An Introduction to Stochastic Calculus (Hull ch.9)

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1. Markov Process
2. Wiener Processes

If z is a Wiener Process, then

(1) $\Delta z = \varepsilon_t \sqrt{\Delta t}$

where ε_t *is a random drawing from a standardized normal distribution, which means* ε_t *follows* $N(0,1)$.

(2) *The value of* Δz *for any two different short time intervals of* Δt *are independent.*

(3) *In any given interval of* T, *the increase in the value of a variable that follows a Wiener process is normally distributed with mean zero and a standard deviation of* T.

3. Generalized Wiener Processes

$$dx = a\,dt + b\,dz,$$

with an expected drift rate of a *and a variance of* b^2.

4. Ito Processes

A generalized Wiener process where the parameters a & b are functions of underlying assets x and time t. An Ito process can be written as:

$$dx = a(x,t)\,dt + b(x,t)\,dz$$

5. An single-variance diffusion process

$$dx(t) = \mu(x,t)\,dt + \sigma(x,t)\,d\omega(t)$$

where $d\omega$ *is a standard process, with* $d\omega(t) = \lim_{\Delta t \to 0} \Delta t^{1/2}\,\tilde{u}$;

$\tilde{u} \sim N(0,1)$, $f(u) = (2\pi)^{-1/2} \cdot e^{-u^2/2}$;

the property of $d\omega$ *follows* $E(d\omega) = 0$, $E[(d\omega)^2] = dt$.

Let $F(x,t)$ *be a twice differential, continuous function. By Ito's lemms,*

$$dF(x,t) = F_x\,dx + F_t\,dt + \tfrac{1}{2} F_{xx}\,(dx)^2,$$

where $dt^2 = dx \cdot dt = 0$, $d\omega^2 = dt$.

6. Ito's Lemma

Ito's Lemma is the Taylor's series expansion with the restricting in setting $(\Delta z)^2 = \Delta t$, and letting other higher order terms go to zero.

Let $f_t = f(S_t, t)$*, then the Taylor's series expansion equals*

$$\Delta f = f(S+\Delta S, t+\Delta t) - f(S_t, t)$$

$$= f_t \cdot \Delta t + f_s \cdot \Delta S + \frac{1}{2!}(f_{tt} \cdot \Delta t^2 + 2 \cdot f_{tS} \cdot \Delta t \Delta S + f_{SS} \cdot \Delta S^2)$$

$$+ \frac{1}{3!}(f_{ttt} \cdot \Delta t^3 + 4 \cdot f_{tts} \cdot \Delta t^2 \Delta S + 4 \cdot f_{tss} \cdot \Delta t \Delta S^2 + f_{SSS} \cdot \Delta S^3)$$

$$+ \frac{1}{4!}(\cdots)$$

Impose the limitations, then

$$\Delta f = f_t \cdot \Delta t + f_s \cdot \Delta S + \frac{1}{2!} f_{SS} \cdot \Delta S^2$$

7. Examples:

a. If $x(t)$ *is a certain process ,* $dx(t) = \mu(x,t)\,dt$*, then* $d\,F(x,t) = (\mu \cdot F_x + F_t)\,dt$.

b. If $\mu(x) = \mu; \sigma(x) = \sigma$ *, then x is said to be a Brownian motion with drift.*

$dw = (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma \cdot d\omega..x$

c. If $\mu(x) = \mu x; \sigma(x) = \sigma x$ *, then x is said to be a geometric Brownian motion. Defineing*

$w = \ln x$*, then*

$dw = (\mu - \frac{1}{2}\sigma^2)\,dt + \sigma \cdot d\omega.$

d. If $\mu(x) = k(\bar{x} - x); \sigma(x) = \sigma x$ *, then x is said to be an Ornstein−Unlenbeck Process. Defineing*

$w = (x - \bar{x}) \exp k(t - t_0)$*, then*

$$dw = \exp k(t-t_0)\,dx + \frac{1}{2} \cdot 0 \cdot dx^2 + k(\bar{x}-x) \exp k(t-t_0)\,dt$$

$$= \exp k(t-t_0)\,\sigma\,d\omega.$$

THE TERM STRUCTURE OF INTEREST RATES

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◎ Term structure of interest rate

◎ **Without Uncertainty / Discrete Time Approach**

(i) Expectations Hypothesis (預期假說)

Main Thoughts: Forward rates equal to the future spot rates. Therefore some propositions hold under expectations hypothesis, such as (a) the return on holding a long-term bond to maturity is equal to the expected return on repeated investment in a series of the short-term bonds. Or (b) the expected rate of return over the next holding period is the same for bonds of all maturities.

(ii)Liquidity Preference Hypothesis(流動性偏好假說):

Risk aversion will cause forward rate to be systematically greater than expected spot rates, usually by an amount increasing with maturity. In another word, term premium is increasing as bond maturity increased..

(iii)Market Segmentation Hypothesis(市場區隔假說):

Individuals have strong maturity preferences and bonds with different maturities trade in separate and distinct markets.

(iv)Preferred Habitat Theory(偏好習性理論):

The theory argues that the riskiness of a bond depends on the holding periods required by investors. A riskless strategy is to hold a discount bond with maturity equal to that of the holding period. Longer-term bonds carry price risk while shorter-term bonds carry reinvestment risk. Under the preferred habitat theory, investor will demand higher expected return on bonds with maturities shorter or longer than their stated holding periods. Term premium decided through investor’s investment horizon, and their preferred habitat.

◎ **With Uncertainty / Continuous Time Approach**

1. Traditional approach

(1) one-factor model (Vasicek (1977), Dothan (1978), Courtadon (1982), CIR (1985b))

(2) multi-factor model

Brennan & Schwartz (1978): uses instantaneous spot rate and yield of console bond as state variables.

CIR (1985b) uses any two unobservable independent state variables.

$$d\,r_1 = \alpha_1(\,\mu_1 - r_1\,)dt + \sigma_1\sqrt{r_1}\,dW_1$$
$$d\,r_2 = \alpha_2(\,\mu_2 - r_2\,)dt + \sigma_2\sqrt{r_2}\,dW_2$$
$$r = r_1 + r_2$$

Chen (1995): uses any two unobservable correlated state variables.

$$d\,r_1 = \alpha_1(\,\mu_1 - r_1\,)dt + \sigma_1\,dW_1$$
$$d\,r_2 = \alpha_2(\,\mu_2 - r_2\,)dt + \sigma_2\,dW_2$$
$$r = r_1 + r_2$$
$$E(\,dW_1 \cdot dW_2\,) = \rho\,dt$$

Closed form solution is found for multi-factor model only if state variables are independent among each other. Closed-form solution can also be found when the correlated state variables follow normal distribution.

2. No arbitrage approach (Arbitrage-free yield curve model of the term structure of interest rates; also called Time Dependent Model).
   (1) use discount bond price as state variable (Ho & Lee (1986), Hull & White (1990))
   (2) useg instantaneous forward rate as state variable (Heath, Jarrow & Morton (1992))
   (3) use short term rates as state variable (Black, Derman, & Toy (1990), Hull & White (1990b), Black & Karasinski (1991))

An Equilibrium Characterization of the Term Structure

Vasicek (1977)

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A. Notation and Assumptions

1. There exist a pure discount bond, $P(t,s),\ t \le s,\ P(s,s)=1$
2. The yield to maturity on bond can be written as

   $$R(t,T)=-\frac{1}{T}\cdot\log P(t,s)\ ,\ where\ s=t+T$$

3. The forward rate will be defined by $F(t,s)=-\frac{\partial}{\partial s}\cdot\left[(s-t)\cdot R(t,s-t)\right]$
4. The spot rate is defined as $r(t)=R(t,0)=\lim_{T\to 0}R(t,T)$.
5. Assumptions:

   (A1)The spot rate is a continuous function of time, and it is assumed that $r(t)$ follows Markov process.

   (A2)The price of bond determined by the spot rate process. Which means $P(t,s)=P(t,s,r(t))$

   (A3)Market is efficient. This assumption implies that investors have homogeneous expectations, and that no profitable riskless arbitrage is possible.

B. The term Structure Equation

Bond price P is the function of time t and instantaneous interest rate r(t).The total difference of bond P can be written as

$$dP = P_t\cdot dt + P_r\cdot dr + \frac{1}{2!}(P_{tt}\cdot(dt)^2 + P_{t\cdot r}\cdot(dt)(dr) + P_{r\cdot r}\cdot(dr)^2 + o(dt))$$

$$= P_t\cdot dt + P_r\cdot dr + \frac{1}{2!}\cdot P_{r\cdot r}\cdot(dr)^2$$

Since $dr = f\cdot dt + \rho\cdot dz$

$$dP = P\cdot\left[\frac{P_t + f\cdot P_r + \frac{1}{2}\cdot\rho^2\cdot P_{rr}}{P}\right]\cdot dt + P\cdot\left[\frac{\rho\cdot P_r}{P}\right]\cdot dz$$

$$= P\cdot\mu\cdot dt + P\cdot\sigma\cdot dz$$

Define $d\tilde{z} = dz + q\cdot dt$

then $dP = P\cdot\mu\cdot dt + P\cdot\sigma\cdot dz$

$$= P\cdot\mu\cdot dt + P\cdot\sigma\cdot(d\tilde{z} - q\cdot dt)$$

$$= P\cdot\left[(\mu-\sigma\cdot q)dt + \sigma\cdot d\tilde{z}\right]$$

$$= P\cdot\left[r\cdot dt + \sigma\cdot d\tilde{z}\right]$$

$$Therefore,\ q = \frac{\mu - r}{\sigma},\ q \text{ means the market price of risk}$$

Since market is complete, there exists no arbitrage opportunity in markets,

$$dP = P\cdot\left[\frac{P_t + f\cdot P_r + \frac{1}{2}\cdot\rho^2\cdot P_{rr}}{P}\right]\cdot dt + P\cdot\left[\frac{\rho\cdot P_r}{P}\right]\cdot dz$$

$$= P\cdot\left[\frac{P_t + f\cdot P_r + \frac{1}{2}\cdot\rho^2\cdot P_{rr} + \rho\cdot q\cdot P_t}{P}\right]\cdot dt + P\cdot\left[\frac{\rho\cdot P_r}{P}\right]\cdot d\tilde{z}$$

$$= r\cdot dt + \rho\cdot P_r\cdot d\tilde{z}$$

we have

$$P_t + f\cdot P_r + \tfrac{1}{2}\cdot\rho^2\cdot P_{rr} + \rho\cdot q\cdot P_t = r\cdot P$$

$$P_t + (f + \rho\cdot q)\cdot P_r + \tfrac{1}{2}\cdot\rho^2\cdot P_{rr} - r\cdot P = 0$$

C. A Special Case

Two more assumptions are given in order to derive the bond price, which states that the market price of risk is a constant and that the spot rate $r(t)$ follows the Ornstein-Uhlenbeck (mean-reverting Wiener) process, O-U with $\alpha > 0$ is called the elastic randon walk, which is a Markov process with normally distributed increments.

$$dr = \alpha(\gamma - r)dt + \rho\cdot dz$$

where $\alpha > 0$,

$$E_t\, r(s) = \gamma + (r(t) - \gamma)\cdot e^{-\alpha(s-t)}$$

$$Var_t\, r(s) = \frac{\rho^2}{2\cdot\alpha}\cdot(1 - e^{-2\alpha(s-t)})$$

Under the above assumptions, the solution to the term structure equation is

$$P(t,s,r)$$

$$= exp\left[-\frac{(1-e^{-\alpha(s-t)})}{\alpha}\cdot r(t) - \left[(s - t - \frac{(1-e^{-\alpha(s-t)})}{\alpha})\cdot(\gamma - \frac{\rho\cdot q}{\alpha} - \frac{\frac{1}{2}\cdot\rho^2}{\alpha^2}) + \frac{\rho^2}{4\cdot\alpha^3}(1-e^{-\alpha(s-t)})^2\right]\right]$$

$$= exp\{-F(t,s)\cdot r(t) - G(t,s)\}$$

where

$$F(t,s) = \frac{(1-e^{-\alpha(s-t)})}{\alpha}$$

$$G(t,s) = \left[(s - t - F(t,s))\cdot(\gamma - \frac{\rho\cdot q}{\alpha} - \frac{\frac{1}{2}\cdot\rho^2}{\alpha^2}) + \frac{\rho^2}{4\cdot\alpha}F(t,s)^2\right]$$

D: The disadvantage of Vasicek (1977) model:

Under certain conditions (constant market price of risk (q), the instantaneous interest rate follows Ornstein-Uhlenbeck stochastic process), the closed form solution for bond price is presented.

1. Constant market price of risk implies bond pricing model is irrelevant to individual's risk attitude, or the utility function of individuals must be CRRA utility form.
2. Under O-U stochastic process, the instantaneous spot rate follows normal distribution, which means the short rate could be negative with the probability of 50%.

In order to avoid the negative interest rate problems, some researches are represented. The general model is given by

$$dr = k(\mu - r)dt + \sigma r^{\gamma} dw$$

When $\gamma = 1$, proportional mean-reverting model is introduced by .

$$dr = k(\mu - r)dt + \sigma r^{0.5} dw$$

When $\gamma = 0.5$, the model is termed the square root model, which is introduced by Cox, Ingersoll and Ross (1985b).

$$dr = k(\mu - r)dt + \sigma r^{1} dw$$

On the other hand, Dothan (1978) also defined a geometric wiener process, to avoid the negative interest rate problem. His model is given by

$$dr = \sigma r dW$$

In such case, interest rate process is purely a random walk, the mean-reversion property no longer existed.

A THEORY OF THE TERM STRUCTURE OF INTEREST RATES

Cox, Ingersoll, and Ross (1985b)

Wang, Yijen

2. Introduction

   In this general equilibrium term structure of interest rates model, anticipations, risk aversion, investment alternatives, and preferences about the timing of consumption all play a role in determining bond prices.

3. The Underlying Equilibrium Model

   From CIR (1985a), there are some assumptions and specialize to be noted.

   (1) Production Possibility

   $$d\eta(t) = I_\eta \cdot \alpha(Y,t)dt + I_\eta \cdot G(Y,t)dw(t), \eta : investments$$

   (2) Technology $dY(t) = \mu(Y,t)dt + S(Y,t)dw(t),$

   (3) Free Entry

   (4) Funding Market Exist

   (5) Contingent Claim Market Exist $dF^i = (F^i \cdot \beta_i - \delta_i)dt + F^i h_i \, dw(t)$

   (6) No Transaction cost

   (7) Individuals seeks to

   $$Max \quad E\int_t^{t'} U(C(s), Y(s), s)ds$$

   $$s.t. \quad dW = \left[\sum_{i=1}^{n} a_i \cdot W(\alpha_i - r) + \sum_{i=1}^{n} b_i \cdot W(\beta_i - r) + r \cdot W - C\right]dt$$

   $$+\sum_{i=1}^{n} a_i \cdot W(\sum_{j=1}^{n+k} g_{ij}\, dw_j) + \sum_{i=1}^{n} b_i \cdot W(\sum_{j=1}^{n+k} h_{ij}\, dw_j)$$

   $$= W\mu(W)dt + W \cdot \sum_{j=1}^{n+k} q_j \, dw_j$$

   (8) The equilibrium interest rate

   $$r(W,Y,t) = a^{*\prime} a - \left(\frac{-J_{WW}}{J_W}\right)\cdot\left(\frac{Var(W)}{J_W}\right) - \sum_{i=1}^{k}\left(\frac{-J_{WW}}{J_W}\right)\cdot\left(\frac{Cov(W,Y)}{J_W}\right)$$

   (9) Specializing the preference structure of constant relative risk aversion utility function, which can be expressed by

$$U(C(s),s)=e^{-\rho s}\left[\frac{C(s)-1}{\gamma}\right]$$

under this special utility form, two important simplifications can be found,

(a) the coefficient of relative risk aversion of the indirect utility function is constant, independent of both wealth and the state variables. Such as, $\frac{-W\cdot J_{WW}}{J_W}=1-\gamma$

(b) the elasticity of the marginal utility of wealth with respect to each of the state variable does not depend on wealth. Such as, $\frac{-J_{WY}}{J_W}=\frac{-f_Y}{f}$

(c) the risk premium depends only on Y.
$\phi_Y=(1-\gamma)a^{*\prime}GS'+(f_Y/f)SS'$

(d) equilibrium interest rate also depends only on Y.

(10) $\gamma=0$, then $f(Y,t)=\frac{1-e^{-\rho(t'-t)}}{\rho}$, the amount of investments can be determined, $a^*=(GG')^{-1}a+\left(\frac{1-1'(GG')^{-1}a}{1'(GG')^{-1}1}\right)(GG')^{-1}1$

(11) The price of contingent claims is determined by following partial differential equation.
$\frac{1}{2}tr(SS'F_{rr})+[\mu'-a^{*\prime}GS']F_Y+\delta-rF=0$

4. A single factor model of the term structure
(A1) Y is described by a single state variable.
(A2) The development of Y can be depressed as

$$dY(t)=[\xi Y+\zeta]dt+\upsilon\sqrt{Y}\,dW(t)$$

(A3) Given $a\equiv\hat{\alpha}Y, GG'\equiv\Omega Y, GS'\equiv\sum Y$, the interest rate follows a diffusion process with $drift\ r=\kappa(\theta-r), var\,r=\sigma^2 r$. The interest rate dynamics can then be expressed as

$$dr=\kappa(\theta-r)dt+\sigma\sqrt{r}\,dz_1.$$

The interest rate behavior has the following properties: (i) Negative interest rates are precluded. (ii) If the interest rate reaches zero, it can subsequently become positive. (iii) The absolute variance of the interest rate increases when the interest rate itself increases. (iv) There is a steady state distributions for the interest rate.

The probability density of the interest rate at time as is given by :

$$f(r(s),s;r(t),t)=c\cdot e^{-u-\upsilon}\left(\frac{\upsilon}{u}\right)^{q/2} I_q(2(u\upsilon)^{1/2}),$$

$$\begin{aligned} where \quad & c\equiv\frac{2\kappa}{\sigma^2(1-e^{-\kappa(t'-t)})}, \\ & u\equiv c\,r(t)e^{-\kappa(t'-t)}, \\ & \upsilon\equiv c\,r(s), \\ & q\equiv\frac{2\kappa\theta}{\sigma^2}-1, \\ & I_q(\cdot)\ \text{is the modified Bessel function of the first kind of order } q. \end{aligned}$$

The distribution function of interest rate follows non-central chi-square distribution.

5. Since bond price P is a function of instantaneous interest rate r(t) and the time t, apply Ito's Lemma, the total differenced of bond price P can be expressed as :

$$dP=P_t\cdot dt+P_r\cdot dr+\frac{1}{2!}\cdot P_{r\cdot r}\cdot(dr)^2$$

$$Since\ dr=k(\theta-r)\cdot dt+\sigma\sqrt{r}\cdot dz_1$$

$$\begin{aligned} dP &= P\cdot\left[\frac{P_t+k(\theta-r)\cdot P_r+\frac{1}{2}\cdot\sigma^2\cdot r\cdot P_{rr}}{P}\right]\cdot dt+P\cdot\left[\frac{\sigma\cdot\sqrt{r}\cdot P_r}{P}\right]\cdot dz_1 \\ &= P\cdot\mu_P\cdot dt+P\cdot\sigma_P\cdot dz \end{aligned}$$

$$where \quad \mu_P=\frac{P_t+k(\theta-r)\cdot P_r+\frac{1}{2}\cdot\sigma^2\cdot r\cdot P_{rr}}{P}$$

$$\sigma_P=\frac{\sigma\cdot\sqrt{r}\cdot P_r}{P}$$

$$Define\ d\tilde{z}_1=dz_1+\frac{\lambda\cdot\sqrt{r}}{\sigma}\cdot dt$$

$$\begin{aligned} then\ dP &= (P_t+k(\theta-r)\cdot P_r+\tfrac{1}{2}\cdot\sigma^2\cdot r\cdot P_{rr})dt+\sigma\cdot\sqrt{r}\cdot P_r\cdot(d\tilde{z}_1-\frac{\lambda\cdot\sqrt{r}}{\sigma}\cdot dt) \\ &= (P_t+k(\theta-r)\cdot P_r+\tfrac{1}{2}\cdot\sigma^2\cdot r\cdot P_{rr}-\lambda\cdot r\cdot P_r)dt+\sigma\cdot\sqrt{r}\cdot P_r\ d\tilde{z}_1 \\ &= rP\,dt+\sigma\cdot\sqrt{r}\cdot P_r\ d\tilde{z}_1 \end{aligned}$$

we have ,

$$P_t + k(\theta - r)\cdot P_r + \tfrac{1}{2}\cdot\sigma^2\cdot r\cdot P_{rr} - \lambda\cdot r\cdot P_r = rP$$

$$P_t + k(\theta - r)\cdot P_r + \tfrac{1}{2}\cdot\sigma^2\cdot r\cdot P_{rr} - \lambda\cdot r\cdot P_r - rP = 0$$

Since $\mu_P = \dfrac{P_t + k(\theta - r)\cdot P_r + \frac{1}{2}\cdot\sigma^2\cdot r\cdot P_{rr}}{P}$, $P_t + k(\theta - r)\cdot P_r + \tfrac{1}{2}\cdot\sigma^2\cdot r\cdot P_{rr} = \mu_P\cdot P$

therefore,

$$\mu_P\cdot P - \lambda\cdot r\cdot P_r = rP$$

$$\mu_P = r + \frac{\lambda\cdot r\cdot P_r}{P}$$

Bond prices is therefore represented as :

$P(r,tT) = A(t,T)e^{-B(t,T)r}$,

where

$$A(t,T) \equiv \left[\frac{2\gamma e^{(\kappa+\lambda+\gamma)(T-t)/2}}{(\kappa+\lambda+\gamma)\cdot(e^{r(T-t)}-1)+2\gamma}\right]^{2\kappa\theta/\sigma^2},$$

$$B(t,T) \equiv \frac{2\gamma(e^{r(T-t)}-1)}{(\kappa+\lambda+\gamma)\cdot(e^{r(T-t)}-1)+2\gamma},$$

$$\gamma \equiv ((\kappa+\lambda)^2 + 2\sigma^2)^{1/2}$$

The property of bond prices:

(1) The bond price is a decreasing convex function of the interest rate.

(2) and an increasing (decreasing) function of time (maturity).

(3) The bond price is a decreasing convex function of the mean interest rate level θ, and an increasing concave function of the speed of adjustment parameter κ if $r > \theta$.

(4) Bond prices are an increasing concave function of the market risk parameter λ.

(5) The bond price is an increasing concave function of the interest rate variance σ^2.

◎ The limitation on CIR (1985b) model:

1. The yield to maturity of different maturity bonds is perfectly correlated.
2. Model yield curve cannot precisely describe the true yield curve.