

Stochastic Volatility Model

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Summary

The stochastic volatility model in *Reditus* allows users to impose an upper and/or lower barriers on the asset, it also allows upper/lower barriers on volatility. Users can specify parameters for the model in term-structure form or as constants, i.e. interest rate, volatility, correlation, kappa, theta, lambda can all be imported in term-structure form or through a data-feed link. This document outlines the implementation of the stochastic volatility model in the *Reditus* finite-element engine, for details on the implementation in the *Reditus* Monte-Carlo engine, please refer to a separate document on *Reditus* Monte-Carlo engine.

The Model

The stochastic volatility model can be formulated as such [1]: a single asset S that follows a log-normal Brownian process:

$$dS = \mu S dt + \sqrt{v} S dZ_1$$

and the variance v evolves according to a mean-reversion stochastic process:

$$dv = \kappa(\theta - v)dt + \sigma\sqrt{v}dZ_2$$

here, μ is the drift of a single asset S , κ is the rate of mean-reversion for variance v , and θ is the long term mean level of the variance, σ is the volatility of the variance v . t is the current time, Z_1 and Z_2 are two Wiener processes which have a correlation coefficient ρ .

Similar governing stochastic equations can be found in references such as [2].

Contingent Claims:

The value V of any contingent claim that is dependent on the asset S now is not only a function of S and time t , it is also a function of the variance v : i.e. $V = V(S, v, t)$. As there are two independent random processes, we need to use asset S and another contingent claim to hedge away the stochastic components. In a procedure similar to deriving equations for bonds, we arrive at the partial-differential equation (PDE) governing the option price $V(S, v, t)$:

$$\frac{\partial V}{\partial t} + \frac{vS}{2} \frac{\partial^2 V}{\partial S^2} + \rho\sigma vS \frac{\partial^2 V}{\partial S \partial v} + \frac{\sigma^2 v}{2} \frac{\partial^2 V}{\partial v^2} + rS \frac{\partial V}{\partial S} + (\kappa(\theta - v) - \lambda v) \frac{\partial V}{\partial v} - rV = 0$$

here λ is the market price of risk. This equation has two independent variables: S and v . There are various ways of solving such equations, in the *Reditus* implementation, we will solve the PDE in the same way as for a typical two-factor model.

Implementation in *Reditus*

In the current *Reditus* set-up, we use S as the first dimension (first factor or the X -coordinate), and the variance v as the second dimension (second factor, or the Y -coordinate). Barriers can be enforced for both the asset S and variance v . In the current version of *Reditus*, only one upper barrier and one lower barrier can be specified for the asset S and variance v . The barriers can be switched on or off during the tenor of the option. However, the barrier levels are fixed from onset.

The scale of asset prices varies in quite different magnitude of numerical values, the variance is however limited to the scale of $[0,1]$. The scaling of S or variance v is extremely important to maintaining high accuracy in pricing contingent claims numerically. *Reditus* provides the facility for users to freely choose or impose scaling rules for either the asset S or variance v . In the current implementation, we keep the absolute value of variance along the Y -coordinate, and scale asset S to make the o - XY computational domain into a square shape. In doing so, we can maintain similar accuracy level for both the asset S and variance v . Of course, users can freely and easily change the scaling of asset S according to their own needs or belief. Users can set up their scaling rules or a formula in the script file *stochastic.prm* (here the file extension ****.prm* stands for the parameter file of a model named as *****) where the cut-off points for both the asset and variance are also set. Alternatively, users can simply go to the *System* page of the Excel interface to change the specific cell for scaling or infinity cutting-off points. (However, if the parameters are only changed in the *System* page, the parameter setting is only used for the current computation, they will not be stored when the same model is invoked next time. To save for future use, users should change the scaling and infinity cutting-off points in the parameter file *stochastic.prm*, or their own ****.prm*).

Pay-Off Functions:

Currently, we have the simple *call* and *put* payoff functions written in the script file *stochastic.prm* and they are the default. However, users can easily change the default pay-off functions into their own specific payoffs, such as digital payoffs, or as a formula dependent on the asset price. Users can change the payoff function by going to the *System* page to change the payoff function cell, or directly change the payoff function in the parameter script file: *stochastic.prm*, which will be discussed later in the document.

Input Market Data in *Reditus*

Reditus accepts all the parameters of the stochastic volatility model as constant or in term structure form. The following parameter names in *Reditus* correspond to those parameters described at the beginning of this document for the stochastic model:

<i>RiskFree</i> =	$r(t)$
<i>Volatility</i> =	$\sigma(t)$
<i>Correlation</i> =	$\rho(t)$
<i>kappa</i> =	$\kappa(t)$
<i>theta</i> =	$\theta(t)$
<i>lambda</i> =	$\lambda(t)$

If users do not click the *Term Structure* button, all the parameters for the volatility model will be constant in time, they will appear in the *Parameters* panel on the *Main Page* of the Excel front-end. Users can simply specify a value for each of the parameters as constant in time, as shown here:

Parameters	
RiskFree	0.0
Volatility	0.01
Correlation	0.5
kappa	2.0
theta	0.01
lambda	0.0

Barriers		Asset S:		Variance nu:	
Start	End	Lower Barrier	Upper Barrier	Lower Barrier	Upper Barrier

If the *Term Structure* button is pushed on the Excel interface, users should go to the extra *MarketData* sheet of this Excel front-end to supply all the parameters in term-structure format as shown here:

Date	RiskFree	Volatility	Correlation	kappa	theta	lambda	
0	23/01/2001	0.1	0.2	0.5	2	0.01	0
42	06/03/2001	0.12	0.25	0.56	3	0.05	0
48	12/03/01	0.12	0.3	0.56	3	0.06	0
101	4/05/01	0.13	0.3	0.57	3	0.07	0
196	7/08/01	0.15	0.4	0.57	3	0.07	0

here, *Date* stores the different dates at which the market parameters are supplied. The first column on the left of the *Date* array is automatically calculated by the Excel front-end, users do not have to input data in this column.

Test Case:

The first test example is a one-year option with the following constant parameters:

T	Strike	spot	σ	θ	κ	λ	ρ	r
1 year	10	10	0.3	0.04	2.0	0.0	0.5	0.1

This set up can be compared with a one-year option with a constant volatility of 0.2 , $r = 0.1$, $strike = 10$, $spot = 10$ in a standard Black-Scholes framework as shown in the following table:

Analytic Solution:

B&S Formula:	Call	Put
Price	1.3270	0.3753
Delta	0.7257	-0.2743
Gamma	0.1666	0.1666
Theta	-0.9263	-0.0214
Vega	3.3322	3.3322

It is interesting to compare the price and greeks from the above standard Black-Scholes framework with those of the stochastic volatility model shown here in the following two tables:

Reditus Stochastic Volatility Model

Call Option:	<i>Finite-Element engine</i>	<i>Monte-Carlo engine</i>
Price	1.2787	1.2909
Delta	0.706076	
Gamma	0.209249	
Theta	-0.904222	
Vega	1.47389	

here, for the Monte-Carlo price, the corresponding implied volatility is 0.18813

Reditus Stochastic Volatility Model

Put Option:	<i>Finite-Element engine</i>	<i>Monte-Carlo engine</i>
Price	0.333265	0.3368
Delta	-0.293996	
Gamma	0.209308	
Theta	0.00167852	
Vega	1.47439	

here, for the Monte-Carlo price, the corresponding implied volatility is 0.1883

The set-up in *Reditus* Finite-element engine for the stochastic volatility model is as the following:

Number of mesh points	2500
Type of mesh:	linear elements
Concentration along strike:	Weighting 0.1, Attenuation: 0.01
Concentration at spot:	Yes

The set-up in *Reditus* Monte-Carlo engine is as the following:

Number of simulations:	50000
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As can be seen from the above three tables, the difference between using a stochastic model and a standard Black-Scholes is not very large for vanilla style options. The option prices for both *Call* and *Put* from the stochastic volatility model is around 10% less than that of the standard Black-Scholes model.

Second Test Case:

The second test example is similar to the first test case, except that a lower barrier and upper barrier exist for the asset during the one-year tenor period. Details of the deal is listed here:

T	Strike	spot	σ	θ	κ	λ	ρ	r	Lower barrier	Upper barrier
1 year	10	10	0.3	0.04	2.0	0.0	0.5	0.1	4	15

This set up can be compared with a one-year double knock-out barrier option with a constant volatility of 0.2, $r = 0.1$, $strike = 10$, $spot = 10$ in a standard Black-Scholes framework using **Reditus** one-factor model (1FactorBarriers model) as shown in the following table:

Computational Results from **Reditus** Standard Black-Scholes Barrier Option Model

Reditus finite-element engine:	Call	(Analytic)	Put	(Analytic)
Price	0.88602	(0.8868)	0.375386	(0.3753)
Delta	0.274016		-0.274267	
Gamma	-0.153997		0.16643	
Theta	0.129926		-0.0213986	
Vega	-3.57672		3.32218	

The following screen shot records the deal set up for the *Call* calculation:

Knock out Multi-barrier option

Value Date: 27/04/2001
Deal Date: 27/04/2001
Maturity Date: 27/04/2002
Spot: 10.0
Strike: 10.0
Face Value: 1.0

Call Put

Price: 0.88602
 Delta: 0.274016
 Gamma: -0.153997
 Theta: 0.129926
 rho: 2.22801
 vega: -3.57672

Use Calendar
 Term Structure

Parameters

Parameter	Value
RiskFreeRate	0.1
DividendRate	0
Volatility	0.2

Barrier Start	Barrier End	Lower Barrier	Lower Rebate	Upper Barrier	Upper Rebate
27/04/2001	27/04/2002	4		15	

As a record, the set-up in **Reditus** Finite-element engine for the one asset model is provided here:

Number of mesh points	1500
Type of mesh:	quadratic elements
Concentration along upper barrier:	Yes for Call, Weighting 0.1, Attenuation: 0.01
Concentration along lower barrier:	Yes for Put, Weighting 0.1, Attenuation: 0.01
Concentration at spot and strike:	Yes

These parameters can be viewed in the *System Page*, they can also be changed dynamically before calculating the price:

The price and greeks from the stochastic volatility model is list in the following table:

Computational Results from *Reditus* Stochastic Volatility Model

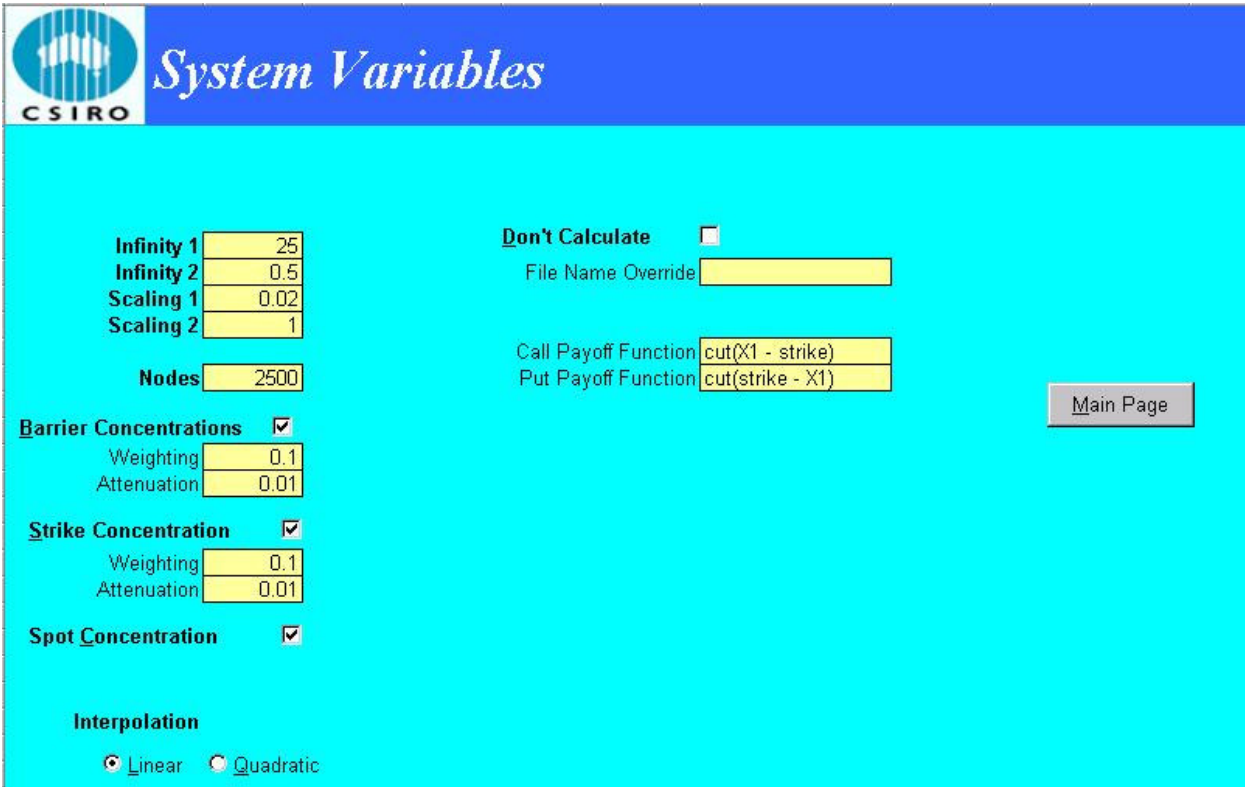
<i>Reditus</i> Finite-Element engine	<i>Call</i>	<i>Put</i>
Price	0.730173	0.332604
Delta	0.298321	-0.293523
Gamma	-0.0322748	0.1953
Theta	0.043426	0.00237265
Vega	-1.1454	1.45766

A screen shot of the *Reditus* interface for the *Call* deal is shown here:

The set-up in **Reditus** Finite-element engine for the stochastic volatility model is as the following:

Number of mesh points	2500
Type of mesh:	linear elements
Concentration along strike:	Weighting 0.1, Attenuation: 0.01
Concentration along barriers:	Weighting 0.1, Attenuation: 0.01
Concentration at spot:	Yes

This information is actually stored and can be changed in the *System Page* of the **Reditus** Excell interface:



As can be seen, the *Call Payoff Function* and *Put Payoff Function* in the *System* page are listed as:

$$cut(X1 - strike)$$

and

$$cut(strike - X1)$$

where, *cut* is a function that will leave positive value intact and set anything negative to zero, *X1* represents the first factor coordinate: i.e. the *X*-coordinate (*X2* is used to represent the second factor, or the *Y*-coordinate). Users can directly change these cells for pay-off functions to whatever formula that uses *X1*, *X2* or **Reditus** system parameters as dependent variables. For example, if we would like to use the variance v to make up the pay-off function for *Call/Put*, they would be:

$$cut(X2 - strike)$$

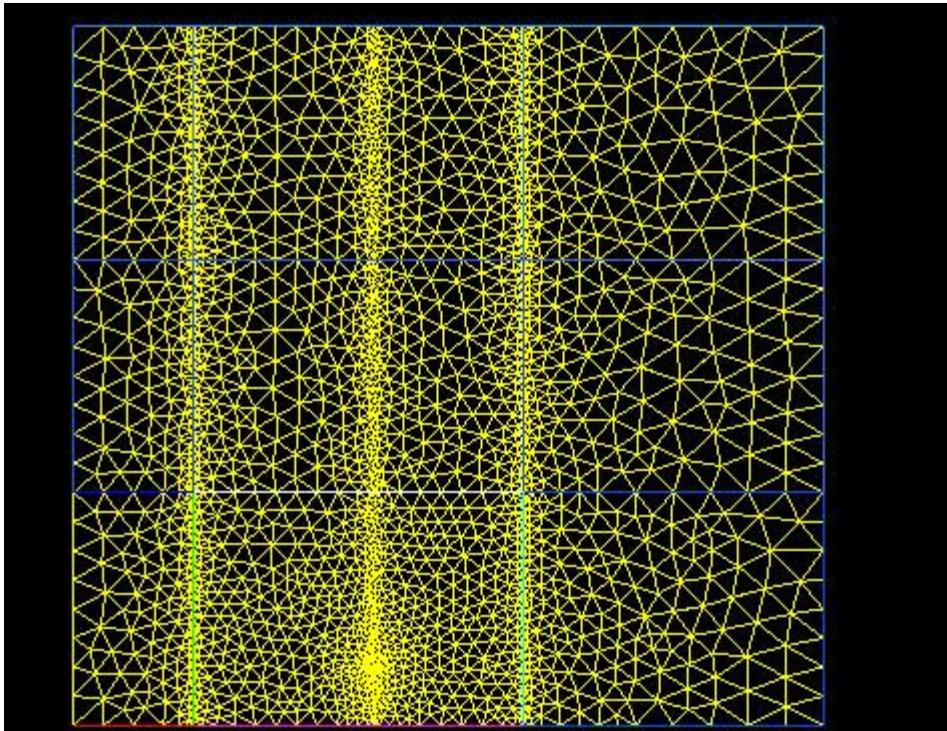
$$cut(strike - X2)$$

Benefits of Using Reditus

The current model can be easily and readily extended or modified to incorporate volatility models such as the Hull & White and Garch (1,1) models. The term structure input for σ , ρ , θ , λ ,

κ gives users the freedom to experiment with different market data. As the Excel interface is dynamically linked with the finite-element engine, any added parameters both constant or in term-structure form are automatically updated and linked to appear on the main page of the Excel interface.

In addition, users can rely on the graphical display of *Reditus* to study and construct option deals intuitively. We will take the double barrier option of the second test case as an example. This double barrier option deal using stochastic volatility model is represented in a two dimensional O - XY domain as shown here in the following figure (the mesh points are automatically generated by the Excel interface program when the deal is set up):

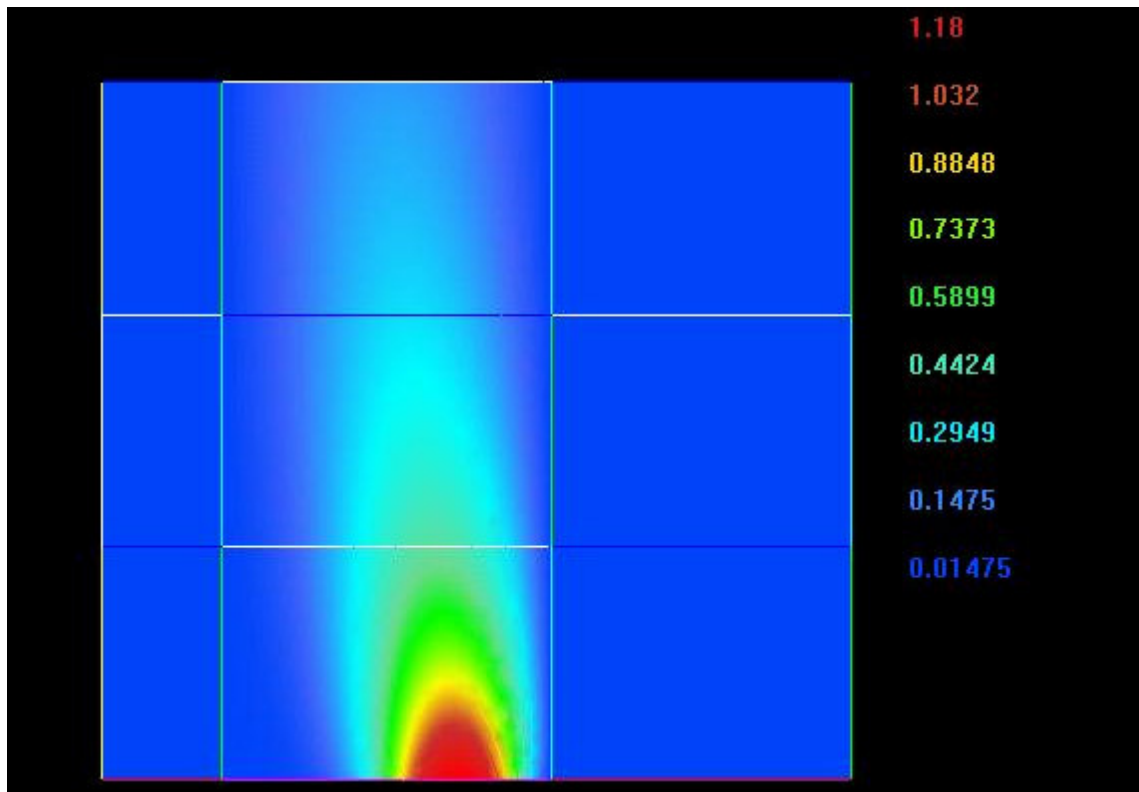


In this figure, the horizontal axis (X -coordinate) is the asset axis and the vertical axis (Y -coordinate) is the axis for the variance v . As a lower barrier and an upper barrier exist for the asset, two vertical lines are set up to represent them. As we have specified in the *System* page, mesh concentration is needed along the barriers, which can be seen here in the figure (more mesh points are concentrated along the barriers to increase the accuracy of computation). As the strike price is between the two barriers, and there is also concentration of mesh points along the strike line (as specified in the *System* page), we have a total of three vertical lines of mesh clustering in the figure. If the barrier concentration cell is not ticked, there will not be any mesh points clustering around the barriers, and in general, the accuracy of the option price might be lower.

As an exercise, users can change the settings in the *System* page and see what difference the change would cause to the option price (the *Save* and *Batch* functions are convenient ways to conduct a series of such experiments). *Reditus* provides a convenient and useful tool for experimenting with different configurations for computing option prices, such as the number of mesh points, infinity cutting-off points, scaling rules and the concentration of mesh points around areas of high price differential.

In the following figure, the colour contour plot of the double barrier option price is shown here together with the colour bar indicating the price/colour correspondence. The area of red colour indicates high option price while the blue colour represents low option price approaching zero.

We can easily see that for the *call* option, the upper barrier cuts sharply into the high price region at small variance values (lower section of the Y-coordinate).



Reference

[1] P. Forsyth, K. Vetzal and R. Zvan: A Finite Element Approach to the Pricing of Discrete Lookbacks with Stochastic Volatility, Technical Report from WWW:<http://yoho.uwaterloo.ca:80/~paforsyt/>, July 11 1997.

[2] Paul Wilmott: Derivatives: The Theory and Practice of Financial Engineering, pp 299-304, University Edition, John Wiley & Sons 1998.