Black-Scholes Model for European vanilla options

Under the assumption

$$\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$$

The option value V=V(S,t) satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0. \quad S > 0 \ t \in [0,T)$$

$$V(S,T) = \begin{cases} (S-X)^+, & \text{for call option,} \\ (X-S)^+, & \text{for put option.} \end{cases}$$

Black-Scholes formulas for European vanilla options

$$V(S,t) = \begin{cases} Se^{-q(T-t)}N(d_1) - Xe^{-r(T-t)}N\left(d_2\right) & \text{for call option} \\ Xe^{-r(T-t)}N(-d_2) - Se^{-q(T-t)}N(-d_1) & \text{for put option} \end{cases}$$

where

$$d_1 = \frac{\log\frac{S}{X} + (r - q + \frac{\sigma^2}{2})(T-t)}{\sigma\sqrt{T-t}},\ d_2 = d_1 - \sigma\sqrt{T-t}$$

Pricing American vanilla options

$$
\begin{aligned}
\min\left\{-\frac{\partial V}{\partial t}-\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}-(r-q)S\frac{\partial V}{\partial S}+rV, V-\varphi\right\} &= 0,\ S>0,\ t\in[0,T) \\
V(S,T) &= \varphi^+
\end{aligned}
$$

Pricing exotic options under Black-Scholes framework

- Multi-asset options
- Barrier options
- Asian options
- Lookback options
- Forward start option, shout option, compound options

Implied volatility

The value for volatility that makes the theoretical option value and the market price the same

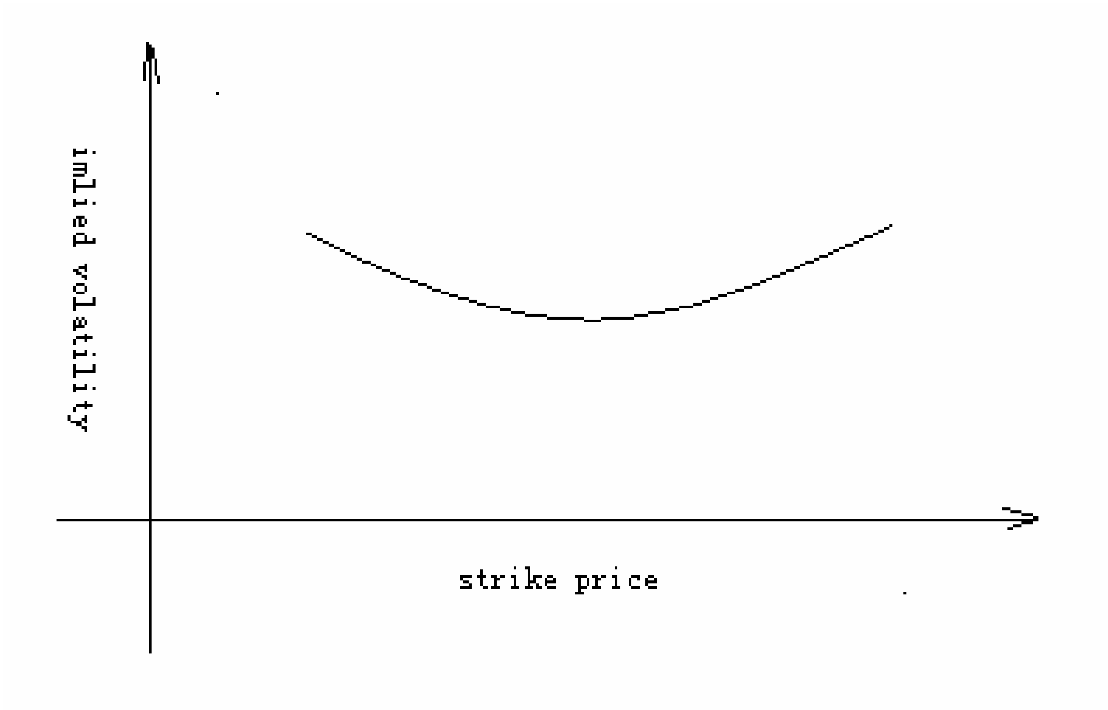
$$V(S^*, t^*; \sigma_{imp}) = V_{market}$$

Volatility smile

- Finance.yahoo.com

$$V(S^*, t^*; \sigma_{imp}, K_i) = V_{market,i}$$

continued



Improved models

- Local volatility model
- Stochastic volatility model
- Jump diffusion model
- Others: discrete hedging, transaction cost

Local volatility model

Suppose

$$\frac{dS_t}{S_t} = \mu dt + \sigma(S,t)dW_t$$

The option value V=V(S,t) satisfies

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(S,t)S^2\frac{\partial^2 V}{\partial S^2} + (r-q)S\frac{\partial V}{\partial S} - rV = 0. \quad S > 0 \ t \in [0,T)$$

$$V(S,T) = \begin{cases} (S-X)^+, & \text{for call option,} \\ (X-S)^+, & \text{for put option.} \end{cases}$$

A special case: $\sigma = \sigma(t)$

The Black-Scholes formulae are still valid when volatility is time-dependent provided we use

$$\sqrt{\frac{1}{T-t}\int_t^T \sigma^2(\tau)d\tau}$$

in place of σ, i.e. now use

$$d_1 = \frac{\ln\frac{S_0}{X} + (r-q)(T-t) + \frac{1}{2}\int_t^T \sigma^2(\tau)d\tau}{\sqrt{\int_t^T \sigma^2(\tau)d\tau}}$$

$$d_2 = \frac{\ln\frac{S_0}{X} + (r-q)(T-t) - \frac{1}{2}\int_t^T \sigma^2(\tau)d\tau}{\sqrt{\int_t^T \sigma^2(\tau)d\tau}}.$$

Identification of $\sigma(t)$

$$\sigma(T) = \sqrt{\sigma^2_{imp}(t^*, T) + 2(T - t^*)\sigma_{imp}(t^*, T)\frac{\partial \sigma_{imp}(t^*, T)}{\partial T}}.$$

In general case: $\sigma = \sigma(S,t)$

- No closed form solution
- How to identify $\sigma(S,t)$?

continued

Let $V(S,t;K,T)$ be the option price with strike K and maturity T. It can be shown (see Wilmott (1998)) that as a function K and T, the function $V(.,.;K,T)$ satisfies

$$\begin{aligned} -\frac{\partial V}{\partial T} + \frac{1}{2}\sigma^2(K,T)K^2\frac{\partial^2 V}{\partial K^2} - (r-q)K\frac{\partial V}{\partial K} - qV &= 0, \text{ for } K > 0,\ T \geq t^*. \\ V(S^*,t^*;K,T) &= (S^*-K)^+ \end{aligned}$$

where t^* is current time, S^* is current asset price. Remember $\sigma(.,.)$ is just a function.

How to use the local volatility model

- Calibration of the model: Identify the volatility function from the market prices of vanilla options
- Price non-traded contracts by using the model

Stochastic volatility model

Assume S satisfies

$$dS = \mu S dt + \sigma S dW_1,$$

and volatility satisfies

$$d\sigma = p(S, \sigma, t)dt + q(S, \sigma, t)dW_2.$$

The two increments dW_1 and dW_2 have a correlation of ρ.

Pricing model

(1) The option value $V = V(S, \sigma, t)$

(2) σ is not a traded asset.

continued

$$\Pi = V - \Delta S - \Delta_1 V_1.$$