MASTER OF SCIENCE BY DISSERTATION PROPOSAL: A COMPARISON OF NUMERICAL TECHNIQUES FOR AMERICAN OPTION PRICING

SEAN RANDELL (9907307X)

(Supervisors: Mr H. Hulley and Prof D.R. Taylor)

1. Introduction to American Option Pricing

A call (or put) option is a contract which affords its holder the right to buy (or sell) a specified underlying asset for a specified price (the strike price K) at or until (depending on the type of option) an expiry date T. European options give the holder only one chance to exercise — at the maturity of the option. An American option can be exercised at any time until maturity.

It is assumed that the price of the underlying asset (referred to from now on simply as the stock) follows the process

$$dS_t = \mu S_t \, dt + \sigma S_t \, dW_t, \tag{1}$$

where μ is the drift rate of the process, σ is its volatility, and W_t is a Wiener process (also known as a Brownian motion). This assumption implies a lognormal distribution of stock prices at any time t.

The payoff function for a European option at maturity is

$$h(S_T) = \begin{cases} \max(S_T - K, 0) & \text{for a call} \\ \max(K - S_T, 0) & \text{for a put.} \end{cases} \tag{2}$$

The payoff function for an American option (at any time up to maturity) is

$$H(S_t, t) = \begin{cases} \max(S_t - K, 0) & \text{for a call} \\ \max(K - S_t, 0) & \text{for a put,} \end{cases} \tag{3}$$

where $t \in [0, T]$. The pricing problem for a European option was solved in [4, 27]. The value of the European option $V(S, t)$ was found to satisfy the following partial differential equation (known as the Black-Scholes PDE):

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} = rV, \tag{4}$$

where r is the constant riskless rate of return available in the market. This equation, together with the initial condition

$$V(S, T) = h(S_T), \tag{5}$$

was solved by [4] to give a closed-form expression for $V(S, t)$.

It was demonstrated by [27] that the price of an American call on a stock whose price is determined by (1) is the same as the price of the corresponding European call. However, the price of an American put is the solution of (4) and (5), plus an additional free boundary condition. Since free boundary problems are notoriously difficult, there is no closed-form expression for the American put price.

No-arbitrage arguments imply that because it can be exercised at any time during its life, the value of an American option must always dominate its intrinsic value. (This is defined as the payoff from exercising the option immediately.) Thus the free boundary condition for an American put is

$$V(S_t, t) \geq \max(K - S_t, 0), \tag{6}$$

for all $t \in [0, T]$.

2. The Partial Differential Equation Approach to American Option Pricing

2.1. The Free Boundary Formulation of the American Put Price.

The American put price is formulated as the solution of a free boundary problem in [31]. The following constraints are imposed there:

- the option value must always exceed its payoff (3);
- the Black-Scholes PDE (4) becomes an inequality;
- the option price must be continuous in S; and
- the partial derivative $\frac{\partial V}{\partial S}$ must be continuous.

This implies the existence of a stock price $S_f(t)$, called the early exercise price, at which it is optimal to exercise. For an American put, if the stock price is below this price, its value is equal to its payoff $H(S_t, t)$; and if the stock price is greater than this price, its value is determined by the Black-Scholes PDE (4). There are thus two possibilities; if $0 < S_t \leq S_f(t)$

$$\begin{aligned} V(S_t, t) &= K - S_t \\ \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV &< 0 \\ \frac{\partial V}{\partial S} &= -1, \end{aligned}$$

and if $S_f(t) < S_t$

$$\begin{aligned} V(S_t, t) &> K - S_t \\ \frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV &= 0 \\ \lim_{S \to S_f(t)^+} \frac{\partial V}{\partial S} &= -1. \end{aligned}$$

The major problem in solving this problem is that the free boundary $S_f(t)$ is unknown.

2.2. The Variational Inequality Formulation of the American Put Price.

Referring to the two possibilities in Section 2.1, the following system of inequalities can be set up:

$$\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV \leq 0 \tag{7}$$

$$V(S_t, t) - H(S_t, t) \geq 0 \tag{8}$$

$$\left(\frac{\partial V}{\partial t} + rS\frac{\partial V}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} - rV\right)(V(S_t, t) - H(S_t, t)) = 0. \tag{9}$$

Equation (9) comes from the fact that if the stock price is below the free boundary, then (8) is an equality; and if the stock price is above the free boundary, then (7) is an equality. Therefore, regardless of the value of the stock price, the product of the expressions in (7) and (8) will always be zero.

The Black-Scholes equation was solved in [4] by first transforming it into the heat equation

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \tag{10}$$

together with a transformed initial condition. Using the same transformations, (7), (8) and (9) can be written as

$$\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \geq 0 \tag{11}$$

$$u(x,\tau) - g(x,\tau) \geq 0 \tag{12}$$

$$\left(\frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2}\right)(u(x,\tau) - g(x,\tau)) = 0, \tag{13}$$

where $g(x,\tau)$ is the transformed payoff function.

2.3. **The Linear Complementarity Formulation of the American Put Price.** A linear complementarity problem is described in [31] as a general system of the form

$$\begin{aligned} \mathcal{A}\mathcal{B} &= 0 \\ \mathcal{A} &\geq 0 \\ \mathcal{B} &\geq 0. \end{aligned}$$

Setting

$$\begin{aligned} \mathcal{A} &= \frac{\partial u}{\partial \tau} - \frac{\partial^2 u}{\partial x^2} \\ \mathcal{B} &= u(x,\tau) - g(x,\tau), \end{aligned}$$

yields the linear complementarity formulation of the American put pricing problem in terms of transformed variables. The finite difference methods discussed in the next section allow this problem to be broken up into a system of discretized linear complementarity problems.

2.4. **Finite Difference Methods.** Finite difference methods are used to approximate solutions of partial differential equations. The idea is to replace partial derivatives with discrete approximations derived from Taylor series expansions. The original PDE and boundary conditions are thus converted into a system of difference equations to be solved on a rectangular grid of points.

There are different finite difference techniques, based on different approximations of the partial derivatives. The explicit, fully implicit and Crank-Nicolson schemes are described in [31]. Another method, the Crandall-Douglas scheme, is described in [25].

The fully-implicit scheme, applied to the heat equation, gives the following approximation of the PDE:

$$\frac{u(x,\tau+\delta\tau)-u(x,\tau)}{\delta\tau}+O(\delta\tau)=$$
$$\frac{u(x+\delta x,\tau+\delta\tau)-2u(x,\tau+\delta\tau)+u(x-\delta x,\tau+\delta\tau)}{(\delta x)^2}+O((\delta x)^2).$$

Using the notation

$$u_n^m = u(n\delta x, m\delta\tau)$$

, where $n=-N^-,\dots,N^+$ and $m=0,\dots,M$, and ignoring error terms, gives

$$-\alpha u_{n-1}^{m+1}+(1+2\alpha)u_n^{m+1}-\alpha u_{n+1}^{m+1} \quad = \quad u_n^m,$$

where $\alpha=\delta\tau/(\delta x)^2$. For each value of m, a system of $(N^+ + N^-)-2$ equations must be solved. A number of methods are available to solve these difference equations. For example, [31] use a projected successive over-relaxation method.

Finite differences were first applied to American option pricing in [6, 7]. In [25], a special property of the American option pricing problem is exploited, resulting in a finite difference scheme where each temporal step only requires $O(n)$ operations.

3. The Monte Carlo Approach to American Option Pricing

3.1. **The Optimal Stopping Problem Formulation of the American Put Price.** [26] first expressed the price of an American option in terms of optimally stopping the stock price process to maximize the expected discounted payoff at the stopping time. This leads to the following expression for the price of an American put:

$$V(S_t,t)=\sup_{\tau}\mathbb{E}\left[e^{-r\tau}\max\left(K-S_\tau,0\right)\right], \tag{14}$$

where the supremum is calculated over all stopping times $\tau \le T$.

In the above, the expected value is computed with respect to the risk-neutral (or equivalent martingale) measure (see [15]). Under this measure S_t satisfies the stochastic differential equation

$$dS_t = rS_t\,dt+\sigma S_t\,d\tilde{W}_t, \tag{15}$$

where $\tilde{W}_t$ is a Brownian motion under the risk-neutral measure. Solving (15) gives the following expression for the price of the stock at time t:

$$S_t = S_0\exp\left(\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\tilde{W}_t\right). \tag{16}$$

This enables one to simulate stock prices by sampling the normal random variable $\tilde{W}_t$.

3.2. **Monte Carlo Valuation of the American Put.** A review of Monte Carlo methods in finance can be found in [5]. The steps for pricing a financial instrument using Monte Carlo simulation are

- simulate the sample paths of the underlying asset over the specified time period using (16);
- calculate the discounted cash flows along each sample path; and
- estimate the price by averaging the discounted cash flows.

An important property of Brownian motion is that its increments are normally distributed, with mean zero and variance equal to the time increment:

$$W_t - W_s \sim \mathcal{N}(0, t - s). \tag{17}$$

Simulation of a Brownian motion increment can therefore be achieved using

$$W_t - Ws \sim \epsilon\sqrt{t - s}, \tag{18}$$

where $\epsilon \sim \mathcal{N}(0, 1)$.

For European options, we only need to sample the stock price at maturity. Therefore the Monte Carlo estimate of a European put price is

$$\hat{v}(S_t, 0) = e^{-rT} \frac{1}{n} \sum_{i=1}^{n} \max \left[K - S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2 \right) T + \sigma\sqrt{T}\,\epsilon_i \right), 0 \right], \tag{19}$$

where $\epsilon_1, \ldots, \epsilon_n \sim \mathcal{N}(0, 1)$.

American options are more difficult to price using Monte Carlo simulation, since they can be exercised at any time before they expire. This is approximated by restricting exercise opportunities to a finite set of times $0 = t_0 < t_1 < \ldots, < t_m = T$. Discrete stock price paths are simulated for each sample $S_0^{(i)}, S_1^{(i)}, \ldots, S_m^{(i)},\ i = 1, \ldots, n$. The discretization of the problem leads to the price being underestimated but this bias decreases as the number of time points is increased.

The next problem is estimating the optimal exercise time of the option. A naive approach to estimating this is to pick the time along the discrete sample path which maximises the payoff and to discount its payoff appropriately:

$$\widehat{V}(S_t, 0) = \frac{1}{n} \sum_{i=1}^{n} \max_{j=1,\ldots,m} \left[e^{-rt_j} \max\left(K - S_0 \exp\left(\left(r - \frac{1}{2}\sigma^2 \right) t_j + \sigma\sqrt{t_j}\,\epsilon_j^i \right), 0 \right) \right]. \tag{20}$$

In reality, an exercise decision is made using only knowledge obtained up to the time of exercise. The naive method uses perfect foresight [5], since one is looking at each path over the entire life of the option and deciding where exercise should have occurred. This leads to overestimation of the price.

There have been a number of recent papers which attempt to estimate the American option price using Monte Carlo simulation. Generally, they attempt to estimate the expected return at exercise conditional only on known information using sophisticated algorithms. The different techniques which will be looked at are given in [30, 3, 9, 23, 29] [1].

4. Research Objectives

4.1. Numerical Analysis. In the analysis of numerical methods, the following are important:

- stability;
- convergence; and
- execution time.

I will implement finite difference and Monte Carlo techniques, and analyze these properties. The dissertation will provide a detailed catalogue of numerical methods available for valuing American options.

[1] This list might expand.

4.2. **Computing the Free Boundary.** The difficulty with the American put is that the free boundary is unknown. If this were not the case, standard finite difference techniques and simpler Monte Carlo techniques could have been applied. In [24] the free boundary for the American put on a dividend-paying stock is expressed implicitly as

$$\int_{S^*}^{S_0} S^{-(1/2\sigma^2)(2D_0-2r+3\sigma^2+\lambda(p))} F(S) dS$$
$$= \frac{1}{4} e^{pT} K^{-(1/2\sigma^2)(2D_0-2r-\sigma^2+\lambda(p))}$$
$$\times \left[\left(\left(\frac{2D_0 - 2r - \sigma^2 - \lambda(p)}{p + D_0} \right) 1 - \left(\frac{r}{D_0} \right)^{-(1/2\sigma^2)(2D_0-2r-\sigma^2+\lambda(p))} \right) \right.$$
$$\left. - \left(\left(\frac{2D_0 - 2r + \sigma^2 - \lambda(p)}{r + p} \right) 1 - \left(\frac{r}{D_0} \right)^{-(1/2\sigma^2)(2D_0-2r+\sigma^2+\lambda(p))} \right) \right],$$

where D_0 is a constant dividend yield, S^* is the free boundary for the corresponding perpetual put, and

$$F(S) = (K - S)e^{pT_f(S)} + [\sigma^2 S^2 - (r - D_0)S(K - S)]T'_f(S) - \frac{\sigma^2 S^2}{2}(K - S)T''_f(S)$$
$$\lambda(p) = \sqrt{4D_0^2 - 8D_0 r + 4D_0\sigma^2 + 4r^2 + 4\sigma^2 r + \sigma^4 + 8\sigma^2 p}.$$

The integral equation above is a nonlinear (Fredholm) equation, which must be solved for the location of the free boundary $T_f(S)$. It is not solved in [24], however.

The planned original contribution of my project will be to solve the above integral equations for the free boundary numerically. The application of simpler finite difference and Monte Carlo techniques can then be considered.

5. Proposed Structure of Dissertation

Introduction. An overview of American option pricing. I will formulate the price of an American put as the solution of an optimal stopping problem and a free boundary problem. I will also consider the associated formulations in terms of variational inequalities and as a linear complementarity problem.

Part 1. Monte Carlo Methods. This will contain an introduction to Monte Carlo methods in general, as well as the detailed analysis of the Monte Carlo algorithms presented in [30, 3, 9, 23, 29].

Part 2. Finite Difference Methods. This will contain an introduction to finite difference methods, as well as detailed analysis of the finite difference methods presented in [6, 7, 25, 31].

Part 3. Comparison of Methods. Here, after having implemented the numerical techniques for American put pricing in Matlab, I will analyze them with respect to the criteria in Section 4.1. My intention is to provide a thorough and up-to-date numerical comparison of what is available for American option pricing today.

Conclusion. A synopsis of the results from the numerical investigation in Part 3.

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