General Restrictions on Prices of Financial Derivatives Written on Underlying Diffusions

Yaacov Z. Bergman*

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*School of Business and the Center for Research in Rationality and Interactive Decision Theory
The Hebrew University, Mount Scopus, Jerusalem•91905, ISRAEL
Voice: (+972)-2-588-3116; Fax: (+972)-2-588-1341
msyberg@mscc.huji.ac.il

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Abstract

It is shown that in any diffusive one-factor model of the term structure, the prices of bonds and of term structure puts decrease as the short-term interest rate increases. However, these prices need not be monotone in the short-term rate, if that rate can experience jumps.

An important comparative statics implication of the monotonicity result for diffusive models is that to a higher short-term interest rate corresponds a yield curve that lies uniformly above the curve that corresponds to a lower short-term rate. Furthermore, if the diffusion that describes the short-term rate is also homogeneous, then two yield curves that are measured at different dates cannot intersect when drawn from the same time origin. If empirically they do intersect, then the short-term rate *cannot* be described by a one-factor homogeneous diffusion.

It is also shown that if the second partial derivative w.r.t. to the short-term interest rate of the drift of the one-factor diffusion describing that rate is less than or equal to 2—special cases being the linear drift models—then the prices of deterministic-coupon bonds and term structure puts are convex in that rate.

The last result is derived using probabilistic representations of solutions to parabolic partial differential equations. The same methodology is used to derive restrictions on prices of European, American, and Asian options when the underlying price follows a stochastic volatility diffusion. Bounds, asymptotic results, and representations are derived for different linear differential transformations of derivative price functions like option's delta, rho, and theta. An example from these results is the fact that the rho of a European call written on a stochastic volatility underlying asset is equal to the price of a digital call with the same exercise price, the same time to expiration, and the same underlying asset as the call, multiplied by the time to expiration and by the exercise price.

The methodology is described in sufficient detail to allow for its ready application in a variety of situations.

General Restrictions on Prices of Financial Derivatives Written on Underlying Diffusion Processes

1 Introduction and Summary

This paper is a sequel to Bergman, Grundy, and Wiener (1996); it further explores general restrictions on prices of financial derivatives written on underlying diffusions.

After a brief review of probabilistic solutions to PDEs which are used extensively in this paper, general restrictions on term structure derivative prices are deduced in Section 3. The setting is that of a diffusive one-factor model of the short-term interest rate, particular instances of which where extensively investigated in the financial literature. See Black and Karasinski (1991); Constantinides (1992); Cox (1975); Cox, Ingersoll, and Ross (1980, 1985); Brennan and Schwartz (1979); Brown and Dybvig (1986); Chan et al. (1992); Courtadon (1982); Dothan (1978); Duffie and Kan (1993); Ho and Lee (1986); Hull and White (1990); Longstaff (1992); Marsh and Rosenfeld (1983); Merton (1973); Pearson and Sun (1994); and Vasicek (1976). It is shown in Section 3 that in such a setting, if the terminal payoff function of an interest rate derivative, as well as the dividend paid to it, are non-increasing in that rate, then its price is also non-increasing in that rate. An explanation is given for why this is no longer true if the short-term interest rate process may experience jumps. Examples for such derivatives are zero-coupon bonds and *term structure puts*, ie,[1] put options written on bonds or directly on the interest rate. An immediate comparative statics implication is that in a diffusive one-factor model of the interest rate, to a higher short-term rate corresponds a yield curve that lies uniformly above the yield curve that corresponds to a lower short-term rate; the two do not intersect. Furthermore, if,

[1] The modern British convention is used, by which "eg" and "ie" are written without the periods. This is kinder to the eye.

as is true of most diffusive one-factor models, the drift and the diffusion functions are also independent of time, ie, the diffusion is *homogeneous*, then two yield curves that are measured at different dates cannot intersect when drawn from the same time origin. If empirically they do intersect, then the short-term rate *cannot* be described by a one-factor homogeneous diffusion.

Another result is that if the second partial derivative w.r.t. to the short-term interest rate of the risk-adjusted drift of the one-factor diffusion describing the short-term rate is less than or equal to 2—special cases being the linear drift models—then the prices of zero-coupon bonds and term structure puts are convex in that rate. The majority of the one-factor models considered hitherto in the literature postulate drifts that are linear in the short-term rate, and, therefore, the prices of bonds and term structure puts are convex in the short-term rate in all these models.

This convexity result is derived using the methodology developed in Bergman (1983). According to it, properties of a linear differential transformation of a financial derivative's price function—like option's delta, rho, or any other "Greek" quantity—are arrived at in the following way. First, the linear differential transformation, or operator, is applied to the partial differential equation (PDE) and to the boundary and end conditions which the derivative's price function satisfies. Notably, the PDE is almost always of the parabolic type. Second, the order of partial derivatives in the PDE is interchanged to get a new parabolic PDE in which the transformed price function serves as the "solution" function. This function can then be represented as a probabilistic solution which uses the information embodied in the coefficients of the new PDE and in the transformed boundary and end conditions.[2] In many important cases, this probabilistic solution allows the derivation of general properties of the transformed price function. A brief review of the mathematical results concerning probabilistic solutions to parabolic PDEs in a context relevant to the applications herein is provided in Section 2.

This methodology, that in Section 3 is used to explore general properties of interest rate derivatives, is used again in Section 4 to investigate general restrictions on American style options written on tradable assets whose prices follow diffusions with stochastic volatility, ie,

[2] Fournié et. al. (1997) arrive at probabilistic representations of derivatives of option prices using the Malliavin calculus. Their purpose is the construction of efficient numerical algorithms for computing these quantities.

volatility that may depend on the concurrent price of the underlying asset. Proposition 7 and 8 extend results in Bergman, Grundy, and Wiener (1996) to American options. The former proposition provides the probabilistic representations of an American option delta and the riskless position in the option's replicating portfolio, namely, the option price minus the product of the underlying price and the option's delta. Proposition 8 uses these representations for delta and for the riskless position in the option's replicating portfolio for placing general bounds on those quantities. Using Lemma 9, Proposition 10 states general asymptotic results concerning European and American delta and their riskless positions. Specifically, it is shown that the limit of those quantities at any time before expiration, as the underlying price increases without limit, is equal to the similar limit taken at expiration. Corollary 11 gives the implications of these results to calls and puts on stochastic volatility underlying assets. Both European and American put deltas approach zero as the underlying price increases without bound. Under the same circumstances, an American call delta approaches one, while a European call delta approaches a "discount factor" with a discount rate that is equal to the difference between the dividend rate paid to the underlying and that which is paid to the call.

Proposition 12 gives a general representation of the rho of a European option in terms of the *price* of another option written on the same *stochastic volatility* underlying asset. A special case of that result states in Corollary 13 that the rho of a European call written on a stochastic volatility underlying asset is equal to the price of a digital call with the same exercise price, the same time to expiration, and the same underlying asset as the call, multiplied by the time to expiration and by the exercise price. Proposition 12 also provides asymptotic results about rho, which, in the case of a call option, translate to the statement that as the underlying price increases beyond bound, its rho approaches the product of the discounted exercise price and the time to expiration. Proposition 12 also states that if the second partial derivative of the final payoff function w.r.t. the underlying price maintains a uniform sign, then that sign is inherited by the partial derivative of rho w.r.t. the underlying price.

Proposition 14 has results about the theta of a call option written on a stochastic volatility underlying asset. For example, it states that at any time before expiration, as the underlying price increases without bound, the call's theta approaches the negative of the product of the discounted exercise price and the interest rate.

To stress that the methodology employed in this paper can be applied to settings with dimension larger than one, Section 5 derives general bounds on derivatives of the prices of Asian type options. Section 6 concludes the paper.

2 Probabilistic solutions to parabolic PDEs: a brief review

The analysis that follows makes extensive use of probabilistic solutions to parabolic PDEs. Therefore, a brief review of the relevant results—an adaptation from Friedman (1975, ch 6, sec 5) or Duffie (1996, Appendix E)—is in order. Define a domain $Q := \{(x,t) : t \in [0,T),\ \underline{x}(t) < x < \overline{x}(t)\}$, where $\underline{x}$ and $\overline{x}$ are continuous functions of time [but $\underline{x}(t) = -\infty$ or $\overline{x}(t) = \infty$, $t \in [0,T]$ are also formally allowed]. A partial differential operator $\mathcal{P}$ defined on the domain Q is called parabolic, if it is of the form

$$\begin{aligned}\mathcal{P}u(x,t) := {} & u_t(x,t) + \tfrac{1}{2}A^2(x,t)u_{xx}(x,t) \\ & + B(x,t)u_x(x,t) + C(x,t)u(x,t) + H(x,t).\end{aligned} \tag{1}$$

Consider the problem

$$\mathcal{P}u(x,t) = 0 \qquad \text{in } Q \tag{2}$$

$$u(x,T) = g(x) \qquad x \in \left(\underline{x}(T),\ \overline{x}(T)\right) \tag{3}$$

$$u(\overline{x}(t),t) = \overline{f}\left(\overline{x}(t), t\right) \qquad t \in [0,T] \tag{4}$$

$$u(\underline{x}(t),t) = \underline{f}\left(\underline{x}(t), t\right) \qquad t \in [0,T]. \tag{5}$$

Equation (2) is called a parabolic PDE; condition (3), where g is a known function, is its end condition; and (4) and (5), with $\overline{f}$ and $\underline{f}$ some known functions, are the upper and the lower boundary conditions, respectively. If this problem has a unique solution, then under some regularity conditions on the functions involved,[3] this solution has a probabilistic representation which is constructed as follows.

For any point (x, t) in the domain Q, reading the functions A and B off the operator $\mathcal{P}$ in (1), define an *auxiliary* Itò process $\{X_s^{x,t}: t \leq s \leq T\}$ in $\mathbb{R}$ as the solution to

$$dX_s = B(X_s, s)ds + A(X_s, s)dW_s, \quad t \leq s \leq T, \quad X_t = x \in (\underline{x}(t), \overline{x}(t)), \tag{6}$$

where $\{W_t: 0 \leq t \leq T\}$ is a one-dimensional Brownian motion.[4] It will be assumed that the drift and the diffusion functions, B and A, are such that the diffusion (6) is regular on $(0, \infty)$, which means that starting from any point in $(0, \infty)$, the process can reach any point in $(0, \infty)$. The end-points of the interval on which the process is regular—the regularity interval—may be either attainable or unattainable. An unattainable boundary is one that cannot be reached in finite time from within the regularity interval of the process. Whether a boundary is unattainable is determined by the behavior of the functions, B and A, near that boundary. See Karlin and Taylor (1981, ch 15) for a description of diffusion boundary classification. For example, the commonly used geometric Brownian motion is regular on $(0, \infty)$, and both zero and infinity are unattainable.[5] In the sequel, it will always be assumed that infinity is unattainable.

For any (x, t) in Q, define $\overline{\tau}^{x,t} := \inf\{s \mid X_s^{x,t} = \overline{x}(s)\}$ to be the first time the auxiliary process $X^{x,t}$ hits the upper boundary of Q. Define $\underline{\tau}^{x,t}$ for the lower boundary in a similar way. The superscripts x and t will usually be omitted form X and τ in what follows; no confusion should arise. To simplify the notation further, $E^{x,t}$ will denote the expectation given that the

[3] Sufficient conditions require that all of σ, B, C, H, and u are continuous, and g, $\overline{f}$, $\underline{f}$ are piecewise continuous; the solution u, and the functions σ, B, H, g, $\overline{f}$, and $\underline{f}$ satisfy a polynomial growth condition in x; C is nonnegative; and σ and B are also Lipschitz in x; see Duffie (1996, p.295).

[4] The Brownian motion is defined on a probability space $\{\Omega, \mathcal{F}, P\}$, where Ω is the state space, $\mathcal{F}$ is the collection of events, P is the probability measure, and $\{\mathcal{F}_t: 0 \leq t \leq T\}$ are the information sets revealed by the Brownian motion. Formally, $\mathcal{F}$ is a σ-algebra, and $\{\mathcal{F}_t: 0 \leq t \leq T\}$ is the Brownian filtration generated by W_t. For the definitions, see, for example, Duffie (1996).

[5] Intuitively, as the GBM approaches zero from above, both the drift and the volatility of the process vanish; motion dies out; and zero remains unattainable. By contrast, in a regular Brownian motion, even with a positive drift which acts to drive the process away from zero, the volatility is unaffected by the approach to zero, and the vigorous wiggling of the process *can* bring it down to zero.

process X starts from x at time t, so that, for example, $E[f(X_s^{x,t}, \tau^{x,t})]$ will be written $E^{x,t}[f(X_s, \tau)]$. Also, $\mathbf{1}_{\text{condition}}$ will denote the indicator function that takes the value 1 if the "condition" is true, and zero otherwise.

The probabilistic representation of the solution is then given by

$$\begin{aligned} u(x,t) \quad = \quad & E^{x,t}\Big\{\exp\Big[\int_t^T C(X_\lambda,\lambda)d\lambda\Big] g(X_T)\,\mathbf{1}_{T\le \min(\underline{\tau},\overline{\tau})}\Big\} \\ & + E^{x,t}\Big\{\exp\Big[\int_t^{\overline{\tau}} C(X_\lambda,\lambda)d\lambda\Big]\,\overline{f}\left(\overline{x}(\overline{\tau})\right)\mathbf{1}_{\overline{\tau}\le \min(\underline{\tau},T)}\Big\} \\ & + E^{x,t}\Big\{\exp\Big[\int_t^{\underline{\tau}} C(X_\lambda,\lambda)d\lambda\Big]\,\underline{f}\left(\underline{x}(\underline{\tau})\right)\mathbf{1}_{\underline{\tau}\le \min(\overline{\tau},T)}\Big\} \\ & + E^{x,t}\Big\{\int_t^{\min(\underline{\tau},\overline{\tau},T)} \exp\Big[\int_t^s C(X_\lambda,\lambda)d\lambda\Big] H(X_s,s)ds\Big\}. \end{aligned} \tag{7}$$

Intuitively, this probabilistic representation of the solution $u(x,t)$ to the problem, (2), (3), (4), and (5), can be interpreted as comprising four contributions. The first term on the RHS of (7) is an average over all the "discounted" end values of u, namely $g(X_T)$, in the event that the auxiliary process X reaches time T neither having hit first the upper nor the lower boundaries. The instantaneous "discount" rate at time λ is $-C(X_\lambda, \lambda)$. The second contribution on the RHS of (7) is the average over all the "discounted" upper boundary values of u, namely $\overline{f}\left(\overline{x}(\overline{\tau})\right)$, in the event that the auxiliary process X hits the upper boundary for the first time at $\overline{\tau}$, provided that time comes before hitting the lower boundary for the first time and before time T. The third contribution on the RHS of (7) is analogous to the second in the obvious way. Finally, the fourth contribution is a "discounted dividend stream" at a rate $H(X_s, s)$, averaged over all sample paths of the auxiliary process X until the first time it hits one of the boundaries or reaches time T having hit neither.

Note that the "discount rate" and "dividend streams" are used here as metaphors that become literal only when the solution is the price of an asset. However, in the analysis that follows, $u(x,t)$ will generally *not* be an asset's price, but, instead, a quantity related to a price, eg, a first partial of a price of a derivative with respect to some state variable.

Suppose that the lower boundary of Q is at $x = 0$ for all $t \in [0, T]$, ie, $\underline{x}(t) \equiv 0$, and suppose also that the boundary is unattainable. This implies that the event $\underline{\tau} < \min(\overline{\tau}, T)$ is null, and therefore the third term on the RHS of (7) equals zero. This, in turn, means that specifying the lower boundary condition (5) has no effect on the solution to PDE (2), and that $\lim_{x \searrow 0} u(x, t)$, being a function of t on the domain $[0, T)$, is forced by the PDE, the end condition (3), and the upper boundary condition (4), provided, of course, that the upper boundary *is* attainable. If the upper boundary is an unattainable boundary, then it too does not affect the solution. See Feller (1951).

3 Term Structure Derivatives

General properties of bonds and other term structure derivatives in a diffusive one-factor model of the term structure are derived in this section. In such a model, the *risk adjusted* process of the short-term interest rate follows a real valued diffusion $\{X_t;\, 0 \le t \le T'\}$, regular on $(0, \infty)$, that solves the stochastic differential equation,

$$dX_t = \mu(X_t, t)dt + \sigma(X_t, t)dW_t\ ;\ 0 \le t \le T', \quad X_0 \in (0, \infty), \tag{8}$$

where $\{W_t :\ 0 \le t \le T'\}$ is a one-dimensional Brownian motion,[6] where μ and σ satisfy Lipschitz conditions in x and have derivatives μ_x, σ_x, μ_{xx}, and σ_{xx} that are continuous and satisfy growth conditions in x. The nature of the boundaries of the risk-adjusted process (8) at zero and at infinity is the same as that of the original (non-adjusted) interest rate diffusion. (Otherwise, the risk-adjusted measure would not be equivalent to the original one.) As discussed in Section 2, the types of boundaries of the diffusion X depend on the behavior of the drift and diffusion functions, $\mu(x, t)$ and $\sigma(x, t)$, near zero and "near" infinity. As stated above, it will be assumed that $x = \infty$ is an unattainable boundary. For $x = 0$, two types of boundary will be considered; a natural unattainable boundary and an absorbing boundary, wherein the interest rate stays after hitting it

[6] The Brownian motion is defined as in footnote 4.

once. A particular diffusive one-factor model of the term structure with absorption of the short-term rate at zero is analyzed by Longstaff (1992), who provides an enlightening discussion of the effects of specifying different boundary types on term structure models. Although absorption at zero may seem less realistic than unattainability of zero, since one-factor models tend to perform empirically best only in short runs, ruling out the stochastic dynamics of the interest rate that may eventually lead to absorption in the longer run, may be overly restrictive.

It should be remarked that the lower boundary of the short-term interest rate need not be set to zero. Any number can serve that role without changing the results, including a negative number or even minus infinity (as long as the latter is unattainable).

An interest rate (European type) derivative is a security that contracts to pay $g(x)$ dollars at some terminal time T $(\leq T')$ if the realization of the interest rate is then $X_T = x$, and also contracts to pay out at any time $t \in [0, T]$ a dividend at the rate $h(x, t)$ dollars if $X_t = x$. The price of such a derivative, $v(x, t)$, is then the solution to the parabolic PDE

$$\begin{aligned} v_t(x,t) + \tfrac{1}{2}\sigma^2(x,t)v_{xx}(x,t) + \mu(x,t)v_x(x,t) \\ -xv(x,t) + h(x,t) = 0, \qquad \text{in } \mathbb{R}_{++}\times[0,T), \end{aligned} \tag{9}$$

with end condition

$$v(x,T) = g(x), \qquad \text{on } \mathbb{R}_{++}, \tag{10}$$

which is the only condition that is both required and that can be imposed in the case that $x = 0$ is an unattainable boundary. On the other hand, if $x = 0$ is an absorption barrier, then a boundary condition along that boundary *is* needed, which equilibrium considerations require to be

$$v(0,t) = g(0) + \int_t^T h(0,s)ds, \qquad t \in [0,T]. \tag{11}$$

The reason is that once the interest rate hits zero at time t and is absorbed there, the owner of the derivative is guaranteed to get $g(0)$ dollars at expiration time T, which is worth $g(0)$ dollars already at t, because the interest rate is sure to stay at zero until expiration T. This is the first term on the RHS of (11). In addition, the owner is guaranteed to get a dividend rate of $h(0, s)$ at any time s in the interval $[t, T]$. Again, because the interest rate is sure to stay at zero level after t, the present value of that dividend stream at t is the integral term on the RHS of (11).

The sufficient regularity conditions of footnote 1 for a probabilistic representation of the solution to problem (9) are assumed here as well. Note that the auxiliary process for that representation is none other than the risk adjusted process (8). The following proposition is well known for the case where zero is unattainable. It is stated for completeness and for the proof when zero is absorbing.

PROPOSITION 1. *In a diffusive one-factor term structure model, when $x = 0$ is an unattainable boundary, the price of an interest rate derivative with terminal payoff function g receiving a dividend rate h is the solution to problem (9) and (10), and can be represented probabilistically as*

$$
\begin{aligned}
v(x,t) \quad &= \quad E^{x,t}\{\exp\left[\int_t^T -X_\lambda d\lambda\right] g(X_T)\} \\
&\quad + E^{x,t}\{\int_t^T \exp\left[\int_t^s -X_\lambda d\lambda\right] h(X_s, s) ds\}.
\end{aligned}
\tag{12}
$$

This is also the probabilistic representation of the solution to (9), (10), *and* (11) *when $x = 0$ is an absorbing boundary.*

Proof: See the Appendix.

PROPOSITION 2. *In a diffusive one-factor model of the term structure, if both the terminal payoff $g(x)$ of an interest rate derivative and the dividend rate $h(x, t)$ paid to it are non-negative and non-increasing in x, then the price of that derivative is non-increasing in the short-term interest rate. If $g(x)$ is also positive on a positive measure real set, then the price of the derivative is strictly decreasing in the short-term rate.*

Proof: Comparing statics, consider two alternative values, x_1 and x_2, for the short-term interest rate at time $t \in [0, T)$, such that $x_1 < x_2$. The aim is to show that $v(x_1, t) \geq v(x_2, t)$. Making explicit the dependence of X on its initial value at time t, it follows from the fact that increasing the initial condition of an SDE increases the entire solution path,[7] that $\forall\lambda \in [t,T)$: $X_\lambda^{x_1,t} \leq X_\lambda^{x_2,t}$ almost surely (where the inequality is strict a.s. in a right neighborhood of t). In particular, $X_T^{x_1,t} \leq X_T^{x_2,t}$ a.s. Therefore, $\exp\left[\int_t^T -X_\lambda^{x_1,t}\,d\lambda\right] > \exp\left[\int_t^T -X_\lambda^{x_2,t}\,d\lambda\right]$ and $g(X_T^{x_1,t}) \geq g(X_T^{x_2,t})$; both almost surely. Taking expectations, it follows that the first summand on the RHS of (12) is no less at (x_1, t) than at (x_2, t). Since $h(x, t)$ is non-increasing in x, it follows that $\forall s \in [0,T]$: $h(X_s^{x_1,t}, s) \geq h(X_s^{x_2,t}, s)$ a.s., hence the second summand is also no less at (x_1, t) than at (x_2, t). □

Examples of term structure derivatives to which Proposition 2 applies are (i) a deterministic-coupon bond; one that pays \$1 at maturity date T, $g_{\text{bond}}(x) = 1$, and also pays a dividend at a preset rate $h(t)$, usually some positive constant [denote the time t price of such a bond by $v_{\text{bond}}(x, t; T)$]; (ii) a European put option on a deterministic-coupon bond, which is a contract to pay $g_{\text{put-on-bond}}(x) = [K - v_{\text{bond}}(x, T''; T)]^+$ dollars at put expiration time $T'' \in [T, T']$ if the short-term interest rate is then x (assuming the bond is priced correctly in the market); (iii) a put option on the interest rate itself, which is a contract to pay $g_{\text{put-on-interest}}(x) = [K - x]^+$ dollars at time $T'' \in [T, T']$ if the short-term interest rate is then x. The last two instruments will be called *term structure puts*, for short. The implications of Proposition 2 to the price behavior of these derivatives and to the term structure of interest rates are summarized in the following two corollaries.

COROLLARY 3. *In a diffusive one-factor model of the term structure, the prices of (i) a deterministic-coupon bond, (ii) a European put option on such a bond, and (iii) a European put option on the short-term interest rate; all strictly decrease in that rate.*[8]

Proof: An immediate application of Proposition 2. □

[7] The argument here is similar to the one given in Duffie (1996, pp. 183, 184) and in Bergman, Grundy, and Wiener (1996, the "no crossing-over property").

[8] Consistent with Corollary 3, in a special diffusive one-factor term structure model, Longstaff (1992) demonstrates that the price of a put on a later time interest rate decreases in the current interest rate.

Let $v_{\text{bond}}(x, t; T)$ denote the price of a zero-coupon bond at time t when the short-term interest rate is x, and which contracts to pay \$1 at time T. The *yield to maturity* is defined, $Y(x,\, t;\, T) := -\frac{\ln v_{\text{bond}}(x,t;T)}{T-t}$. The *yield curve,* as of time t when the short-term interest rate equals x, is the graph of $Y(x,\, t;\, T)$ as a function of maturity T.

COROLLARY 4. *In a diffusive one-factor model of the term structure, the whole yield curve shifts upwards with an increase in the short-term interest rate. Formally, let* x_1 and x_2 *be two levels of the short-term interest rate with* $x_1 < x_2$. *Then at any given time t,* $Y(x_1,\, t;\, T) < Y(x_2,\, t;\, T)$ *for all* T. *If, in addition, the drift and diffusion functions of the short-term interest rate process are time-independent, then for all* t, T, *and* Δ, $Y(x_1,\, t;\, T) < Y(x_2,\, t+\Delta;\, T+\Delta)$.

Proof: An increase in the short-term interest rate implies a decrease in v_{bond} by Corollary 3, which, by definition, implies an increase in Y for every T. The last inequality follows from the fact that in the case of a homogeneous diffusion the transition density function depends only on the time difference between the initial and the final states. Therefore, by (12), a zero-coupon bond price depends only on the time remaining to its maturity, so that $v_{\text{bond}}(x,\ t;\ T) = v_{\text{bond}}(x,\, t+\Delta;\, T+\Delta)$. $\square$

The first part of Corollary 4 is a comparative static result. In addition, the latter part of Corollary 4 has an important empirical implication. Suppose that two yield curves are measured at two different dates, and then are drawn starting from the same time origin. If the two curves intersect, then the short-term interest rate *cannot* be described by a homogeneous one-factor diffusion model.

The proof of Propsition 2 depends crucially on the increase of entire diffusion sample paths with an increase in their initial values, which, in turn, follows from the almost sure continuity of diffusion sample paths. This implies that counter examples to Propsition 2 and Corollaries 3 and 4 may be found when the interest rate process has discontinuous sample paths with positive probability. Indeed, suppose that the interest rate process is such that when it reaches a value r_0 it jumps to a higher level. Then a comparative static decrease in the interest rate towards r_0 may result in a *decrease* in the price of a zero-coupon bond. Intuitively, the decrease in the short-term rate has two effects. It acts to increase the price of the bond, but at the same time it implies—in the current example—an increase in the probability of a jump upwards in future spot rates, which acts to decrease the bond price. Given a large enough jump amplitude and a large enough probability for that jump, the latter effect may overwhelm the former.

In contrast, when the short-term interest rate follows a diffusion, the continuity of its sample paths implies that a decrease, say, in the interest rate indicates that it is likely to stay in its new lower locale for a while. Therefore, the only effect is an increase in the bond price.

The next proposition represents as expectations the first and the second partial derivatives of the price of a term structure derivative. Those expectations are then used to derive general properties of the derivatives.

PROPOSITION 5. *In a diffusive one-factor model of the term structure, the first partial derivative of the price of a European type term structure derivative w.r.t. the short-term interest rate has a probabilistic representation*[9] *in* $\mathbb{R}_{++}\times[0,T)$,

$$v_x(x,t) = E^{x,t}\Big\{\int_t^T \varphi_{t,s}[h_x(X_s,s) - v(X_s,s)]ds + \varphi_{t,T}g'(X_T)\Big\}, \tag{13}$$

where $\varphi_{t,s} := \exp(\int_t^s [\mu_x(X_\lambda,\lambda) - X_\lambda]d\lambda$; *where the auxiliary process* X *in* (13) *satisfies* $dX_s = B(X_s,s)ds + \sigma(X_s,s)dW_s$; $t \le s \le T$, $X_t = x$; *where* $B(X_s,s) := \sigma(X_s,s)\sigma_x(X_s,s) + \mu(X_s,s)$; *and where it is assumed that* $B(X_s,s)$ *and* $\sigma^2(X_s,s)$ *make* $x=0$ *and* $x=\infty$ *unattainable boundaries for the process* X.

The second partial derivative also has a probabilistic representation in $\mathbb{R}_{++}\times[0,T)$,

$$v_{xx}(x,t) = E^{r,t}\Big\{\int_t^T \varphi_{t,s}\{h_{xx}(X_s,s) + [\mu_{xx}(X_s,s) - 2]v_x(X_s,s)\}ds + \varphi_{t,T}g''(X_T)\Big\}, \tag{14}$$

where $\varphi_{t,s} := \exp(\int_t^s [\sigma(X_\lambda,\lambda)\sigma_{xx}(X_\lambda,\lambda) + \sigma_x^2(X_\lambda,\lambda) + 2\mu_x(X_\lambda,\lambda) - X_\lambda]d\lambda$; *where the auxiliary process* X *in* (14) *satisfies* $dX_s = B(X_s,s)ds + \sigma(X_s,s)dW_s$; $t \le s \le T$, $X_t = x$; *where*

[9] Provided that the sufficient conditions, as in footnote 3, for the existence of probabilistic representations are met.

$B(X_s,\, s) := 2\sigma(X_s,\, s)\sigma_x(X_s,\, s) + \mu(X_s,\, s)$; *and where it is assumed that* $B(X_s,\, s)$ *and* $\sigma^2(X_s, s)$ *make* $x = 0$ *and* $x = \infty$ *unattainable boundaries for the process* X.

Proof: See the Appendix.

Remark 1: The assumption that $x = 0$ is an unattainable boundary is necessary for simplicity of exposition. Other types of barrier at zero, like absorption, require more detailed treatment, but imply most of the results herein.

Remark 2: Sufficient conditions for zero to be an unattainable boundary for the auxiliary processes in (13) and in (14) are (i) that zero be an unattainable boundary for the risk-adjusted short-term interest rate process (8), and (ii) that σ be non-decreasing in the first argument. The reason is that while the volatilities of the three processes are identical, the drifts of (13) and (14) are not smaller than the drift of the risk-adjusted process.

Remark 3: The functions g' and g'' may also be generalized functions. For example, if $g(x) = (x - K)^+$, then g' is the Heavyside function, and g'' is Dirac's delta function.

Inspection of (13) reconfirms that the conditions for a non-positive sign of $v_x(x,t)$ are those of Proposition 2. Moreover, The integral term in the RHS of (13) is always non-positive, but the sign of the other term depends on that of the random variable $g'(X_T)$. In the case of a call option on a bond or on the interest rate itself, it is non-negative, and it is positive on a positive measure event. This implies that the sign of $v_x(x,\, t)$ is indeterminate in the case of a call option; see Longstaff (1990) who demonstrates the change of sign of an interest rate call option price slope in a particular diffusive one-factor model of the term structure.

COROLLARY 6. *Let* μ *and* σ *be the drift and the volatility functions of the risk-adjusted short-term interest rate. If* $\mu_{xx}(x,\, \tau) \le 2$ *on* $\mathbb{R} \times [0, T)$ [eg, drifts linear in x] *and if zero is unattainable by the diffusion* $dX_s = \left[2\sigma(X_s,\, s)\sigma_x(X_s,\, s) + \mu(X_s, s)\right]ds + \sigma(X_s, s)dW_s$; $t \le s \le T$, $X_t = x$, *then the price of a deterministic-coupon bond is (quasi) convex in the short-term interest rate, and so are the prices of a put option on such a bond and of a put on the interest rate.*

Proof: The conditions of Proposition 5 are satisfied in the corollary, whence $v_x(x, s) \le 0$ for the bond and the puts. The conditions of the corollary also imply $h_{xx} \equiv 0$ and $g''(x) \ge 0$ for all three derivative assets. Using all this in (14) yields $v_{xx}(x,t) \ge 0$ on $\mathbb{R}_{++}\times[0,T)$. □

In most extant diffusive one-factor models of the term structure (see the Introduction), the drift is linear in the short-term interest rate, ie, $\mu_{xx}(x, t) \equiv 0$, and the condition that makes zero an unattainable boundary for the auxiliary process in (14)—the probabilistic representation of the second derivative of the price—are met. Therefore, in all those models, the prices of bonds, puts on bonds, and interest rate puts are all convex in the interest rate.

Example. Consider the "square-root" risk adjusted process for the short-term interest rate in Cox, Ingersoll, and Ross (1985), where $\mu(x,t) := \alpha - \beta x$, with positive α, and $\sigma^2(X_s, s) := \eta^2 x$. The auxiliary process in (14) has then a drift $\eta^2 + \alpha - \beta x$ and volatility $\theta^2 x$. Since $0 < \eta^2 < 2(\alpha + \eta^2)$ for all real η, therefore zero is always unattainable for the auxiliary process in (14) for the current example, *irrespective of whether the risk-adjusted interest rate process has zero as an unattainable boundary as well*. Therefore, since $\mu(x,t)$ is linear in x, the price of a zero-coupon bond (and those of the puts) is convex in the interest rate, both when zero is unattainable and when it is attainable by the "square-root" risk-adjusted interest rate process. The first case corresponds to Cox, Ingersoll and Ross's (1985) bond price formula in their equation (23). The second case corresponds to Longstaff's (1992) bond price formula which he gets in his equation (6) for the case that the risk-adjusted interest rate process is absorbed at zero. It can be verified that both price formulas, that of CIR and that of Longstaff, are convex in the interest rate as required by Corollary 6.

4 General restrictions on option prices: Further results

Bergman, Grundy, and Wiener (1996) derive general properties of European option prices. In this section, which shares the same methodology with the previous one, I extend their results to American type options and derive further results about the delta, rho, and theta of an option.

The standard assumptions of the Black-Merton-Scholes option pricing model are employed in the sequel, except that the underlying price process is not restricted to be a geometric Brownian motion, but instead its volatility function may be stochastic. Specifically, the underlying price is the real valued Itò process $\{S_t;\, 0 \le t \le T\}$ which solves the stochastic differential equation[10]

$$dS_t = \mu(S_t, t)dt + \sigma(S_t, t)dW_t \,;\; 0 \le t \le T\,, \quad S_0 \in \mathbb{R}. \tag{15}$$

It will be assumed that μ and σ are such that the process (15) is regular on $(0, \infty)$, and that zero and infinity are unattainable boundaries (see above).

The underlying asset pays out a deterministic dividend yield $\delta^{\mathcal{U}}$. Price taking agents can borrow and lend at a deterministic riskless interest rate r in a frictionless capital market, and arbitrage opportunities are not possible. To simplify the notation, $\delta^{\mathcal{U}}$ and r are assumed constant.

Both American and European type options will be treated. Specifically, consider an American type option with an expiration date T which pays out a constant dividend yield $\delta^{\mathcal{O}}$. If the owner of the American option decides to exercise it at time t $(\le T)$ when the underlying price has a realization $S_t = x$, he gets the contractual payoff $g(x)$ dollars, where $g:\mathbb{R}_+ \to \mathbb{R}_+$ is assumed to be continuous and piecewise twice boundedely differentiable. Consideration of general payoff functions is motivated by the existence of contingent claims whose payoff functions are not piecewise linear, like the "turbo" or "power" options with $g(x) := [x^n - k]^+$; n rational. Like above, the functions g' and g'' may also be generalized functions. As a technical condition, it is assumed that both the interest rate and the underlying's dividend yield are not smaller than the dividend yield on the option, ie, $d^{\mathcal{O}} \le r$ and $\delta^{\mathcal{O}} \le \delta^{\mathcal{U}}$.

Assuming the *American Regularity Conditions* [Duffie (1996) p. 173], the function $v:\mathbb{R}_{++} \times [0, T] \to \mathbb{R}_+$, which assigns to the American option its no-arbitrage price $v(x, t)$ when the time t underlying price is x, can be characterized as follows. Let $\mathcal{C} \subseteq \mathbb{R}_{++} \times [0, T)$ be the *continuation region* and let $\mathcal{E} \subseteq \mathbb{R}_+ \times [0, T)$ be the *exercise region* of the option. Together, $\mathcal{E}$ and $\mathcal{C}$ partition

[10] Here, W is a Brownian motion like above. The drift and the diffusion functions, μ and σ are assumed to be Borel measurable, continuous, Lipschitzian in their first argument, and satisfy a growth condition, so that a unique solution to the SDE is guaranteed.

$\mathbb{R}_{++} \times [0,T]$. The *optimal exercise boundary* is the boundary between the two regions where the option is optimally exercised, if and when the underlying price hits it from within the continuation region. In general, depending on the payoff function, the optimal exercise boundary can have more than one branch, but for ease of notation it will be assumed that it is the graph of a twice differentiable function x_f:$[0,T) \to \mathbb{R}$; $t \mapsto x_f(t)$. According to this assumption, one region lies "above" the other. For example, the exercise region of an American call lies above its continuation region, and the reverse is true of an American put.

Define the Black-Merton-Scholes parabolic differential operator[11] $\mathcal{L}_{\text{BMS}}$ by

$$\mathcal{L}_{\text{BMS}}v(x,t) := v_t(x,t) + \tfrac{1}{2}\sigma^2(x,t)v_{xx}(x,t) + (r - \delta^{\mathcal{U}})xv_x(x,t) + (\delta^{\mathcal{O}} - r)v(x,t). \tag{16}$$

Then on the continuation region $v(x,t)$ satisfies the Black-Merton-Scholes PDE

$$\mathcal{L}_{\text{BMS}}v(x,t) = 0, \tag{17}$$

with the end condition,

$$v(x,T) = g(x), \text{ for } x \in \mathbb{R}_{++}, \tag{18}$$

and a free boundary condition,

$$v_x(x_f(t), t) = g'(x_f(t)), \text{ for } t \in [0,T], \tag{19}$$

which implicitly defines the optimal exercise boundary [see Brekke, K. A. and B. Oksendal (1991)]. By the maintained assumption that zero is an unattainable barrier, boundary conditions along $x = 0$ should not be specified, [Feller (1951), see above]. Completing the characterization of the option price is the requirement that $v(x,t) = g(x)$ hold on the exercise region $\mathcal{E}$.

A European option will also be considered that is exercisable only at expiration time T with a payoff function g and that pays out a constant dividend yield $\delta^{\mathcal{O}}$. It is characterized similarly to an American option, except that the continuation region $\mathcal{C}$ extends over all of $\mathbb{R}_+ \times [0,T]$. The

[11] The mathematics that follows is standard, but *operator* notation simplifies the exposition considerably.

end condition is then $v(x,T) = g(x)$, $x \in \mathbb{R}_{++}$. Again, boundary conditions along $x = 0$ are not to be specified.

The next propositions are valid for two different interpretations; each needs its own auxiliary stochastic process

$$dX_s = B(X_s,s)ds + \sigma(X_s,s)dW_s, \quad t \le s \le T, \quad X_t = x,\ (x,s) \in \mathcal{C}, \tag{20}$$

where σ is taken from (15), and where B is defined separately for each interpretation below.

(a) *Delta interpretation*

Define

$$\mathcal{D}v(x,\, t) := v_x(x,\, t)$$
$$\mathcal{D}g(x) := g'(x)$$
$$\delta := \delta^{\mathcal{U}} - \delta^{\mathcal{O}}$$
$$B(x,\, s) := \sigma(x,\, s)\sigma_x(x,\, s) + (r - \delta^{\mathcal{O}})x.$$

If $v(x,\, t)$ is the option price at time t when the underlying price is x, then $\mathcal{D}v(x,\, t)$ is the option's *delta*, hence the designation "delta interpretation." Note that by the definition of B the corresponding auxiliary process (20) is *not* the risk neutralized underlying process.

(b) *Riskless position interpretation*

Define

$$\mathcal{D}v(x,\, t) := v(x,\, t) - xv_x(x,\, t)$$
$$\mathcal{D}g(x) := g(x) - xg'(x)$$
$$\delta := r - \delta^{\mathcal{O}}$$
$$B(x,\, s) := \sigma(x,\, s)\sigma_x(x,\, s) - \sigma^2(x,\, s)/x + (r - \delta^{\mathcal{U}})x.$$

If $v(x,\, t)$ is the option price at time t when the underlying price is x, then $\mathcal{D}v(x,\, t)$ is the number of dollars invested in the riskless asset in the portfolio that replicates the option, hence the designation "riskless position interpretation." Again, in general, the corresponding auxiliary process (20) is *not* the risk neutralized process.

For both interpretations, if the continuation region is below the exercise region, it will be assumed that B and σ in (20) are such that zero is unattainable by the auxiliary process.

PROPOSITION 7. *For both the delta and the riskless position interpretations* [$\mathcal{D} := \dfrac{\partial}{\partial x}$ and $\mathcal{D} := 1 - x\dfrac{\partial}{\partial x}$, respectively]: *let* $v(x,\, t)$ *be the solution to the Black-Merton-Scholes PDE (17) with end and boundary conditions (18) and (19). Then at any point* (x, t) *in the continuation region, the following holds for an American option,*

$$\begin{aligned}\mathcal{D}v(x,\, t) = {} & E^{x,\, t}\left\{\mathrm{e}^{-\delta(\tau - t)}\mathcal{D}g\big(x_f(\tau)\big)\mathbf{1}_{\tau<T}\right\} \\ & + \mathrm{e}^{-\delta(T - t)}E^{x,\, t}\left\{\mathcal{D}g(X_T)\mathbf{1}_{\tau=T}\right\},\end{aligned} \tag{21}$$

and the following holds for a European option

$$\mathcal{D}v(x,\, t) = \mathrm{e}^{-\delta(T - t)}E^{x,\, t}\left\{\mathcal{D}g(X_T)\right\}, \tag{22}$$

where the expectation is taken w.r.t. the auxiliary process (20).

Under the delta interpretation (with the respective form of the B function), Proposition 7 states that the deltas of both an American and a European[12] options are equal to the discounted expectation of their deltas on the boundaries of the continuation regions, where the expectation is taken at time t with respect to the auxiliary process X which starts then at x, and where the discounting is at a rate that is equal to the difference between the dividend yields on the underlying and on the option. For example, writing equation (22) explicitly yields that a European option's delta is

$$\Delta(x,\, t) := v_x(x,\, t) = \mathrm{e}^{-(\delta^{\mathcal{U}} - \delta^{\mathcal{O}})(T - t)}E^{x,\, t}\left\{g'(X_T)\right\}, \tag{23}$$

where X satisfies $dX_s = [\sigma(X_s,\, s)\sigma_x(X_s,\, s) + (r - \delta^{\mathcal{O}})X_s]ds + \sigma(X_s, s)dW_s$; $t \le s \le T$, $X_t = x$. Because of the drift, X is generally *not* the risk-neutralized process of the underlying.

[12] The European part of Proposition 7 was proven in Bergman (1983).

Under the alternative interpretation, Proposition 7 states—for American and European options—that the riskless position is equal to the discounted expectation of the riskless position on the boundaries of the continuation regions. For a European option, equation (22) yields

$$v(x,\,t) - xv_x(x,\,t) = \mathrm{e}^{-(r\,-\,\delta^{\mathcal{O}})(T\,-\,t)}E^{x,\,t}\{g(X_T) - xg'(X_T)\}, \tag{24}$$

where the auxiliary process X satisfies $dX_s = [\sigma(X_s, s)\sigma_x(X_s, s) - \sigma^2(X_s, s)/X_s + (r - \delta^{\mathcal{O}})X_s]ds + \sigma(X_s, s)dW_s;\ t \le s \le T,\ X_t = x$. Again, X is usually *not* the risk-neutralized process of the underlying.

The proof is constructive, and can be used in different applications, including the propositions below about option *rho* and *theta*. It will therefore be given in the form of an algorithm. The general idea is the following. In order to analyze a transformation $\mathcal{D}v$ of the price function v, where the latter is the solution to a known parabolic PDE (like the Black-Merton-Scholes equation), it is very useful to represent $\mathcal{D}v$ as an expectation of a known function of a random variable with a known distribution. This can be accomplished if another parabolic PDE can be found which $\mathcal{D}v$ satisfies, because then $\mathcal{D}v$ can be represented as a probabilistic solution of that other PDE, ie, as an expectation. To find this other PDE, apply the operator $\mathcal{D}$ to the original PDE and get a second PDE in the function v. Then try to interchange the order of the operator $\mathcal{D}$ throughout that second PDE, "pushing" it to the right until it ends up just to the left of all occurrences of the function v in that PDE. If this can be done, you've got a *third* PDE with $\mathcal{D}v$ playing the role of the unknown function. In most cases, this third PDE would turn out to be parabolic, because the original PDE is. This method has already been applied in the proof of Proposition 5.

Proof: A proof in the form of a structured algorithm is given in the Appendix.

Since $\mathcal{L}_{\text{BMS}}$ is itself parabolic, Proposition 7 holds for a third interpretation as well, in which $\mathcal{D}$ is the identity operator. In that case, $B(x,\,t) = (r - \delta^{\mathcal{U}})x,\ C(x,\,t) = (\delta^{\mathcal{O}} - r) = \delta$, and $H(x,\,t) = 0$. The auxiliary process X *is* then the risk-neutralized underlying price process (adjusted for the option's dividend payout) satisfying,

$$dX_s = [(r - \delta^{\mathcal{O}})X_s]ds + \sigma(X_s, s)dW_s; \quad t \le s \le T, \ \ X_t = x, \tag{25}$$

and the probabilistic representation of the solution is none other than the risk-neutral valuation of the option.

The bounds in the next proposition follow almost directly from the probabilistic representations of Proposition 7.

PROPOSITION 8. *For both the delta and the riskless position interpretations, denote* $\underline{\mathcal{D}g} := \inf_{z\in\mathbb{R}} \mathcal{D}g(z)$ and $\overline{\mathcal{D}g} := \sup_{z\in\mathbb{R}} \mathcal{D}g(z)$.

For an American type claim,

(i) *If* $0 \le \underline{\mathcal{D}g} \le \overline{\mathcal{D}g}$, *then* $e^{-\delta(T-t)}\,\underline{\mathcal{D}g} \le \mathcal{D}v(x,t) \le \overline{\mathcal{D}g}$,

(ii) *If* $\underline{\mathcal{D}g} < 0 \le \overline{\mathcal{D}g}$, *then* $\underline{\mathcal{D}g} \le \mathcal{D}v(x,t) \le \overline{\mathcal{D}g}$,

(iii) *If* $\underline{\mathcal{D}g} \le \overline{\mathcal{D}g} < 0$, *then* $\underline{\mathcal{D}g} \le \mathcal{D}v(x,t) \le e^{-\delta(T-t)}\,\overline{\mathcal{D}g}$,

where the inequalities on the RHS hold at any (x,t) *in the continuation region.*[13]

For a European type claim,

$$e^{-\delta(T-t)}\,\underline{\mathcal{D}g} \le \mathcal{D}v(x,t) \le e^{-\delta(T-t)}\,\overline{\mathcal{D}g}. \tag{26}$$

Proof: See the Appendix.

An American call's delta corresponds to case (i) of the proposition. It implies that this delta is always bounded between zero and one, irrespective of the dividend yields on the underlying and on the call. An American put's delta corresponds to case (ii); its delta is bounded between zero and negative one. In particular, Proposition 7 implies that monotonicity w.r.t. the underlying of

[13] In the exercise region, the option value function is identical to the payoff function. Obviously, the properties of the former are those of the latter.

both American and European contingent claim prices is a property inherited from the monotonicity of their payoff function.[14]

Under the riskless position interpretation, case (ii) applies to an American call implying that the number of dollars invested in the riskless asset in the portfolio that replicates it is always bounded between zero and minus the exercise price. For an American put, case (i) translates to a similar boundedness between zero and the exercise price. It is easy to verify that if the underlying follows a geometric Brownian motion, then in all these examples the bounds are the tightest possible.

Bergman, Grundy, and Wiener (1996) have shown that convexity (concavity) in the underlying price is a European option price function property inherited from the convexity (concavity) of its payoff function. Using different methods, El Karoui, Jeanblanc-Picqué, and Shreve (1996) have shown that the convexity inheritance is true of an American option as well. This result can also be derived using the methods of the current paper. In fact, defining the operator $\mathcal{D}v(x,\, t) :=$ $v_{xx}(x,\, t)$ and $\mathcal{D}g(x) := g''(x)$, and using the same algorithm as in the proof of Proposition 7, it is straightforward to show that option's gamma can be represented probabilistically as

$$v_{xx}(x,\, t) = \ E^{x,\, t}\{\varphi_{t,\, \tau} v_{xx}(x_f(\tau),\, \tau)\mathbf{1}_{\tau<T}\}$$

$$+\, E^{x,\, t}\{\varphi_{t,T}\, g''(X_T)\mathbf{1}_{\tau=T}\}, \tag{27}$$

where $B(x,\, t) = 2\sigma(x,\, t)\sigma_x(x,\, t) + (r - \delta^{\mathcal{U}})x$, $C(x,\, t) = \partial[\sigma(x,\, t)\sigma_x(x,\, t)]/\partial x + (r + \delta^{\mathcal{U}} - 2\delta^{\mathcal{U}})$, and where $\varphi_{t,\, \tau} = \exp[\int_t^\tau C(X_s, s)ds]$. The quantity $v_{xx}(x_f(t),\, t)$ can be shown to be non-negative for all $t \in [0, T]$, if g is convex. Therefore, using (27), $v_{xx}(x,\, t)$, option's *gamma*, is non-negative for all (x, t) in the continuation region. Noteworthy, unlike convexity, concavity of an American type option price is generally *not* inherited from the payoff function. The reason is that the value of an American claim is derived by maximizing over stopping times, and the "max" function is convex.

[14] For European options, this has been shown in Bergman (1983) and in Bergman, Grundy, and Wiener (1996). The European results are given here for completeness.

For asymptotic results concerning option prices and transformations thereof, the following lemma is needed. It states that if a "well-behaved" diffusion is started at ever increasing levels, the probability that it reaches *below* a fixed value in finite time tends to zero. (Somewhat picturesquely, after smoke is puffed out from ever increasing altitudes, a ground-level measurement of smoke concentration will detect vanishingly decreasing quantities.) The lemma is then used to deduce that when a diffusion is started out at a high altitude, the value of an expectation of a function of its level at any subsequent time gets almost all of its contributions from high altitude realizations of that diffusion.

LEMMA 9. *Let* $X^{x,t}$ *be the solution to* $dX_s = B(X_s, s)ds + \sigma(X_s, s)dW_s$; $t \le s \le T'$, $X_t = x$, *where* B *and* σ *are functions with suitable regularity properties* (footnote 3). *Then* $\forall T \in [t, T']$, $\forall a$: $\lim_{x\uparrow\infty} Pr\left[X_T^{x,t} \le a\right] = 0$.

The proof of the lemma is technical and uninformative; it is therefore omitted. The asymptotic result in the next proposition follows from the probabilistic representations of Proposition 7 and from Lemma 9.

PROPOSITION 10. *Provided that* $\lim_{z\uparrow\infty}\mathcal{D}g(z)$ *exists, the following equalities are true for a European type option for both the delta and the riskless position interpretations*

$$\forall t \in [0, T]:\ e^{\delta(T-t)}\lim_{x\uparrow\infty}\mathcal{D}v(x, t) = \lim_{x\uparrow\infty}E^{x,t}\left\{\mathcal{D}g(X_T)\right\} = \lim_{z\uparrow\infty}\mathcal{D}g(z), \tag{28}$$

The equalities are also true for an American type option, provided that the continuation region is above the exercise region (eg, put). If the exercise region is above the continuation region (eg, call), then (28) is trivially true with $\delta = 0$.

Proof: See the Appendix.

The following corollary is an immediate consequence of Proposition 10 under the delta interpretation.

COROLLARY 11. *At any time before expiration, the delta of a put option, either an American or a European, written on a stochastic volatility underlying asset, tends to zero as the underlying price increases without bound. Formally,* $\forall t \in [0, T]$: $\lim_{x\uparrow\infty} p_x(x,\ t) = 0$, *where* p *is the put's price. Similarly, the delta of a European call written on a stochastic volatility underlying asset tends to* $\exp[-(\delta^{\mathcal{U}} - \delta^{\mathcal{O}})(T-t)]$, *and that of an American call tends to* 1.

In the next proposition the operator $\mathcal{D} := \partial/\partial r$ (partial derivative w.r.t. the interest rate) is applied to the Black-Merton-Scholes PDE to produce an expectation representation of an option's *rho* $[\rho(x,t) := v_r(x,t;\, r)]$.[15]

PROPOSITION 12. *The rho of a European option, with payoff function* $g(x)$ *and with dividend rate* $\delta^{\mathcal{O}}$, *that is written on a stochastic volatility underlying asset that pays a dividend rate* $\delta^{\mathcal{U}}$, *is equal to the time to expiration multiplied by the negative of the price of a European option also paying a dividend rate* $\delta^{\mathcal{O}}$ *but with a payoff function* $g(x) - xg'(x)$ *written on the same underlying asset,*

Formally, At any point (x,t) *in* $\mathbb{R}_+ \times [0,T]$,

$$\rho(x,t) = -(T-t)\mathrm{e}^{-(r-\delta^{\mathcal{O}})(T-t)}E^{x,t}\big[g(X_T) - X_T g'(X_T)\big], \tag{29}$$

where X_T *is the random value, as of time* T, *of the risk-neutralized underlying (auxiliary) process (25).*

Therefore, rho always lies in the interval

$$\Big((t-T)\mathrm{e}^{-(r-\delta^{\mathcal{O}})(T-t)}\sup_{z\in\mathbb{R}}[g(z)-zg'(z)], (t-T)\mathrm{e}^{-(r-\delta^{\mathcal{O}})(T-t)}\inf_{z\in\mathbb{R}}[g(z)-zg'(z)]\Big), \tag{30}$$

and,

$$\forall t\in[0,T]:\ \lim_{x\uparrow\infty}\rho(x,t) = -(T-t)\mathrm{e}^{-(r-\delta^{\mathcal{O}})(T-t)}\lim_{z\uparrow\infty}[g(z) - zg'(z)]. \tag{31}$$

[15] One cannot naively compute rho by taking the partial w.r.t. the interest rate "under the expectation sign" in the risk-neutralized representation of the price function, because a small change in the interest rate also affects the auxiliary (risk neutralized) process.

If $g''(x)$ *maintains a uniform sign on* $\mathbb{R}_+$ (eg, call or put), then $\frac{\partial \rho}{\partial x}$ (which is equal to $\frac{\partial \Delta}{\partial r}$) *inherits that sign on* $\mathbb{R}_+ \times [0, T]$.

Proof: See the Appendix.

Corollary 13 below gives the immediate implications of Proposition 12 to calls and puts. In the following, $\mathcal{H}$ denotes the Heavyside function that maps the negatives to zero and the non-negatives to one.

COROLLARY 13. (i) *The rho of a European call written on a stochastic volatility underlying asset is equal to the price of a digital call with the same exercise price, the same time to expiration, and the same underlying asset as the call, multiplied by the time to expiration and by the exercise price.*

Formally, at any point (x, t) *in* $\mathbb{R}_+ \times [0, T]$,

$$\begin{aligned} \rho_{\text{call}}(x, t) &= (T - t)K\mathrm{e}^{-(r - \delta^{\mathcal{O}})(T - t)}E^{x,t}\big[\mathcal{H}(X_T - K)\big] \\ &= (T - t)K \cdot (\textit{Price of a digital call}), \end{aligned}$$

and,

$$\begin{aligned} \rho_{\text{put}}(x, t) &= -(T - t)K\mathrm{e}^{-(r - \delta^{\mathcal{O}})(T - t)}E^{x,t}\big[\mathcal{H}(K - X_T)\big] \\ &= -(T - t)K \cdot (\textit{Price of a digital put}), \end{aligned}$$

where X_T *is the random value, as of time* T, *of the risk-neutralized underlying process (25).*

(ii) *The rho of a call option and that of a put are bounded:*

$$0 \le \rho_{\text{call}}(x, t) \le (T - t)\mathrm{e}^{-(r - \delta^{\mathcal{O}})(T - t)}K,$$

$$-(T - t)\mathrm{e}^{-(r - \delta^{\mathcal{O}})(T - t)}K \le \rho_{\text{put}}(x, t) \le 0.$$

(iii) *Asymptotically,*

$$\forall t \in [0, T]\text{: } \lim_{x \uparrow \infty} \rho_{\text{call}}(x, t) = (T - t)\mathrm{e}^{-(r - \delta^{\mathcal{O}})(T - t)}K,$$

$$\forall t \in [0, T]\text{: } \lim_{x \uparrow \infty} \rho_{\text{put}}(x, t) = 0.$$

Proof: A direct application of Proposition 12. □

The next proposition provides an expectation representation of *theta*, the time derivative of the price of a contingent claim [$\Theta(x,\,t) := v_t(x,\,t)$], which is then used to derive asymptotic results of a call's theta. For simplicity, it is assumed that the volatility parameter depends only on the underlying price but not on time, ie, $\sigma = \sigma(x)$.

PROPOSITION 14. *Let the underlying asset pay no dividend, and let its volatility parameter be time independent* [$\sigma = \sigma(x)$]. *Then at any point* $(x,\,t)$ *in* $\mathbb{R}_+ \times [0, T]$, *the theta of a European option with payoff function* g *can be represented as*

$$\Theta(x, t; T) = \mathrm{e}^{-r(T\,-\,t)} E^{x,\,t}\big\{-\tfrac{1}{2}\sigma^2(X_T)g''(X_T) + r\big[g(X_T) - X_T\, g'(X_T)\big]\big\}, \tag{32}$$

where X_T *is the random value, as of time* T, *of the risk-neutralized underlying process.*

In particular, the theta of a call option on such an underlying is given by

$$\Theta_{\text{call}}(x,\,t; T) = -\mathrm{e}^{-r(T\,-\,t)}\big[\tfrac{1}{2}\sigma^2(K)\phi(K,\,T;\,x,\,t) + rK\int_K^\infty \phi(y,\,T;\,x,\,t)dy\big], \tag{33}$$

where $\phi(\cdot,\,T;\,x,\,t)$ *is the probability density of* $X_s^{x,t}$.[16]

Therefore, (i) *a call's theta is always nonpositive;* (ii)

$$\forall t \in [0, T)\colon \lim\nolimits_{x\uparrow\infty}\Theta_{\text{call}}(x, t; T) = -\mathrm{e}^{-r(T\,-\,t)} rK; \tag{34}$$

and (iii) *if zero is an unattainable boundary of the underlying risk-neutralized process, then* $\forall t \in [0, T)\colon \Theta_{\text{call}}(0, t;\, T) = 0$.

Furthermore,

$$\forall x \in \mathbb{R}_+ : \lim\nolimits_{(T\,-\,t)\uparrow\infty}\Theta_{\text{call}}(x, t;\, T) = 0; \tag{35}$$

[16] In other words, $\phi(y,\,T;\,x,\,t)$ is the transition density function of the risk-neutral process from level x at time t, to level y at time T.

$$\lim_{(T-t)\downarrow 0} \Theta_{\text{call}}(x,t;T) = \begin{cases} 0 & x < K \\ -\infty & x = K \\ -rK & K < x \end{cases} \tag{36}$$

Proof: See the Appendix.

Expectedly, the results in (34), (35), and (36) which are true for an underlying asset with time independent, stochastic volatility, are consistent with Figures 14.5 and 14.6 in Hull (1997), which were drawn for the special case of a geometric Brownian motion.

5 Restrictions on prices of Asian options

The methodology used above generalizes readily to more than one dimension. As an illustration I'll consider Asian contingent claims that depend on the average of the underlying price taken over some time interval. Formally, define $I_t := \int_0^t S_\tau d\tau$, where $\{S_t;\ 0 \le t \le T\}$ is the process defined in (15). Note that since the underlying can generally take on negative values—as when it is the value of a forward contract—so can its time integral, I_t. An Asian claim is a contract that promises to pay $g(S_T,\ I_T)$ dollars at expiration time T, where g is some real function on $\mathbb{R}^2$. Examples include the *average strike call*, $g(S_T,\ I_T) = (S_T - I_T/T)^+$ and the *average rate call*, $g(S_T,\ I_T) = (I_T/T - K)^+$.

Bergman (1981, 1985) has derived a PDE which has to be satisfied by prices of general path-contingent claims. In particular, he showed that the no-arbitrage value of an Asian option at any time $t \in [0, T)$ is $v(S_t, I_t, t)$, where $v(x, i, t)$ is the solution to the parabolic PDE

$$\mathcal{L}_{\text{Asian}} v(x, i, t) = 0, \tag{37}$$

defined on $\mathbb{R}^2 \times [0, T]$ with end condition

$$v(x, i, T) = g(x,\ i) \text{ for } (x,\ i) \in \mathbb{R}^2, \tag{38}$$

and where the parabolic differential operator $\mathcal{L}_{\text{Asian}}$ is defined (assuming zero dividends) by

$$\mathcal{L}_{\text{Asian}} v(x,i,t) := \mathcal{L}_{\text{BMS}} v(x,i,t) + x v_i(x,i,t)$$

$$= v_t(x,i,t) + \tfrac{1}{2}\sigma^2(x,t) v_{xx}(x,i,t) + r x v_x(x,i,t) + x v_i(x,i,t) - r v(x,i,t). \tag{39}$$

The next proposition brings the methodology developed herein to bear on PDE (37) to produce an expectation representation of the first partial of the price of an Asian claim w.r.t. the underlying price (the hedge ratio or number of underlying units in the replicating portfolio), and to produce an expectation representation of the first partial w.r.t. the updated time average of the underlying.

PROPOSITION 15. *The following representations hold for an Asian claim that is written on a stochastic volatility underlying asset.*

$$v_2(x,i,t) = E^{x,\,i,\,t}\left\{ \mathrm{e}^{-r(T-t)} g_2(X_T, J_T) \right\}, \tag{40}$$

$$v_1(x,i,t) = E^{x,\,i,\,t}\left\{ \int_t^T v_2(X_s, J_s, s)\,ds + g_1(X_T, J_T) \right\}, \tag{41}$$

where X is the risk-neutralized underlying process; where $J_t := \int_0^t X_\tau d\tau$; and where $E^{x,\,i,\,t}$ denotes expectation given that $X_t = x$ and that $J_t = i$. These representations imply that at any point $(x,i,t) \in \mathbb{R}^2 \times [0,T]$, the following bounds hold for an Asian claim.

$$\mathrm{e}^{-r(T-t)} \inf_{(y,j)\in\mathbb{R}^2} g_2(y,j) \le v_2(x,i,t) \le \mathrm{e}^{-r(T-t)} \sup_{(y,j)\in\mathbb{R}^2} g_2(y,j)\ , \tag{42}$$

$$\inf_{(y,j)\in\mathbb{R}^2} g_1(y,j) + \frac{1-e^{-r(T-t)}}{r} \inf_{(y,j)\in\mathbb{R}^2} g_2(y,j) \le$$

$$v_1(x,i,t) \le$$

$$\sup_{(y,j)\in\mathbb{R}^2} g_1(y,j) + \frac{1-e^{-r(T-t)}}{r} \sup_{(y,j)\in\mathbb{R}^2} g_2(y,j)\ . \tag{43}$$

The proof uses the same principles as the previous proofs, and is therefore omitted.

The implications of Proposition 15 to specific Asian options are given in the next corollary.

COROLLARY 16. (i) *For the average strike call option,* $g(S_T, I_T) = (S_T - I_T/T)^+$, *the following bounds hold at any point* $(x, i, t) \in \mathbb{R}^2 \times [0, T]$

$$-[1 - \mathrm{e}^{-r(T - t)}]/rT \le v_1(x, i, t) \le 1,$$

$$-\mathrm{e}^{-r(T - t)}/T \le v_2(x, i, t) \le 0.$$

(ii) *For the average rate call option,* $g(S_T, I_T) = (I_T/T - K)^+$, *that is written on a stochastic volatility underlying asset, the following bounds hold at any point* $(x, i, t) \in \mathbb{R}^2 \times [0, T]$

$$0 \le v_1(x, i, t) \le [1 - \mathrm{e}^{-r(T - t)}]/rT,$$

$$0 \le v_2(x, i, t) \le \mathrm{e}^{-r(T - t)}/T,$$

and at any $(x, i) \in \mathbb{R}^2$: $\lim_{t\uparrow T} v_1(x, i, t) = 0$.

Proof: A direct application of Proposition 15. □

6 Conclusions

It was shown that in a diffusive one-factor model of the term structure, the prices of bonds and of term structure puts decrease as the short-term interest rate increases. However, these prices need not be monotone in the short-term rate, if that rate can experience jumps.

An important comparative statics implication is that to a higher short-term interest rate corresponds a yield curve that lies uniformly above the curve that corresponds to a lower short-term rate. Furthermore, if the diffusion that describe the short-term rate is also homogeneous, then two yield curves that are measured at different dates cannot intersect when drawn from the same time origin. If empirically they do intersect, then the short-term rate *cannot* be described by a one-factor homogeneous diffusion.

It was also shown that if the second partial derivative w.r.t. to the short-term interest rate of the drift of the one-factor diffusion describing that rate is less than or equal to 2—special cases being the linear drift models—then the prices of deterministic-coupon bonds and term structure puts are convex in that rate.

The last result was derived using probabilistic representations of solutions to parabolic partial differential equations. The same methodology was used to derive restrictions on prices of European, American, and Asian options when the underlying price follows a stochastic volatility diffusion. Bounds, asymptotic results, and representations were derived for different linear differential transformations of derivative price functions like option's delta, rho, and theta. An example from these results is the fact that the rho of a European call written on a stochastic volatility underlying asset is equal to the price of a digital call with the same exercise price, the same time to expiration, and the same underlying asset as the call, multiplied by the time to expiration and by the exercise price.

The methodology used in this paper was described in sufficient detail to allow for its ready application in a variety of situations.

Appendix

Proof of Proposition 1.

That $x = \infty$ is formally equivalent to $\overline{\tau} = \infty$, so the second term in (7) is zero. That $x = 0$ is also an unattainable boundary is formally equivalent to $\underline{\tau} = \infty$. In that case, $\mathbf{1}_{T \le \min(\underline{\tau}, \overline{\tau})} = 1$ almost surely; the second and third terms in (7) are equal to zero; and $\min(\underline{\tau},\ \overline{\tau},\ T) = T$. The result follows.

When $x = 0$ is an absorption barrier, noting that $\exp\big[\int_{\underline{\tau}}^{s} -X_\lambda d\lambda\big] = 1$, (7) becomes

$$
\begin{aligned}
v(x,t) \;=\;\; & E^{x,\,t}\Big\{\exp\Big[\int_t^T -X_\lambda d\lambda\Big]\, g(X_T)\mathbf{1}_{T \le \underline{\tau}}\Big\} \\
& + E^{x,\,t}\Big\{\exp\Big[\int_t^{\underline{\tau}} -X_\lambda d\lambda\Big]\Big[g(0) + \int_{\underline{\tau}}^{T} h(0,s)ds\Big]\,\mathbf{1}_{\underline{\tau}<T}\Big\} \\
& + E^{x,\,t}\Big\{\int_t^{\min(\underline{\tau},T)} \exp\Big[\int_t^{s} -X_\lambda d\lambda\Big] h(X_s, s)ds\Big\} \\
\\
= \;\; & E^{x,\,t}\Big\{\exp\Big[\int_t^T -X_\lambda d\lambda\Big]\, g(X_T)\Big\} \\
& + E^{x,\,t}\Big\{\exp\Big[\int_t^{\underline{\tau}} -X_\lambda d\lambda\Big]\Big[\int_{\underline{\tau}}^{T} h(0,s)ds\Big]\,\mathbf{1}_{\underline{\tau}<T}\Big\} \\
& + E^{x,\,t}\Big\{\int_t^{\min(\underline{\tau},T)} \exp\Big[\int_t^{s} -X_\lambda d\lambda\Big] h(X_s, s)ds\big(\mathbf{1}_{\underline{\tau}<T} + \mathbf{1}_{\underline{\tau}\ge T}\big)\Big\} \\
\\
= \;\; & E^{x,\,t}\Big\{\exp\Big[\int_t^T -X_\lambda d\lambda\Big]\, g(X_T)\Big\} \\
& + E^{x,\,t}\Big\{\int_{\underline{\tau}}^{T} \exp\Big[\int_t^{s} -X_\lambda d\lambda\Big]\big[h(0,s)ds\big]\,\mathbf{1}_{\underline{\tau}<T}\Big\} \\
& + E^{x,\,t}\Big\{\int_t^{\underline{\tau}} \exp\Big[\int_t^{s} -X_\lambda d\lambda\Big] h(X_s, s)ds\,\mathbf{1}_{\underline{\tau}<T}\Big\} \\
& + E^{x,\,t}\Big\{\int_t^{T} \exp\Big[\int_t^{s} -X_\lambda d\lambda\Big] h(X_s, s)ds\,\mathbf{1}_{\underline{\tau}\ge T})\Big\}.
\end{aligned}
$$

This immediately yields (12). □

Proof of Proposition 5.

It will prove useful to denote by ∂_x, ∂_{xx}, and ∂_t the operations of taking partial derivatives in the obvious way. Take the partial derivative of the PDE (9), then interchange the order of partial differentiation as necessary to get

$$\begin{aligned}&\partial_t v_x(x,t) + \tfrac{1}{2}\sigma^2(x,\, t)\partial_{xx} v_x(x,t) + [\sigma(x,s)\sigma_x(x,s) + \mu(x,s)]\partial_x v_x(x,t)\\ &\quad + [\mu_x(x,s) - x]v_x(x,t) - v(x,t) + h_x(x,s) = 0, \quad \text{in } \mathbb{R}_{++}\times[0,T).\end{aligned} \tag{A1}$$

Differentiating the end condition (10) w.r.t. x as well yields the end condition for (A1):

$$v_x(x,T) = g'(x), \quad \text{on } \mathbb{R}_{++}. \tag{A2}$$

Treating $v(x,t)$ as a known function, equation (A1) is a parabolic PDE that must be satisfied by $v_x(x,\, t)$ with end condition (A2). Since, by assumption, $x = 0$ is an unattainable boundary for (A1), and therefore specification of a boundary condition there, is neither possible nor needed. Therefore, the solution to (A1) and (A2) can be represented as in (13). The probabilistic representation (14) is proven similarly by taking yet another derivative w.r.t. x of (A1) and its end condition (A2). $\square$

Proof of Proposition 7.

The proof is described as an algorithm in four steps.

Step 1. Apply the operator $\mathcal{D}$ to the Black-Merton-Scholes PDE (17) to get the PDE $\mathcal{D}\mathcal{L}_{\text{BMS}}v(x,\, t) = 0$ on the option's continuation region $\mathcal{C}$. Apply $\mathcal{D}$ to both sides of the end condition (18) to get

$$\mathcal{D}v(x,T) = \mathcal{D}g(x),\ x\in\mathbb{R}, \tag{A3}$$

and using (19), get that

$$\mathcal{D}v(x_f(t),t) = \mathcal{D}g(x_f(t)) \text{ for } t\in[0,T]. \tag{A4}$$

Step 2. Rewrite $\mathcal{D}\mathcal{L}_{\mathrm{BMS}}v(x,\, t)$ as $\mathcal{P}\mathcal{D}v(x,\, t)$, interchanging the order of derivatives where necessary.[17] In fact, $\mathcal{P}$ is *identified* as that operator which satisfies the identity $\mathcal{D}\mathcal{L}_{\mathrm{BMS}}v(x,\, t) = \mathcal{D}v(x,t)$ on $\mathcal{C}$. This last identity and $\mathcal{D}\mathcal{L}_{\mathrm{BMS}}v(x,t) = 0$ imply that

$$\mathcal{P}\mathcal{D}v(x,t) = 0 \text{ on } \mathcal{C}. \tag{A5}$$

Denoting $u(x,t) := \mathcal{D}v(x,t)$ and $f(x) := \mathcal{D}g(x)$, equations (A5), (A3), and (A4) can be rewritten as

$$\mathcal{P}u(x,t) = 0 \text{ on } \mathcal{C} \tag{A6}$$

$$u(x,T) = f(x),\ x \in \mathbb{R}, \tag{A7}$$

$$u\big(x_f(t),t\big) = f\big(x_f(t)\big),\ t \in [0,T]. \tag{A8}$$

This means that the function $\mathcal{D}v(x,\, t)$ is a solution of PDE (A6) with end condition (A7) and boundary condition (A8).

Step 3. Check whether $\mathcal{P}$ is a *parabolic* differential operator, ie, check if it is of the form

$$\begin{aligned}\mathcal{P}u(x,t) := \;& u_t(x,t) + \tfrac{1}{2}\sigma^2(x,\, t)u_{xx}(x,t) \\ & + B(x,t)u_x(x,t) + C(x,t)u(x,t) + H(x,t).\end{aligned}$$

Step 4. If, indeed, $\mathcal{P}$ is parabolic, then, under regularity conditions,[18] the results described in Section 2 can be applied to write the probabilistic solution to the problem (A6) , (A7), (A8) in the form

$$\begin{aligned}u(x,t) \;=\; & E^{x,\,t}\big\{\varphi_{t,\tau}f\big(x_f(\tau)\big)\mathbf{1}_{\tau<T}\big\} \\ & + E^{x,\,t}\big\{\varphi_{t,T}f(X_T)\mathbf{1}_{\tau=T}\big\} \\ & + E^{x,\,t}\Big\{\int_t^\tau \varphi_{t,s}H(X_s,s)ds\Big\}, \quad (x,t) \in \mathcal{C}\end{aligned} \tag{A9}$$

[17] It is assumed that all the relevant (mixed) partial derivatives of v are continuous, so that interchanging the order of the partials is allowed.

[18] The regularity conditions [(see Duffie (1996, p.295)] require that all of σ, B, C, H, f, and u are continuous; the solution u, and the functions σ, B, H and f satisfy a polynomial growth condition in x; C is nonnegative; and σ and B are also Lipschitz in x.

where X is the auxiliary process

$$dX_s = B(X_s, s)ds + \sigma(X_s, s)dW_s, \quad t \le s \le T, \quad X_t = x,$$

where $\{W_t : 0 \le t \le T\}$ is a one-dimensional Brownian motion, and where $\varphi_{t,\,s} := \exp[\int_t^s C(X_\lambda, \lambda) d\lambda]$.

Indeed, $\mathcal{P}$ turns out to be parabolic under both interpretations. This is usually the case, since $\mathcal{L}_{\text{BMS}}$ is itself parabolic and $\mathcal{D}$ is a linear differential operator. For example, under the riskless position interpretation $\mathcal{P}$ *is* parabolic with $B(x,\,t) = \sigma(x,\,t)\partial_x\sigma(x,\,t) - \sigma^2(x,\,t)/x + (r - \delta^{\mathcal{U}})x]$, $C(x,\,t) = \delta^{\mathcal{O}} - r$, and $H(x,\,t) = 0$. Substituting these and $u = \mathcal{D}v$ and $f = \mathcal{D}g$ in (A9) yields (21). For a European option, early exercise is impossible, which formally means *Pr*$[\mathbf{1}_{\tau<T} = 1] = 0$, and therefore the first term in (A9) is also zero. This yields (22). □

Proof of Proposition 8.

Using (21) and the assumptions that $0 \le \delta$ and $0 \le \underline{\mathcal{D}g}$ yields

$$\mathcal{D}v(x,t) \ge \underline{\mathcal{D}g}\, E^{x,\,t}\{\mathrm{e}^{-\delta(\tau - t)}\mathbf{1}_{\tau<T} + \mathrm{e}^{-\delta(T - t)}\mathbf{1}_{\tau=T}\}$$

$$\ge \underline{\mathcal{D}g}\, E^{x,\,t}\{\mathrm{e}^{-\delta(\tau - t)}\mathrm{e}^{-\delta(T - \tau)}\mathbf{1}_{\tau<T} + \mathrm{e}^{-\delta(T - t)}\mathbf{1}_{\tau=T}\}$$

$$\ge \mathrm{e}^{-\delta(T - t)}\underline{\mathcal{D}g}\, E^{x,\,t}\{\mathbf{1}_{\tau<T} + \mathbf{1}_{\tau=T}\} = \mathrm{e}^{-\delta(T - t)}\underline{\mathcal{D}g}\,. \square$$

This proves the left-hand inequality of (i). The other inequalities are proven similarly. □

Proof of Proposition 10.

The proof will be given for a European option. Denote $L := \lim_{z\uparrow\infty}\mathcal{D}g(z)$ and let $\Phi_T^{x,t}$ be the distribution function of $X_T^{x,t}$, the value of the auxiliary process at time T. By definition of L as a limit, $\forall\varepsilon > 0$, $\exists a_\varepsilon$ such that $\forall y > a_\varepsilon$, $|\mathcal{D}g(z) - L| < \varepsilon$. Therefore, using (22), for a given ε,

$$\inf_{z\in(-\infty,a_\varepsilon]} \mathcal{D}g(z)\int_{-\infty}^{a_\varepsilon} d\Phi_T^{x,t}(y) + (L-\varepsilon)\int_{a_\varepsilon}^{\infty} d\Phi_T^{x,t}(y)$$

$$\leq \int_{-\infty}^{a_\varepsilon} \mathcal{D}g(y)d\Phi_T^{x,t}(y) + \int_{a_\varepsilon}^{\infty} \mathcal{D}g(y)d\Phi_T^{x,t}(y)$$

$$= E^{x,\,t}\{\mathcal{D}g(X_T)\} = \mathrm{e}^{\delta(T\,-\,t)}\mathcal{D}v(x,\,t)$$

$$\leq \sup_{z\in(-\infty,a_\varepsilon]} \mathcal{D}g(z)\int_{-\infty}^{a_\varepsilon} d\Phi_T^{x,t}(y) + (L+\varepsilon)\int_{a_\varepsilon}^{\infty} d\Phi_T^{x,t}(y).$$

By Lemma 9, $\lim_{x\uparrow\infty}\int_{-\infty}^{a_\varepsilon} d\Phi_T^{x,t}(y) = 0$ and $\lim_{x\uparrow\infty}\int_{a_\varepsilon}^{\infty} d\Phi_T^{x,t}(y) = 1$. Assuming that the “inf” and the “sup” above are finite (the payoff function of most derivatives is bounded on any ray extending to the left), sending x to infinity yields

$$L-\varepsilon \leq \lim_{x\uparrow\infty}E^{x,\,t}\{\mathcal{D}g(X_T)\} = \mathrm{e}^{\delta(T\,-\,t)}\lim_{x\uparrow\infty}\mathcal{D}v(x,\,t) \leq L+\varepsilon,$$

which is true for any ε. Taking $\varepsilon\downarrow 0$ concludes the proof. When the “inf” and the “sup” above are not both finite, the proof needs to be modified by a suitable generalization of Lemma 9. The proof for the American case is similar to that of the European. □

Proof of Proposition 12.

Following the algorithm outlined in the proof of Proposition 7, apply the operator $\frac{\partial}{\partial r}$ (the first partial w.r.t the interest rate r) to the Black-Merton-Scholes PDE (17) (with dividend yields set to zero) to get

$$\frac{\partial}{\partial r}\mathcal{L}_{\text{BMS}}v(x,t) = 0. \tag{A10}$$

Recall that usually $v(x,t)$ depends on r as a parameter. This dependence is not shown for brevity. Rewrite (A10), bringing in the operator $\frac{\partial}{\partial r}$ to the immediate left of the v throughout, to get

$$\frac{\partial}{\partial t}v_r(x,t) \; + \tfrac{1}{2}\sigma^2(x,\,t)\frac{\partial^2}{\partial x^2}v_r(x,t)$$

$$+\,(r-\;\delta^{\mathcal{U}})x\,\frac{\partial}{\partial x}v_r(x,t) - (r-\delta^{\mathcal{O}})v_r(x,t) + \ell(x,t) = 0, \tag{A11}$$

where $\ell(x,t) := -[v(x,t) - x\partial_x v(x,t)]$. PDE (A11) is parabolic with $v_r(x,t)$—which is rho, by definition—as the solution function. To get the end condition for this PDE, apply the $\frac{\partial}{\partial r}$ operator to the end condition (18) and get the end condition

$$v_r(x,T) = \frac{\partial}{\partial r}g(x) = 0 \text{ on } \mathbb{R}_+.$$

Reading the coefficient functions off PDE (A11) and using the end condition, yields the probabilistic representation

$$\rho(x,t) = \; v_r(x,t) = -E^{x,\,t}\Big\{\int_t^T \mathrm{e}^{-(r\,-\,\delta^{\mathcal{O}})(s\,-\,t)}\big[v(X_s,s) - X_s v_x(X_s,s)\big]ds\Big\}$$

$$= -\int_t^T \mathrm{e}^{-(r\,-\,\delta^{\mathcal{O}})(s\,-\,t)}E^{x,\,t}\{E^{X_s,\,s}\mathrm{e}^{-(r\,-\,\delta^{\mathcal{O}})(T\,-\,s)}\big[g(X_T) - X_T g'(X_T)\big]ds\Big\}$$

$$= -E^{x,\,t}\big[g(X_T) - X_T g'(X_T)\big]\mathrm{e}^{-(r\,-\,\delta^{\mathcal{O}})(T\,-\,t)}\int_t^T ds$$

$$= -(T-t)\mathrm{e}^{-(r\,-\,\delta^{\mathcal{O}})(T\,-\,t)}E^{x,\,t}\big[g(X_T) - X_T g'(X_T)\big],$$

where result (24) was used to obtain the second line, and the law of iterated expectations – to obtain the third.

The bounds in (30) follow immediately from (29). Equation (31) follows from (28) and (29).

To get the last result of the proposition, take the partial derivative of both sides of (A10) w.r.t. x to get

$$\frac{\partial}{\partial x}\frac{\partial}{\partial r}\mathcal{L}_{\text{BMS}}v(x,t)$$

$$= \frac{\partial v_{xr}}{\partial t} + \frac{1}{2}\sigma^2(x,t)\frac{\partial^2 v_{xr}}{\partial x^2} + [rx + \sigma(x,t)\sigma_1(x,t)]\frac{\partial v_{xr}}{\partial x} + xv_{11}(x,t) = 0.$$

Similarly, the end condition for this PDE is $v_{xr}(x,T) = \dfrac{\partial}{\partial x}\dfrac{\partial}{\partial r}g(x) = 0$. Whence, the probabilistic solution is

$$\frac{\partial \Delta}{\partial r} = \frac{\partial \rho}{\partial x} = v_{xr}(x,t) = E^{x,\,t}\Big[\int_t^T X_s v_{11}(X_s,s)ds\Big], \tag{A12}$$

where X_s is the appropriate auxiliary process. As shown by Bergman (1983) and by Bergman, Grundy, and Wiener (1996), for European options with payoff function g, if $g''(x)$ maintains uniformly the same sign on $\mathbb{R}_+$, then the option's gamma, $v_{11}(x,s)$, maintains the same uniform sign on $\mathbb{R}_+ \times [0,T)$. Therefore, by (A12), so does $v_{xr}(x,t)$. □

Proof of Proposition 14.

Taking the partial derivative of the Black-Merton-Scholes PDE (17) w.r.t time yields $\mathcal{L}_{\text{BMS}}v(x,\,t)$ $\equiv \mathcal{L}_{\text{BMS}}v_2(x,\,t) = 0$, ie,

$$\mathcal{L}_{\text{BMS}}\Theta(x,\,t;\,T) = 0, \qquad \text{on } \mathbb{R}_+ \times [0,T). \tag{A13}$$

To get the end condition for (A13), note that (17) implies that $\Theta(x,\,T;\,T) = -\frac{1}{2}\sigma^2(x)\partial_{xx}v(x,\,T)$ $+\, r[v(x,\,T) - x\partial_{xx}v(x,\,T)]$. But since $v(x,\,T) = g(x)$, it follows that the end condition for PDE (A13) is

$$\Theta(x,\,T;\,T) = -\sigma^2(x)g''(x) + r[g(x) - xg'(x)], \qquad \text{on } \mathbb{R}_+. \tag{A14}$$

Therefore, (32) is the probabilistic representation of the solution to (A13) and (A14).

For a call option, $g(y) = (y-K)^+$, $g'(y) = \mathcal{H}(y-K)$, $g''(y) = \delta(y-K)$, and $g(y) - yg'(y)$ $= -K\mathcal{H}(y-K)$, where $\mathcal{H}$ is the Heavyside function and δ is Dirac's delta. Substituting these in (32) yields (33). By Lemma 9, $\lim_{x\uparrow\infty}\int_K^\infty \phi(y,\,T;\,x,\,t)dy = \lim_{x\uparrow\infty} Pr[X_T^{x,t} > K] = 1$, and $\lim_{x\uparrow\infty}\phi(K,$ $T;\,x,\,t) = 0$ (otherwise, the Lemma is contradicted). Noting this and taking $\lim_{x\uparrow\infty}$ of (33) yields (34). If zero is a boundary from which the underlying risk-neutralized process cannot reach positive values, then $\phi(y,\,T;\,0,\,t) = 0$, almost everywhere. Using this in (33) gives $\Theta_{\text{call}}(0,\,t;\,T)$ $= 0$.

For diffusions with "well behaved" $\sigma(x)$, the expression in square brackets in (33) is finite as long as $T - t > 0$. This observation yields the limit in (35).

Finally, observe that for regular diffusions, $\phi(y,\;T;\;x,\;T) = \delta(y\;-\;x)$ [Dirac's delta]. Therefore, by (33) $\lim_{(T-t)\downarrow 0}\Theta_{\text{call}}(x,\,t;\;T) = -\left[\frac{1}{2}\sigma^2(K)\delta(K\;-\;x) + \;rK\int_K^{\infty}\delta(y\;-\;x)dy\right]$, which immediately implies (36). $\square$

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