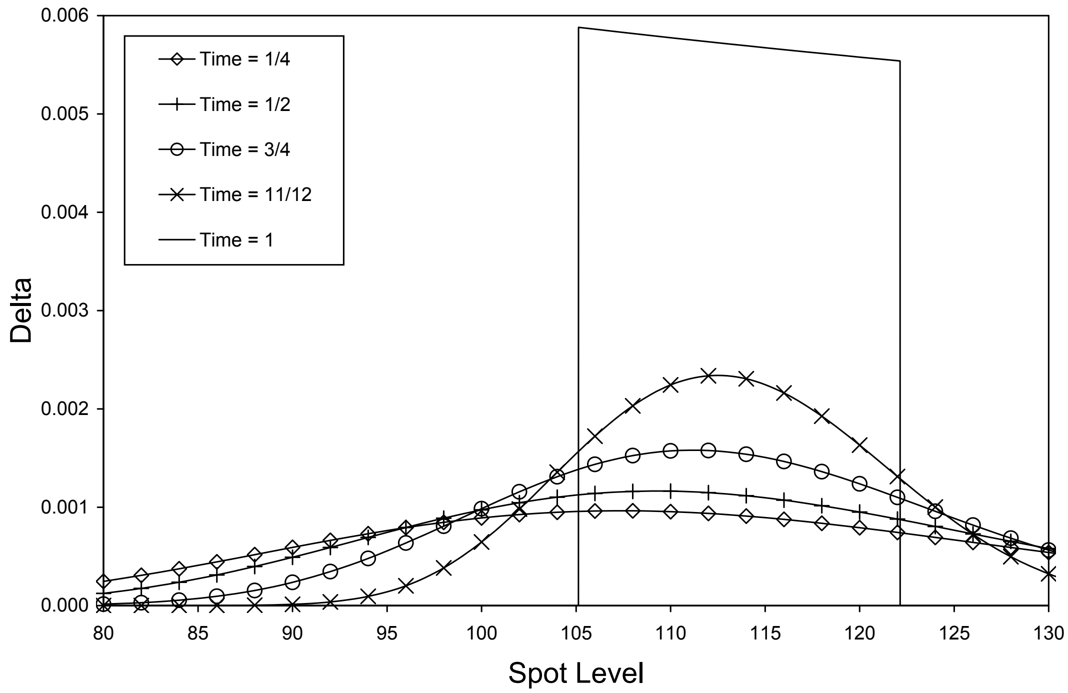


Fig. C.4. These diagrams depict the time sensitivity of the Delta of a Cliquet design EIA with 1 year maturity and one reset. The top panel shows the results for the VG model while the bottom contains the Black-Scholes results. The contract specifications are as follows: $\alpha = 60\%$, $\gamma = 3\%$, $\kappa = 12\%$ and the VG model parameters are: $\sigma = 20\%$, $\theta = -20\%$ and $\nu = \frac{1}{2}$. The interest rate is set at 5% and the dividend yield at 1%.

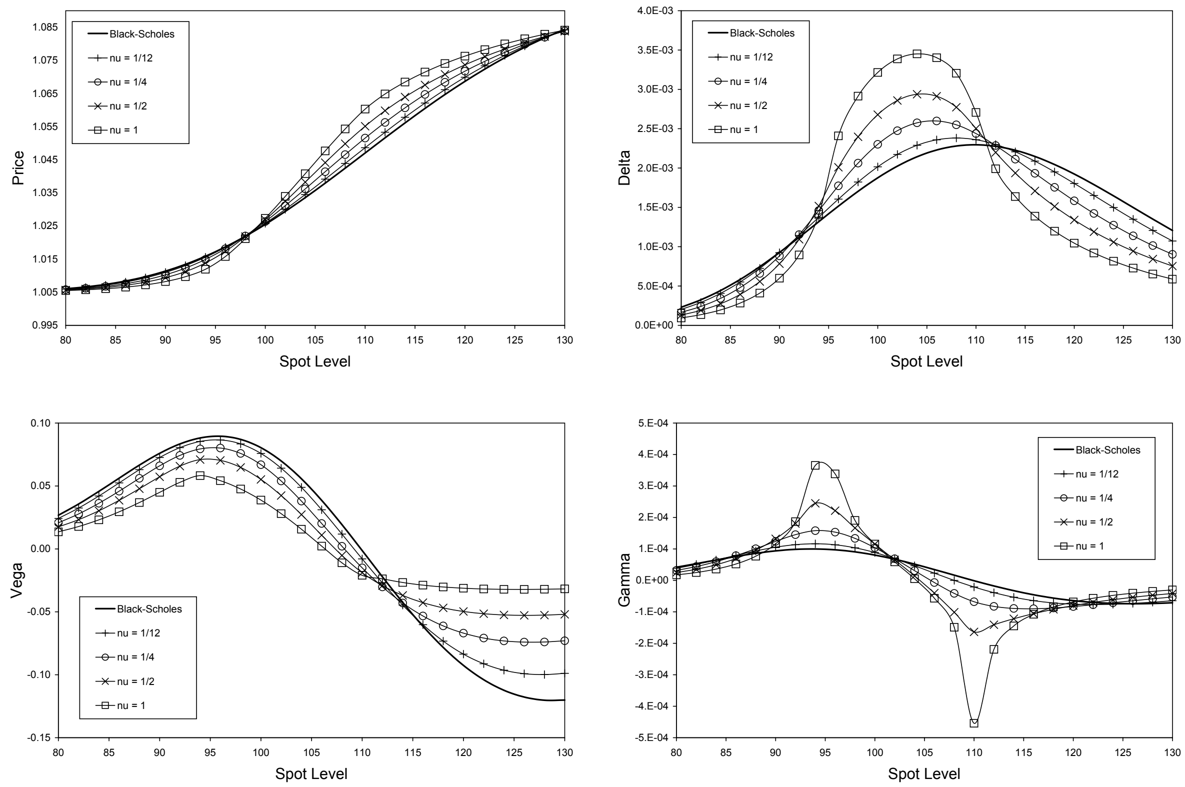


Fig. C.5. These diagrams depict the sensitivity of the Price, Delta, Gamma and Vega of a 1-year capped Cliquet, with a single reset, on the excess kurtosis parameter. The contract and model specifications are as follows: $\alpha = 60\%$, $\gamma = 3\%$, $\kappa = 12\%$, $\sigma = 20\%$, $\theta = -20\%$, $t = \frac{1}{2}$ year, $r = 5\%$ and $\delta = 1\%$.

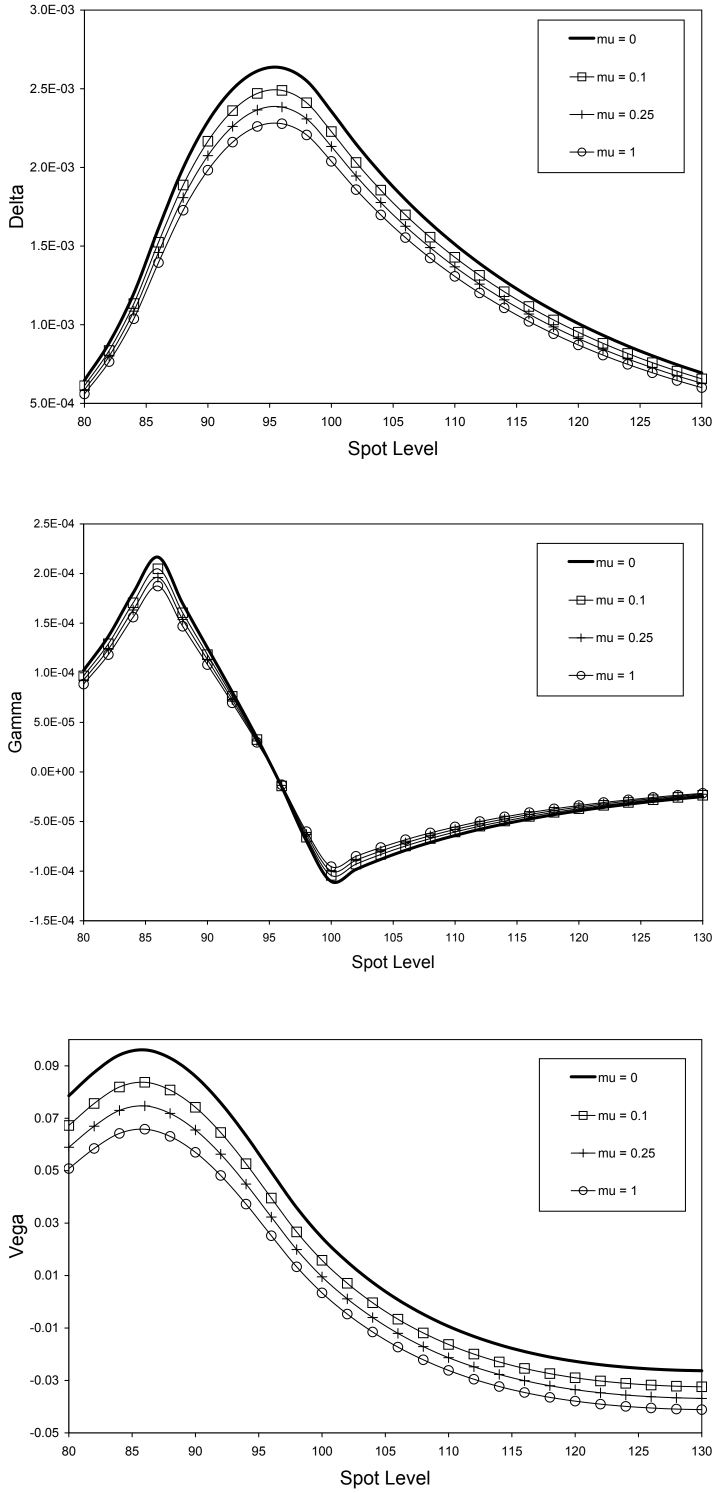


Fig. C.6. These diagrams depict the sensitivity of the Delta, Gamma and Vega of a 10-year capped Cliquet, with 10 resets, on the hazard rate μ. The contract and model specifications are as follows: $\alpha = 60\%$, $\gamma = 3\%$, $\kappa = 12\%$, $\sigma = 20\%$, $\theta = -20\%$, $\nu = 1$, $t = \frac{1}{2}$ year, $r = 5\%$ and $\delta = 1\%$.