Exact Pricing of Asian Options: An Application of Spectral Theory

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Abstract

Arithmetic Asian or *average price (rate) options* deliver payoffs based on the average underlying price over a pre-specified time period. Asian options are an important family of derivative contracts with a wide variety of applications in currency, equity, interest rate, commodity, energy, and insurance markets. We derive two analytical formulae for the price of the arithmetic Asian option when the underlying asset price follows geometric Brownian motion. The mathematics of the Asian option turns out to be related to the Schrödinger equation with Morse (1929) potential. Our derivation relies on the spectral theory of singular Sturm-Liouville (Schrödinger) operators and associated eigenfunction expansions. The first formula is an infinite series of terms involving Whittaker functions M and W. The second formula is a single real integral of an expression involving Whittaker function W plus (for some parameter values) a finite number of additional terms involving incomplete Gamma functions and Laguerre polynomials. The two formulae allow exact computation of Asian option prices.

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1 Introduction

Arithmetic Asian or *average price (rate) options* deliver payoffs based on the average underlying price or financial variable over a pre-specified time period. Asian-style derivatives constitute an important family of derivative contracts with a wide variety of applications in currency, equity, interest rate, commodity, energy, and insurance markets. To the best of our knowledge, Asian options have been first introduced to the academic literature by Boyle and Emanuel (1980) (see Boyle (1996) for the history of Asian options).

There are many economic reasons why Asian options are so popular. First, when developing long-term financial projections, the corporate treasurer is often concerned with the average foreign exchange rate or commodity price realized over the accounting period. Thus the average rate (or average price) options are a natural corporate financial risk management tool. In addition, they are typically less expensive than standard options since the volatility of the average exchange rate or commodity price is lower than the volatility of the rate or price itself. To give a specific example, according to the recent RISK magazine interview with Microsoft treasurer Jean-Francois Heitz (Falloon (1998)), Microsoft uses average-rate foreign exchange options as a main tool for hedging its net revenue exposures. In 1995, 1996 and 1997, approximately 37%, 34% and 32% of Microsoft's revenue respectively was collected in foreign currencies. A sizable currency exposure is the result, and Microsoft typically hedges a percentage of the planned net revenue in major currencies. For the fiscal year ending June 30, 1997, for example, the notional amount of the currency options outstanding totaled $2.1 billion. Heitz says it is company policy to hedge the entire budgeted revenue in a given quarter. Hedging its foreign exchange exposures with average rate options helps Microsoft maintain stable earnings quarter after quarter.

Second major class of applications exploit the fact that, while it may be possible for a large market participant to manipulate the price of a thinly traded commodity on any given day, it is much harder to manipulate its average price over a period of time. For this reason, many commodity contracts are Asian-style and are settled based on the average price of a commodity over a specified time period preceding contract expiration (e.g., crude oil futures and options on the NYMEX and copper and aluminum options on the London Metals Exchange). For the same reason, many corporate mergers and takeovers are also average price contingent (e.g., the mergers of Dow Chemical with Marion Laboratories and Rhone-Poulenc with the Rhorer Group). Another important class of applications of average rate contracts are found in the insurance industry (e.g., catastrophe loss options; see Bakshi and Madan (2001) and references therein for more details and further applications).

Accurate pricing of Asian options is an important practical problem in Financial Engineering. This problem raises several interesting methodological issues. First, these options are path-dependent. The price of an Asian option at any point in time is a function of both the price of the underlying asset at that time and also the average of the underlying prices up to that time. Second, the arithmetic average is not lognormally distributed when the underlying follows the geometric Brownian motion process assumed in the standard

Black-Scholes-Merton option pricing framework. In fact, this distribution turns out to be quite complicated to characterize analytically.

Different approaches to this problem can be subdivided into three broad classes: analytical pricing formulae, analytical approximations, and numerical methods. Currently three analytical pricing formulae are available for continuously-averaged arithmetic Asian options. In a pioneering contribution, Yor (1992) expresses the arithmetic Asian option price as a *triple integral*. Unfortunately the triple integral formula is of limited practical use, since the triple integral has to be evaluated numerically. As an alternative, Geman and Yor (1992), (1993) derive a closed-form expression for the *Laplace transform* of the arithmetic Asian call option price. Their formula can be expressed in terms of the *Kummer confluent hypergeometric function* (see Donati-Martin, Ghomrasni, and Yor (1999) and Yor (2001b) for further references and alternative derivations of the celebrated Geman-Yor result). This Laplace transform has to be inverted numerically by applying a suitable numerical Laplace inversion algorithm. Several authors report computational difficulties associated with this Laplace inversion (Geman and Eydeland (1995), Fu, Madan and Wang (1997), Craddock, Heath and Platen (2000)). In an interesting recent contribution, Dufresne (2000) obtains an alternative formula that expresses the Asian option price as an infinite series of Laguerre polynomials with each coefficient of the series given by a single integral that needs to be computed numerically (see also Dufresne (2001) for further details and some simplifications). Thus at present the situation is not entirely satisfactory: to compute Asian option prices one either needs to compute the triple integral, invert the Laplace transform, or compute the infinite series of integrals.

A number of authors have suggested various analytical approximations to approximate the distribution of the average. Among them are the lognormal approximation with matched first and second moments of Turnbull and Wakeman (1992) and the reciprocal Gamma approximation of Milevsky and Posner (1998). In general, the problem with analytical approximations is that in many cases no reliable error bounds or estimates are available and, thus, the approximation error is uncontrolled. The approximation may work well for some parameter values, while it may fail for others. The practical advantage of analytical approximations is in their simplicity and ease of implementation.

Monte Carlo simulation (Boyle and Emanuel (1980), Kemna and Vorst (1990), Boyle, Broadie, and Glasserman (1997)) and numerical finite-difference PDE methods (Rogers and Shi (1995)[1], Zvan, Forsyth, and Vetzal (1997), Vecer (2000) and (2001)) are among the numerical approaches to the pricing of Asian options. The practical importance of numerical methods is in their flexibility, as many market realities that are typically difficult to incorporate in analytical models (e.g., discrete dividend payments, discrete price observations, day count conventions, time-dependent parameters) can usually be incorporated with relative ease.

From the practitioner's perspective, the ideal situation is the availability of a closed-form analytical solution that allows to produce exact prices under some simplifying assumptions, combined with a stable general-purpose numerical method that is flexible

[1]Rogers and Shi (1995) also obtain interesting analytical bounds for Asian option prices.

enough to incorporate various complications and market realities not available in the analytical model. Then the analytical formulae can serve as benchmarks and control variates for the numerical method.

In this paper we revisit the continuously-averaged arithmetic Asian option problem and take a different approach. Our goal is to obtain, as explicitly as possible, analytical formulae for the price of the Asian option that do not involve multiple integrals or Laplace transform inversion and allow exact computation of Asian option prices. First, using Dufresne's (1989) identity in law for exponential Brownian functionals, we show that the problem of pricing Asian options on geometric Brownian motion is equivalent to the problem of pricing vanilla European options on an auxiliary one-dimensional Markov diffusion process. Next, the Liouville transformation transforms the Kolmogorov backward partial differential equation for this Markov process into the Schrödinger equation with Morse (1929) potential. We analyze this problem using the spectral theory of singular Sturm-Liouville (Schrödinger) operators. This leads us to the two alternative analytical expressions for the Asian option price. The first formula is an infinite series: an eigenfunction expansion of the Asian option pricing function in the basis of Whittaker functions. This series formula does not require any numerical integrations. Each term in the series is characterized in terms of known special functions. The second formula is a single real integral of an expression involving the Whittaker function W related to the Tricomi confluent hypergeometric function plus (for some parameter values) a finite number of terms involving incomplete Gamma functions and Laguerre polynomials. Our derivation relies on the spectral resolution of the Schrödinger operator with Morse potential. We give a real-variable proof of this spectral resolution. The resulting formulae can be programmed in *Mathematica* and allow exact calculation of Asian option prices. We give numerical examples of computing Asian option prices with the precision of ten significant digits.

Methodologically, this paper is a sequel to our paper Davydov and Linetsky (2001b) where the method of eigenfunction expansions associated with Sturm-Liouville problems has been applied to pricing options on scalar diffusions (see also an interesting recent paper by Lewis (1998)). In Davydov and Linetsky (2001b) we have studied regular problems and singular problems with purely discrete spectra. The Asian option problem furnishes a striking example of a singular problem with the spectrum containing a continuous portion plus (for some parameter values) a finite number of discrete eigenvalues. In this paper we employ the real-variable approach of Levitan and Levinson to problems with continuous spectrum (see Levitan and Sargsjan (1975)) and do not rely on Laplace transform inversion. To the best of our knowledge, this is the first application of this approach to financial engineering problems. Previous calculations, including Davydov and Linetsky (2001b) and Lewis (1998), relied on the complex-variable approach of Titchmarsh (1962).

To conclude this introduction, we point out that the practically important problem of pricing Asian options has a very rich mathematical structure and is closely related to a number of problems in probability theory (exponential Brownian functionals, Wong's stationary diffusion), quantum physics (Morse oscillator, disordered systems), and geometry (harmonic analysis on the hyperbolic plane). We refer the reader to Yor (2001a), Donati-Martin, Matsumoto, and Yor (2001) and Ikeda and Matsumoto (1999) where some of these connections are explored.

The remainder of the paper is organized as follows. Section 2 describes continuous arithmetic Asian options, sets up the valuation problem in the Black-Scholes-Merton environment, and introduces our notation. Section 3 states the main results of the paper. Section 4 re-states the problem as Markovian in one state variable, and points out some connections with related problems. Section 5 outlines our calculation strategy based on the real-variable approach to the spectral theory of singular Sturm-Liouville (Schrödinger) operators. Section 6 explicitly solves the problem on the interval $(0, b]$ and presents the first valuation formula as an infinite series (eigenfunction expansion) of Whittaker functions. Section 7 explicitly solves the problem on the infinite interval $(0, \infty)$ by means of passing to the limit $b \to \infty$ and presents the second valuation formula as a single integral of an expression involving the Whittaker function W. Section 8 presents computational results. Section 9 concludes the paper.

2 Setting of the Problem: Continuous Arithmetic Asian Options

2.1. Asian options in the Black-Scholes-Merton economy. We assume that, under the risk-neutral probability measure $\mathbf{Q}$, the underlying asset price follows a geometric Brownian motion process

$$S_t = S_0 e^{\sigma B_t + (r-q-\sigma^2/2)t}, \;\; t \geq 0, \tag{1}$$

where $\{B_t, t \geq 0\}$ is a standard Brownian motion defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t, t \geq 0\}, \mathbf{Q})$, $\sigma > 0$ is the constant volatility, $r \geq 0$ is the constant risk-free interest rate, $q \geq 0$ is the constant dividend yield, and $S_0 > 0$ is the initial asset price at $t = 0$.

Define a continuous arithmetic average price process $\{\mathcal{A}_t, t \geq 0\}$:

$$\mathcal{A}_0 = S_0, \;\; \mathcal{A}_t = \frac{1}{t}\int_0^t S_u \, du, \;\; t > 0. \tag{2}$$

On applying Fubini's theorem, the risk-neutral expectation of the average price up to time $T > 0$ is:

$$\mathbf{E}[\mathcal{A}_T] = \begin{cases} \frac{(e^{(r-q)T}-1)}{(r-q)T} S_0, & r \neq q \\ S_0, & r = q \end{cases}. \tag{3}$$

A continuous arithmetic Asian call (put) with the strike price $K > 0$ and expiration date $T > 0$ delivers the payoff $(\mathcal{A}_T - K)^+$ $((K - \mathcal{A}_T)^+)$ at time T $(x^+ \equiv \max\{x, 0\})$. As an immediate consequence of Eq.(3), Asian put and call prices are related by the well-known put-call parity relationship for Asian options (see Geman and Yor (1993)):

$$e^{-rT}\mathbf{E}[(\mathcal{A}_T-K)^+]=e^{-rT}\mathbf{E}[(K-\mathcal{A}_T)^+]+\begin{cases}\frac{(e^{-qT}-e^{-rT})}{(r-q)T}S_0-e^{-rT}K, & r\neq q\\ e^{-rT}(S_0-K), & r=q\end{cases}. \tag{4}$$

Thus, it is sufficient to price Asian puts. Asian call prices can be recovered by appealing to the put-call parity.

2.2. Seasoned Asian options. So far we have discussed *newly-written* Asian options. Suppose the Asian put option has been initiated at time zero, and we are interested in pricing it at some time $0<t<T$ during the option's life (*seasoned Asian option*). Let $\mathcal{A}_{t,T}$ be the average price between t and T, $\mathcal{A}_{t,T}=\frac{1}{T-t}\int_t^T S_u\,du$. From the Markov property and time homogeneity of Brownian motion we have:

$$\mathbf{E}[(K-\mathcal{A}_{0,T})^+|\mathcal{F}_t]=\mathbf{E}\left[\left(K-\frac{t}{T}\mathcal{A}_{0,t}-\frac{T-t}{T}\mathcal{A}_{t,T}\right)^+\middle|\mathcal{F}_t\right]$$
$$=\frac{T-t}{T}\mathbf{E}[(K^*-\mathcal{A}_{t,T})^+|\mathcal{F}_t]=\frac{T-t}{T}\mathbf{E}_{0,S_t}[(K^*-\mathcal{A}_{0,T-t})^+],$$

where the *modified strike* is $K^*=\frac{T}{T-t}K-\frac{t}{T-t}\mathcal{A}_{0,t}$, $\mathcal{A}_{0,t}$ is the average price up to time t (known at t), $\mathcal{A}_{0,t}=\frac{1}{t}\int_0^t S_u du$, and the subscript $0,S_t$ in the expectation operator E_{0,S_t} in the last equality indicates that the initial asset price is S_t at time zero. If $\mathcal{A}_{0,t}$ is sufficiently large, K^* may become negative. For $K^*\leq 0$, the seasoned Asian put option has no chance of finishing in-the-money and delivering a positive payoff at expiration and, thus, vanishes identically. The call price is then given by the second term on the righ-hand side of Eq.(4) with the strike K^* (the present value at time t of an *Asian-style forward contract* with delivery price K^*). For $K^*>0$, pricing a seasoned Asian option at time $0<t<T$ is equivalent to pricing a quantity $\frac{T-t}{T}$ of newly-written Asian options with time to expiration $T-t$ and strike $K^*>0$. Thus, it is enough to consider newly-written Asian put options for positive strikes.

3 Main Results: Two Analytical Pricing Formulae

3.1. Notation. Here we state without proof the main results of the paper: two analytical pricing formulae for the arithmetic Asian put. The proofs will be presented in Sections 4–7. Following Geman and Yor (1993), introduce the following normalized parameters τ (dimensionless time to expiration), k, and ν:

$$\tau:=\frac{\sigma^2T}{4},\ \ k:=\frac{\tau K}{S_0},\ \ \nu:=\frac{2(r-q)}{\sigma^2}-1. \tag{5}$$

Then the arithmetic Asian put price is expressed in terms of a function $P^{(\nu)}(k,\tau)$ of these three parameters:

$$e^{-rT}\mathbf{E}[(K-\mathcal{A}_T)^+]=e^{-rT}\left(\frac{4S_0}{\sigma^2T}\right)P^{(\nu)}(k,\tau). \tag{6}$$

The analytical formulae for the function $P^{(\nu)}(k,\tau)$ will be expressed in terms of several well-known special functions: first and second Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ (Abramowitz and Stegun (1972, p.505), Slater (1960, pp.9-10), and Buchholz (1969, pp.9-20)), Gamma function $\Gamma(z)$ (Abramowitz and Stegun (1972, p.255)), the incomplete Gamma function $\Gamma(a,z)$ (Abramowitz and Stegun (1972, p.260)), and the generalized Laguerre polynomials $L_n^{(a)}(z)$ (Abramowitz and Stegun (1972, p.773)).

3.2. The Series Formula. Fix a large number $b >> k$. Let $\{p_{n,b}, n = 0, 1, 2, ...\}$ be the zeros on the positive real line $p > 0$ of the Whittaker function $W_{\kappa,\mu}(z)$ with the fixed first index $\kappa = \frac{1-\nu}{2}$, fixed argument $z = \frac{1}{2b}$, and the purely imaginary second index $\mu = \frac{ip}{2}$, $p > 0$ ($i = \sqrt{-1}$). That is, the $p_{n,b}$ are the positive roots of the equation[2]

$$W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2b}\right) = 0. \tag{7}$$

This equation has an infinite set of simple roots on the positive real axis. If the roots are ordered by $0 < p_{0,b} < p_{1,b} < ...$, then $p_{n,b} \to \infty$ as $n \to \infty$. Furthermore, let $m_\nu(b) \geq 0$ be the number of roots of the equation

$$W_{\frac{1-\nu}{2},\frac{q}{2}}\left(\frac{1}{2b}\right) = 0 \tag{8}$$

in the interval $0 \leq q < |\nu|$ and, for $m_\nu(b) > 0$, let $\{q_{n,b}, n = 0, ..., m_\nu(b)\}$ be the corresponding roots. Then the function $P^{(\nu)}(k,\tau)$ entering the Asian put valuation formula (6) is approximated by the following series ($\sum_{n=0}^{-1} \equiv 0$ by convention):

$$P_b^{(\nu)}(k,\tau) = \sum_{n=0}^{\infty} e^{-\frac{(\nu^2+p_{n,b}^2)\tau}{2}} \left\{ \frac{p_{n,b}(2k)^{\frac{\nu+3}{2}} e^{-\frac{1}{4k}} W_{-\frac{\nu+3}{2},\frac{ip_{n,b}}{2}}\left(\frac{1}{2k}\right)\Gamma(\frac{\nu+ip_{n,b}}{2}) M_{\frac{1-\nu}{2},\frac{ip_{n,b}}{2}}\left(\frac{1}{2b}\right)}{4\Gamma(1+ip_{n,b})\left(\frac{\partial W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2b}\right)}{\partial p}\right)\Big|_{p=p_{n,b}}} \right\}$$

$$+ \sum_{n=0}^{m_\nu(b)-1} e^{-\frac{(\nu^2-q_{n,b}^2)\tau}{2}} \left\{ \frac{q_{n,b}(2k)^{\frac{\nu+3}{2}} e^{-\frac{1}{4k}} W_{-\frac{\nu+3}{2},\frac{q_{n,b}}{2}}\left(\frac{1}{2k}\right)\Gamma(\frac{\nu+q_{n,b}}{2}) M_{\frac{1-\nu}{2},\frac{q_{n,b}}{2}}\left(\frac{1}{2b}\right)}{4\Gamma(1+q_{n,b})\left(-\frac{\partial W_{\frac{1-\nu}{2},\frac{q}{2}}\left(\frac{1}{2b}\right)}{\partial q}\right)\Big|_{q=q_{n,b}}} \right\}. \tag{9}$$

In the limit $b \to \infty$ we have $\lim_{b\to\infty} P_b^{(\nu)}(k,\tau) = P^{(\nu)}(k,\tau)$, and the series formula (9) yields the integral formula (10) for the function $P^{(\nu)}(k,\tau)$ given below. However, for typical parameter values encountered in financial applications, the series formula approximates the exact integral formula so well that already for $b = 1$ the numerical Asian

[2] Note that $W_{\kappa,\frac{ip}{2}}(z)$ is a real function of p since the Whittaker function is even in the second index, $W_{\kappa,\mu}(z) = W_{\kappa,-\mu}(z)$ (Slater (1960), p.11, Eq.(1.7.11)).

option prices computed using the series formula agree with the prices computed using the integral formula at the precision level of ten significant digits or better (see Section 8).

3.3. The Integral Formula. In the limit $b \to \infty$ we obtain the exact integral formula ($[x]$ denotes the integer part of x):

$$P^{(\nu)}(k,\tau) = \frac{1}{8\pi^2}\int_0^\infty e^{-\frac{(\nu^2+p^2)\tau}{2}}(2k)^{\frac{\nu+3}{2}}e^{-\frac{1}{4k}}W_{-\frac{\nu+3}{2},\frac{ip}{2}}\left(\frac{1}{2k}\right)\left|\Gamma\left(\frac{\nu+ip}{2}\right)\right|^2 \sinh(\pi p)\,p\,dp$$

$$+\mathbf{1}_{\{\nu<0\}}\frac{1}{2\Gamma(|\nu|)}\{2k\Gamma(|\nu|,1/(2k)) - \Gamma(|\nu|-1,1/(2k))\}$$

$$+\mathbf{1}_{\{\nu<-2\}}e^{-2(|\nu|-1)\tau}\frac{(|\nu|-2)}{\Gamma(|\nu|)}\Gamma(|\nu|-2,1/(2k))$$

$$+\mathbf{1}_{\{\nu<-4\}}\sum_{n=2}^{\left[\frac{|\nu|}{2}\right]} e^{-2n(|\nu|-n)\tau}\frac{(-1)^n(|\nu|-2n)}{2n(n-1)\Gamma(1+|\nu|-n)}(2k)^{\nu+n+1}e^{-\frac{1}{2k}}L_{n-2}^{(|\nu|-2n)}\left(\frac{1}{2k}\right). \tag{10}$$

The analytical formulae (9) and (10) allow exact calculation of the Asian put price (6).

4 Reformulation of the Asian Option Problem as Markovian

4.1. Asian options and Brownian exponential functionals. Consider a slightly more general put payoff:

$$(K - wS_T - (1-w)\mathcal{A}_T)^+,$$

where $0 \le w < 1$ is the weight parameter. For $w = 0$ this payoff reduces to the standard Asian put. Similar to Section 3.1, it is convenient to standardize the problem as follows:

$$e^{-rT}\mathbf{E}[(K - wS_T - (1-w)\mathcal{A}_T)^+] = e^{-rT}\left(\frac{4S_0(1-w)}{\sigma^2 T}\right)P^{(\nu)}(x,k,\tau), \tag{11}$$

where the function $P^{(\nu)}(x,k,\tau)$ is defined by

$$P^{(\nu)}(x,k,\tau) := \mathbf{E}\left[\left(k - xe^{2(B_\tau+\nu\tau)} - A_\tau^{(\nu)}\right)^+\right], \tag{12}$$

$A_\tau^{(\nu)}$ is a *Brownian exponential functional* (see Yor (2001a))

$$A_\tau^{(\nu)} := \int_0^\tau e^{2(B_u+\nu u)}du,$$

and the normalized parameters τ, ν, k and x are:

$$\tau := \frac{\sigma^2 T}{4},\;\; \nu := \frac{2(r-q)}{\sigma^2} - 1,\;\; k := \frac{\tau K}{S_0(1-w)},\;\; x := \frac{\tau w}{1-w}. \tag{13}$$

The reduction (11)-(12) follows from an identity in law

$$\int_0^T e^{\sigma B_t + (r-q-\sigma^2/2)t} dt \stackrel{(law)}{=} \frac{4}{\sigma^2} \int_0^{\frac{\sigma^2 T}{4}} e^{2(B_u + \nu u)} du,$$

which is an immediate consequence of the scaling property of Brownian motion: $\sigma B_t \stackrel{(law)}{=} 2B_{\frac{\sigma^2 t}{4}}$.

Our goal is to compute the function $P^{(\nu)}(x, k, \tau)$ as explicitly as possible. By Eq.(11) it will give the price of the generalized Asian put. The standard Asian put with the payoff $(K - \mathcal{A}_T)^+$ is then recovered in the limit $w \to 0$ ($x = 0$ and $k = \frac{\tau K}{S_0}$):

$$P^{(\nu)}(k, \tau) \equiv P^{(\nu)}(0, k, \tau) = \mathbf{E}[(k - A_\tau^{(\nu)})^+].$$

4.2. Dufresne's identity and Wong's diffusion. Our starting point is an identity in law for any fixed $t \geq 0$ due to Dufresne (1989), (1990):

$$xe^{2(B_t + \nu t)} + A_t^{(\nu)} \stackrel{(law)}{=} X_t, \;\; t \geq 0, \tag{14}$$

where the process $\{X_t, t \geq 0\}$ is defined by

$$X_t = e^{2(B_t + \nu t)} \left(x + \int_0^t e^{-2(B_u + \nu u)} du \right), \;\; t \geq 0,$$

has an Ito differential

$$dX_t = [2(\nu + 1)X_t + 1]dt + 2X_t dB_t, \tag{15}$$

and starts at x, $X_0 = x$. To show (14), write $X_t = xe^{2(B_t + \nu t)} + \int_0^t e^{2(B_t - B_u) + 2\nu(t-u)} du$, and use time reversal to establish $\int_0^t e^{2(B_t - B_u) + 2\nu(t-u)} du \stackrel{(law)}{=} \int_0^t e^{2B_s + 2\nu s} ds$.

Thus, for any fixed $t \geq 0$, the process $xe^{2(B_t + \nu t)} + A_t^{(\nu)}$ has the same distribution as a one-dimensional Markov diffusion process on $(0, \infty)$ starting at $x \geq 0$. From (15) it has the infinitesimal generator

$$\mathcal{G} = 2x^2 \frac{d^2}{dx^2} + [2(\nu + 1)x + 1] \frac{d}{dx}, \tag{16}$$

or, in the equivalent form,

$$\mathcal{G} = \frac{1}{\mathfrak{m}(x)} \frac{d}{dx} \left[\frac{1}{\mathfrak{s}(x)} \frac{d\bullet}{dx} \right], \tag{17}$$

where $\mathfrak{s}(x)$ and $\mathfrak{m}(x)$ are the scale and speed densities[3]

$$\mathfrak{s}(x) = x^{-\nu-1} e^{\frac{1}{2x}}, \;\; \mathfrak{m}(x) = \frac{1}{2} x^{\nu-1} e^{-\frac{1}{2x}}. \tag{18}$$

[3]See Karatzas and Shreve (1991, p.343) and Borodin and Salminen (1996, p.17) for discussions of scale and speed densities for one-dimensional diffusions.

For $\nu < 0$ this diffusion first appeared in Wong (1964). In this paper we consider this process for all $\nu \in \mathbb{R}$ and term it *Wong's diffusion*. For all $\nu \in \mathbb{R}$ the process has an entrance boundary at zero. If started at zero, the process rapidly enters the state space $(0, \infty)$ under the influence of positive drift (+1 in the drift rate in Eq.(15)). Infinity is a natural boundary, attracting for $\nu > 0$ and non-attracting for $\nu \le 0$. For $\nu < 0$ the process has a stationary distribution with the density

$$\frac{\mathfrak{m}(x)}{\int_0^\infty \mathfrak{m}(y)dy} = \frac{2^\nu}{\Gamma(-\nu)} x^{\nu-1} e^{-\frac{1}{2x}}.$$

The function (12) can now be re-written in terms of the process X

$$P^{(\nu)}(x, k, \tau) = \mathbf{E}_x[(k - X_\tau)^+], \tag{19}$$

where the subscript x in the expectation operator $\mathbf{E}_x$ signifies that X is starting at x at time zero. In the limit $x \to 0$ we have

$$P^{(\nu)}(k, \tau) \equiv P^{(\nu)}(0, k, \tau) = \mathbf{E}_0[(k - X_\tau)^+].$$

Thus, the problem of pricing the (generalized) arithmetic Asian put on the geometric Brownian motion (1) is reduced to the problem of pricing a put on the one-dimensional diffusion X.

In addition to the put (19), we are also interested in *up-and-out puts* on the process X. For $b > x \vee k$ ($x \vee y \equiv \max\{x, y\}$), consider the function

$$P_b^{(\nu)}(x, k, \tau) := \mathbf{E}_x[\mathbf{1}_{\{\mathcal{T}_b > \tau\}}(k - X_\tau)^+], \tag{20}$$

where $\mathcal{T}_b := \inf\{t \ge 0 : X_t = b\}$ is the first hitting time of b. Our strategy is to first explicitly calculate the function $P_b^{(\nu)}(x, k, \tau)$ and then pass to the limit $b \to \infty$ to recover the function $P^{(\nu)}(x, k, \tau)$. We also use the following notation for the limiting case $x \to 0$:

$$P_b^{(\nu)}(k, \tau) := P_b^{(\nu)}(0, k, \tau) = \mathbf{E}_0[\mathbf{1}_{\{\mathcal{T}_b > \tau\}}(k - X_\tau)^+].$$

4.3. Morse Potential. The function $P^{(\nu)}(x, k, \tau)$ solves the PDE

$$2x^2 P_{xx} + [2(\nu + 1)x + 1]P_x = P_\tau, \quad x \in (0, \infty), \quad \tau \in (0, \infty), \tag{21}$$

with the initial condition $P^{(\nu)}(x, k, 0) = (k - x)^+$. The Liouville transformation

$$\xi = \frac{1}{\sqrt{2}} \ln x, \quad P^{(\nu)}(x, k, \tau) = x^{-\frac{\nu}{2}} e^{\frac{1}{4x}} p^{(\nu)}(\xi(x), k, \tau)$$

reduces the PDE (21) to the *Liouville normal form* (the coefficient in front of the second derivative term is equal to one and the first derivative term p_ξ is absent):

$$p_{\xi\xi} - Q(\xi)p = p_\tau, \quad \xi \in \mathbb{R}, \quad \tau \in (0, \infty), \tag{22}$$

with the *potential function*

$$Q(\xi) = \frac{1}{8}e^{-2\sqrt{2}\,\xi} + \frac{\nu - 1}{2}e^{-\sqrt{2}\,\xi} + \frac{\nu^2}{2} \tag{23}$$

and the initial condition $p^{(\nu)}(\xi, k, 0) = \exp\left(\frac{\nu}{\sqrt{2}}\xi - \frac{1}{4}e^{\sqrt{2}\,\xi}\right)(k - e^{\sqrt{2}\,\xi})^+$. The PDE (22) with potential (23) has the form of the heat equation for the *Sturm-Liouville* (or *Schrödinger*) *operator with the Morse potential*:

$$-\frac{d^2}{d\xi^2} + \frac{1}{8}e^{-2\sqrt{2}\,\xi} + \frac{\nu - 1}{2}e^{-\sqrt{2}\,\xi} + \frac{\nu^2}{2}. \tag{24}$$

Remark 4.1. Markovian formulation. Rogers and Shi (1995) obtained a PDE of the type (21) by changing the variables in the two state variable PDE for Asian options (one state variable for the underlying asset price at the current time and the other — for the average accumulated up to the current time). The resulting PDE formulation has been used by a number of authors (see, e.g., Lipton (1999), Zvan, Forsyth, and Vetzal (1997), He and Takahashi (2000), and Vecer (2000)). Dufresne's (1989), (1990) identity in law (14) gives a probabilistic interpretation of this PDE formulation. It shows that the Brownian exponential functional has the same distribution as a one-dimensional Markov diffusion. Hence, the arithmetic Asian option problem is essentially Markovian in one state variable (see Donati-Martin, Ghomrasni, and Yor (1999) for further details and an alternative derivation of the original Geman and Yor (1993) result using this reduction to the Markovian problem). Finally, we remark that the identity (14) is a particular case of some general identities for Levý processes (see Carmona, Petit, and Yor (1997) and other articles in the same volume).

Remark 4.2. Schrödinger equation with Morse potential and hyperbolic geometry. The Sturm-Liouville or Schrödinger operator with potential of the form $ae^{-2\beta x} - be^{-\beta x}$ first appeared in quantum mechanics in the classic paper of Morse (1929) on the spectra of diatomic molecules. The potential function of this form is called *Morse potential* and the corresponding one-dimensional quantum system — *Morse oscillator* (see, e.g., Landau and Lifshitz (1965)).[4]

The Schrödinger operator with Morse potential is closely related to another classical differential operator — *Maass Laplacian* or *Schrödinger operator on the Poincaré upper half-plane in magnetic field*. Let H^2 be the upper half-plane with rectangular coordinates (x, y), $x \in \mathbb{R}$, $y > 0$, and with the Poincaré metric (hyperbolic plane). It is a space of constant negative curvature and has an isometry group $SL_2(\mathbb{R})/\{\mathrm{Id}, -\mathrm{Id}\}$ that acts by the linear fractional transformations (Id is the identity element). Consider the Schrödinger

[4]On a historical note, at this point it is appropriate to recall that, in addition to his contributions to quantum mechanics (see, e.g., the classic Morse and Feshbach (1953)), Philip Morse is also widely considered to be the father of the discipline of Operations Research in the U.S. In 1952 he was the founding president of ORSA, now INFORMS, — the publisher of *Operations Research*. The 1929 paper we have cited was a part of his doctoral studies in quantum physics.

operator with a uniform magnetic field on H^2 ($B \in \mathbb{R}$):

$$H_B = -\frac{1}{2}y^2\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) + iBy\frac{\partial}{\partial x} + \frac{B^2}{2}.$$

This is a standard Laplace-Beltrami operator on H^2 plus magnetic field terms. Introduce a new variable $\eta = -\ln y$, so that

$$H_B = -\frac{1}{2}\frac{\partial^2}{\partial \eta^2} - \frac{1}{2}\frac{\partial}{\partial \eta} - \frac{1}{2}\left(e^{-\eta}\frac{\partial}{\partial x} - iB\right)^2.$$

The operator H_B acts on functions of the form $u(\eta, x) = e^{-ipx-\frac{1}{2}\eta}v(\eta)$ ($p \in \mathbb{R}$) according to:

$$H_B u(\eta, x) = -\frac{1}{2}\frac{\partial^2 u}{\partial \eta^2} + \left(\frac{p^2}{2}e^{-2\eta} + pBe^{-\eta} + \frac{B^2}{2} + \frac{1}{8}\right)u.$$

Thus, on functions of the form $e^{-ipx-\frac{1}{2}\eta}v(\eta)$ the operator H_B reduces to the Morse operator. Harmonic analysis on H^2 can now be applied. Connections with hyperbolic geometry have been explored both in the physics (Comtet (1987), Comtet and Monthus (1996), Grosche (1988)) and mathematics literature (Alili and Gruet (1997), Alili, Matsumoto, and Shiraishi (2000), Ikeda and Matsumoto (1999)). Here we will not pursue this approach further, instead working with the infinitesimal generator (16) directly.

5 Spectral Analysis of the Operator $-\mathcal{G}$

The analytical tool we will use to calculate the functions $P_b^{(\nu)}(x, k, \tau)$ and $P^{(\nu)}(x, k, \tau)$ is the spectral theory of second-order differential operators (Sturm-Liouville or Schrödinger operators) and associated eigenfunction expansions. In particular, we will subject the negative of the infinitesimal generator $\mathcal{G}$ (16) to spectral analysis and explicitly construct associated eigenfunction expansions. Detailed treatments of the theory of Sturm-Liouville operators are given in Dunford and Schwartz (1960), Coddington and Levinson (1955, Chapter 9), and Levitan and Sargsjan (1975) and (1991). Fulton, Pruess, and Xie (1996) and Fulton and Pruess (1998) give a summary of the main results particularly convenient for our purposes. In this paper we follow the real variable approach of Levitan (1950) and Levinson (1951). Ito and McKean (1974, Section 4.11) develop applications to diffusion processes. For applications to options pricing see our companion paper Davydov and Linetsky (2001b) and Lewis (1998).

Consider a second-order ODE (*Sturm-Liouville ODE*)

$$-\mathcal{G}u = \lambda u, \tag{25}$$

where $\mathcal{G}$ is the infinitesimal generator of some diffusion process on the (finite or infinite) interval with the left and right end-points $\mathfrak{l}$ and $\mathfrak{r}$ written in the form (17) with some scale and speed densities $\mathfrak{s}(x)$ and $\mathfrak{m}(x)$. If the interval is finite and the scale and speed densities are absolutely integrable near both end-points $\mathfrak{l}$ and $\mathfrak{r}$, then the Sturm-Liouville problem is *regular*. Otherwise, the problem is *singular*. Regular problems are discussed in detail in Section 2 of Davydov and Linetsky (2001b).

A complete classification scheme for singular Sturm-Liouville problems based on the celebrated Weyl's (1910) *limit point/limit circle alternative* and *oscillatory/non-oscillatory classification* can be found in Fulton, Pruess, and Xie (1996) and Zwillinger (1998, pp.97-8). The analysis proceeds by first transforming the singular problem to the Liouville normal form, as we have done in Eqs.(22)-(23) for the operator (16). Then one investigates the behavior of the potential function Q near the singular end-points and applies the known criteria to determine the character of each singular end-point according to the limit point/limit circle and oscillatory/non-oscillatory classifications. When the character of each singular end-point is determined, one can apply the spectrum determination criteria collected in Fulton, Pruess, and Xie (1996) and Zwillinger (1998, pp.97-8).

Consider the Sturm-Liouville ODE (25) on the interval with the endpoints $\mathfrak{l}$ and $\mathfrak{r}$, and λ — an arbitrary complex number. There are two fundamental disjoint types into which the Sturm-Liouville equation is classified at each end-point: (1) limit point or limit circle which is independent of $\lambda \in \mathbb{C}$, and (2) non-oscillatory or oscillatory for real value of λ, which can vary with λ. For simplicity we give the definitions at $\mathfrak{l}$ only, as the definitions for $\mathfrak{r}$ are entirely similar. The Sturm-Liouville equation is said to be *limit circle* at $\mathfrak{l}$ if and only if every solution $u(x)$ is square-integrable with the weight $\mathfrak{m}(x)$ near the end-point $\mathfrak{l}$, i.e., $\int_{\mathfrak{l}}^{\mathfrak{l}+\epsilon} |u(x)|^2\mathfrak{m}(x)dx < \infty$. Otherwise the equation is called *limit point* at $\mathfrak{l}$. This classification due to Weyl is mutually exclusive and independent of λ. That is, if the ODE (25) has two linearly independent solutions which are square-integrable for one value of λ, then it will have two linearly independent solutions for all $\lambda \in \mathbb{C}$. When the limit point case occurs, for $\mathrm{Im}(\lambda) \neq 0$ there exists only one (up to a multiplicative factor independent of x) solution that is square-integrable with $\mathfrak{m}$, while for real values of λ there may be one or no solution which is square integrable. To generate self-adjoint operators in the Hilbert space $L^2((\mathfrak{l}, \mathfrak{r}), \mathfrak{m})$, whenever the limit circle occurs, a boundary condition must be imposed at that end-point, while no boundary condition is required at a limit point end-point.

The oscillatory/non-oscillatory classification is of fundamental importance in determining the qualitative nature of the spectrum. For a given real λ, the Sturm-Liouville ODE is *oscillatory* at $\mathfrak{l}$ if and only if every solution has infinitely many zeros clustering at $\mathfrak{l}$. Otherwise it is called *non-oscillatory* at $\mathfrak{l}$. This classification is mutually exclusive for a fixed λ, but can vary with λ. All regular end-points are limit circle and non-oscillatory.

We now come back to the operator (17), (18). Fix a positive number b and consider the Sturm-Liouville problem (25) on the interval $(0, b]$ with the Dirichlet boundary condition at b, $u(b) = 0$. The end-point 0 is singular since the scale density (18) is not integrable near zero. It is limit point and non-oscillatory for all real λ.[5] Then, by Theorem 18 in Fulton, Pruess, and Xie (1996), there exists a solution $\phi(x, \lambda)$ of the ODE (25) which

[5]This follows from Theorems 4 and 5 in Fulton, Pruess, and Xie (1996) or Corollary 38, p.1464 and Theorem 23, p.1414 in Dunford and Schwartz (1960) applied to the potential function (23).

is square integrable with respect to the speed density in a neighborhood of zero for all complex λ, is entire in λ for each fixed $x \in (0,b)$, and $\phi(x,\lambda)$ and $d\phi/dx$ are continuous in x and λ in $(0,b) \times \mathbb{C}$.

Let $\mathcal{H}_b = L^2((0,b],\mathfrak{m})$ be the Hilbert space of real-valued speed-measurable functions on $(\mathfrak{l},b]$ square-integrable with the speed density $\mathfrak{m}$ and with the inner product $\langle f,g\rangle_b = \int_0^b f(x)g(x)\mathfrak{m}(x)dx$. The problem on $(0,b]$ falls into the spectral category 1 of Fulton, Pruess, Xie (1996) (see also Zwillinger (1998, p.97)). The spectrum is simple, purely discrete, and positive. If the eigenvalues are ordered by $0 < \lambda_{0,b} < \lambda_{1,b} < \ldots < \lambda_{n,b} < \ldots$, then $\lambda_{n,b} \to \infty$ as $n \to \infty$, and the eigenfunction associated with $\lambda_{n,b}$ is $\phi(x,\lambda_{n,b})$. It has exactly n zeros in $(0,b]$. For any $f \in \mathcal{H}_b$, we have the associated eigenfunction expansion ($\|f\|_b = \sqrt{\langle f,f\rangle_b}$):

$$f(x) = \sum_{n=0}^{\infty} c_{n,b} \frac{\phi(x,\lambda_{n,b})}{\|\phi(\cdot,\lambda_{n,b})\|_b^2}, \tag{26}$$

$$c_{n,b} = \langle f, \phi(\cdot,\lambda_{n,b})\rangle_b, \tag{27}$$

and convergence is in the norm of $\mathcal{H}_b$. The normalized eigenfunctions are $\frac{\phi(x,\lambda_{n,b})}{\|\phi(\cdot,\lambda_{n,b})\|_b}$, and the Parseval equality holds:

$$\|f\|^2 = \sum_{n=0}^{\infty} \frac{c_{n,b}^2}{\|\phi(\cdot,\lambda_{n,b})\|_b^2}. \tag{28}$$

The eigenvalues $\lambda_{n,b}$ can be determined as follows. Since zero is limit point, for any λ the function $\phi(x,\lambda)$ is the unique (up to a multiplicative factor independent of x) solution of the ODE (25) in $\mathcal{H}_b$. Thus, if λ is an eigenvalue, the corresponding eigenfunction is necessarily $\phi(x,\lambda)$ (up to a multiplicative factor independent of x). At the same time, the eigenfunction must satisfy the Dirichlet boundary condition at b:

$$\phi(b,\lambda) = 0. \tag{29}$$

Thus, the eigenvalues are zeros of $\phi(b,\lambda)$. From the theory cited above, all eigenvalues (and, hence, zeros of $\phi(b,\lambda)$) are simple and occur in an infinite set $0 < \lambda_{0,b} < \lambda_{1,b} < \ldots < \lambda_{n,b} < \ldots$. From the practical stand-point, to explicitly determine the eigenfunction expansion of any $f \in \mathcal{H}_b$, we need to find explicitly the solution $\phi(x,\lambda)$, find the zeros $\{\lambda_{n,b}, n = 0,1,2,...\}$ of $\phi(b,\lambda)$, calculate the norms $\|\phi(x,\lambda_{n,b})\|_b$ and coefficients $c_{n,b}$

Now consider the problem on the positive real line $(0,\infty)$. The associated Sturm-Liouville problem (25) is singular at both end-points 0 and ∞. At infinity, the problem is limit point and oscillatory for $\lambda > \frac{\nu^2}{2}$ and non-oscillatory for $\lambda \leq \frac{\nu^2}{2}$.[6] Since both end-points are limit point, no boundary conditions are necessary and the operator $-\mathcal{G}$

[6] This follows from Theorems 6, 7, and 20 in Fulton, Pruess, and Xie (1996) applied to the potential function (23).

is self-adjoint in the Hilbert space $\mathcal{H} := L^2((0,\infty),\mathfrak{m})$ with the inner product $\langle f,g\rangle = \int_0^\infty f(x)g(x)\mathfrak{m}(x)dx$. It is also non-negative semi-definite. This problem falls into the spectral category 2A of Fulton, Pruess, and Xie (1996) (see also Zwillinger (1998, p.97)), and the spectrum of this operator is simple, non-negative, contains a continuous part on $[\nu^2/2,\infty)$ and, possibly, a finite number m_ν of discrete eigenvalues in $[0,\nu^2/2)$. For each function $f \in \mathcal{H}$, we have the associated eigenfunction expansion ($\sum_{n=0}^{-1} \equiv 0$ by convention):

$$f(x) = \int_{\frac{\nu^2}{2}}^{\infty} F(\lambda)\phi(x,\lambda)d\rho_c(\lambda) + \sum_{n=0}^{m_\nu-1} c_n \frac{\phi(x,\lambda_n)}{\|\phi(\cdot,\lambda_n)\|^2}, \tag{30}$$

where the integral is taken over the continuous spectrum, $\phi(x,\lambda)$ is the (unique up to a multiplicative factor independent of x) solution of the ODE (25) square-integrable with the speed density near zero for all complex λ and entire in λ for fixed x, $\rho_c(\lambda)$ is the continuous part of the *spectral function*, the sum is over the finite number of discrete eigenvalues λ_n, $n = 0,\dots,m_\nu-1$ (if $m_\nu > 0$), the corresponding normalized eigenfunctions are $\frac{\phi(x,\lambda_n)}{\|\phi(\cdot,\lambda_n)\|}$, $\|f\| = \sqrt{\langle f,f\rangle}$ is the norm in $\mathcal{H}$, and the coefficients of the expansion are

$$F(\lambda) = \int_0^\infty f(x)\phi(x,\lambda)\mathfrak{m}(x)dx, \tag{31}$$

$$c_n = \langle f, \varphi(\cdot,\lambda_n)\rangle \tag{32}$$

with the Parseval equality

$$\|f\|^2 = \int_{\frac{\nu^2}{2}}^{\infty} F^2(\lambda)d\rho_c(\lambda) + \sum_{n=0}^{m_\nu-1} \frac{c_n^2}{\|\phi(\cdot,\lambda_n)\|^2}. \tag{33}$$

Note that on the finite interval $(0,b]$ the spectrum is purely discrete, while it has a continuous portion on the infinite interval $(0,\infty)$. The real-variable approach of Levitan (1950) and Levinson (1951) to problems with continuous spectrum is to first consider the problem on the finite interval and then pass to the limit $b\to\infty$[7]. We will first construct the eigenfunction expansion (26) associated with the operator (17), (18) on the interval $(0,b]$ with the Dirichlet boundary condition at b, and then pass to the limit $b\to\infty$ to recover the eigenfunction expansion (30) of the original problem on $(0,\infty)$.

Consider the eigenfunction expansion (26). Introduce a monotonically non-decreasing right-continuous step function (the *spectral function* of the problem on $(0,b]$):

$$\rho_b(\lambda) = \sum_{n=0}^{\infty} \frac{1}{\|\phi(\cdot,\lambda_{n,b})\|_b^2} \mathbf{1}_{\{\lambda_{n,b}\le\lambda\}}, \tag{34}$$

[7] An alternative is the complex-variable approach of Titchmarsh (1965).

where $\mathbf{1}_{\{\lambda_{n,b}\leq\lambda\}} = 1(0)$ if $\lambda_{n,b} \leq \lambda$ ($\lambda_{n,b} > \lambda$). It jumps by $\frac{1}{\|\phi(\cdot,\lambda_{n,b})\|_b^2}$ at an eigenvalue $\lambda_{n,b}$. The eigenfunction expansion (26) can now be re-written in the equivalent form as a Stieltjes integral:

$$f(x) = \int_0^\infty F_b(\lambda)\phi(x,\lambda)d\rho_b(\lambda), \tag{35}$$

$$F_b(\lambda) = \int_0^b f(x)\phi(x,\lambda)\mathfrak{m}(x)dx, \tag{36}$$

with the Parseval equality

$$\|f\|^2 = \int_0^\infty F_b^2(\lambda)d\rho_b(\lambda).$$

By theorems in Chapter 2 of Levitan and Sargsjan (1975) (see also Levitan and Sargsjan (1991) and Fulton and Pruess (1998)), the limit $\rho(\lambda) = \lim_{b\to\infty}\rho_b(\lambda)$ exists and is a monotonically non-decreasing right-continuous function which is bounded on every finite interval — the *spectral function* of the problem on $(0,\infty)$. Then the eigenfunction expansion for the problem on $(0,\infty)$ can be derived from (35) by letting $b\to\infty$, and the end result is in the form of the Stieltjes integral transform pair:

$$f(x) = \int_0^\infty F(\lambda)\phi(x,\lambda)d\rho(\lambda), \tag{37}$$

$$F(\lambda) = \int_0^\infty f(x)\phi(x,\lambda)\mathfrak{m}(x)dx, \tag{38}$$

with the Parseval equality

$$\|f\|^2 = \int_0^\infty F^2(\lambda)d\rho(\lambda). \tag{39}$$

6 The Up-and-Out Problem on $(0,b]$: An Eigenfunction Expansion

The goal of this Section is to explicitly calculate the function $P_b^{(\nu)}(x,k,\tau)$ defined by Eq.(20). Our strategy consists of the following steps. Consider an up-and-out claim on the process X with the payoff $\mathbf{1}_{\{\mathcal{T}_b>\tau\}}f(X_\tau)$ for some $f\in\mathcal{H}_b$. The first step is to explicitly develop the payoff f into an eigenfunction expansion (26). Next, we take the expectation of the eigenfunction expansion. Then, we specify to the put payoff $f(x)=(k-x)^+$ and explicitly calculate the function $P_b^{(\nu)}(x,k,\tau)$. For large $b >> x\vee k$, $P_b^{(\nu)}(x,k,\tau)$ closely approximates $P^{(\nu)}(x,k,\tau)$. Substituting it in place of $P^{(\nu)}(x,k,\tau)$ in Eq.(11) we obtain an analytical approximation formula for the generalized Asian put price. Finally, passing to the limit $w\to 0$ ($x=0$, $k=\frac{\tau K}{S_0}$), we obtain the analytical approximation for the standard Asian put price (9), (6). In the next Section we explicitly take the limit $b\to\infty$ and obtain the exact Asian option pricing formula (10), (6).

Proposition 1 *For any $\lambda \in \mathbb{C}$, $\nu \in \mathbb{R}$ and $b > 0$, the unique (up to a multiplicative factor independent of x) solution of the ODE (25), (16) such that*

$$\int_0^b |\phi(x,\lambda)|^2 \mathfrak{m}(x)dx < \infty$$

is given by:[8]

$$\phi(x,\lambda) = x^{\frac{1-\nu}{2}} e^{\frac{1}{4x}} W_{\frac{1-\nu}{2},\frac{1}{2}\sqrt{\nu^2-2\lambda}}\left(\frac{1}{2x}\right), \tag{40}$$

where $W_{\kappa,\mu}(z)$ is the second Whittaker function (Abramowitz and Stegun (1972, p. 505), Slater (1960, p.10), and Buchholz (1969, p.19)). This solution is entire in λ for fixed $x > 0$.

Proof. The transformation

$$z = \frac{1}{2x}, \quad u(x) = x^{\frac{1-\nu}{2}} e^{\frac{1}{4x}} v(z(x)) \tag{41}$$

reduces the ODE (25) with the operator (16) to the ODE for $v(z)$:

$$v_{zz} + \left(-\frac{1}{4} + \frac{\kappa}{z} + \frac{\frac{1}{4} - \mu^2}{z^2}\right) v = 0, \quad z \in (0,\infty), \tag{42}$$

where $\kappa = \frac{1-\nu}{2}$ and $\mu = \frac{1}{2}\sqrt{\nu^2 - 2\lambda}$. This is the Whittaker's form of the confluent hypergeometric equation (Abramowitz and Stegun (1972, p.505), Slater (1960, p.9), and Buchholz (1969, p.11)). The Whittaker functions $W_{\kappa,\mu}(z)$ and $W_{-\kappa,\mu}(e^{i\pi}z)$ provide two linearly independent solutions of the Whittaker equation for any values of κ and μ (real or complex) with the Wronskian $e^{-i\kappa\pi}$ (Buchholz (1969, p.25)). Since zero is a limit point end-point of the original ODE (25), for any complex λ there is only one square-integrable solution. Using the asymptotics of the Whittaker function (Slater (1960, p.65))

$$W_{\kappa,\mu}(z) \sim z^{\kappa} e^{-\frac{z}{2}} \quad \text{as} \quad z \to \infty, \tag{43}$$

we establish that the solution (40) is square-integrable with the speed density (18) on $(0, b]$ for any $\lambda \in \mathbb{C}$, $\nu \in \mathbb{R}$. Since $W_{\kappa,\mu}(z)$ is entire in μ for fixed $\kappa \in \mathbb{R}$ and $z > 0$ and is even in μ, $\phi(x,\lambda)$ is entire and single-valued in λ for fixed $x > 0$.$\square$

The eigenvalues are found as zeros of the Whittaker function $W_{\frac{1-\nu}{2},\frac{1}{2}\sqrt{\nu^2-2\lambda}}\left(\frac{1}{2b}\right)$ considered as a function of λ for fixed ν and b. First, from the Sturm-Liouville theory we know that all eigenvalues (and, hence, zeros) are simple and positive. Thus, we are looking for zeros on the positive real line. Consider two cases: $0 < \lambda \leq \frac{\nu^2}{2}$ and $\lambda > \frac{\nu^2}{2}$. In the first

[8] Note that, despite of the square root in the second index of the Whittaker function in Eq.(40), $\phi(x,\lambda)$ is a single-valued function of λ. This follows from the fact that $W_{\kappa,\mu}(z)$ is even in the second index, $W_{\kappa,\mu}(z) = W_{\kappa,-\mu}(z)$ (Slater (1960, p.11)).

case, without loss of generality set $\lambda = \frac{\nu^2 - q^2}{2}$, $0 \leq q < |\nu|$. Let $m_\nu(b) \geq 0$ be the number of roots of the equation

$$W_{\frac{1-\nu}{2}, \frac{q}{2}}\left(\frac{1}{2b}\right) = 0 \tag{44}$$

in the interval $0 \leq q < |\nu|$. Then, for $m_\nu(b) > 0$, the eigenvalues corresponding to the roots $\{q_{n,b}, n = 0, ..., m_\nu(b)\}$ are:

$$\lambda_{n,b} = \frac{\nu^2 - q_{n,b}^2}{2}, \quad n = 0, ..., m_\nu(b) - 1. \tag{45}$$

To get precise numerical values of the zeros $q_{n,b}$, we need to find the roots of Eq.(44) numerically. However, for large b we can get estimates by using the following asymptotics of the Whittaker function for $\mu > 0$ and $z > 0$ (Slater (1960)):

$$W_{\kappa,\mu}(z) \sim \frac{\Gamma(2\mu)}{\Gamma(\frac{1}{2} + \mu - \kappa)} z^{\frac{1}{2}-\mu} e^{-\frac{z}{2}} \text{ as } z \to 0. \tag{46}$$

This gives the large-b asymptotics

$$\phi\left(b, \frac{\nu^2 - q^2}{2}\right) = b^{\frac{1-\nu}{2}} e^{\frac{1}{4b}} W_{\frac{1-\nu}{2}, \frac{q}{2}}\left(\frac{1}{2b}\right) \sim 2^{\frac{\nu-1}{2}} \frac{\Gamma(q)}{\Gamma(\frac{\nu+q}{2})} \left(\frac{1}{2b}\right)^{\frac{\nu-q}{2}}. \tag{47}$$

For $\nu < 0$, the reciprocal of the Gamma function $1/\Gamma\left(\frac{\nu+q}{2}\right)$ has zeros at the non-positive integers $\frac{\nu+q}{2} = -n$, $n = 0, ..., [|\nu|/2]$. Thus, for $\nu < 0$ and large b the total number of eigenvalues less than or equal to $\nu^2/2$ is equal to $m_\nu := \lim_{b\to\infty} m_\nu(b) = [|\nu|/2] + 1$, and we have an estimate:

$$q_{n,b} = (|\nu| - 2n) + o(1), \quad \lambda_{n,b} = 2n(|\nu| - n) + o(1), \quad n = 0, ..., [|\nu|/2]. \tag{48}$$

For $\nu \geq 0$ the right-hand side of Eq.(47) never vanishes for any value of $q \in [0, \nu)$, and thus $m_\nu(b) = 0$ in this case (i.e., there are no eigenvalues less or equal to $\nu^2/2$).

Now consider the second case $\lambda > \frac{\nu^2}{2}$. Without loss of generality set $\lambda = \frac{\nu^2 + p^2}{2}$, $p > 0$. Then the eigenvalues above $\nu^2/2$ are given by

$$\lambda_{m_\nu(b)+n,b} = \frac{\nu^2 + p_{n,b}^2}{2}, \quad n = 0, 1, 2, ..., \tag{49}$$

where $\{p_{n,b}, n = 0, 1, 2, ...\}$ are the positive roots of the equation ($W_{\kappa, \frac{ip}{2}}(z)$ is a real function of p since the Whittaker function is even in the second index)

$$W_{\frac{1-\nu}{2}, \frac{ip}{2}}\left(\frac{1}{2b}\right) = 0. \tag{50}$$

To get precise numerical values of the $p_{n,b}$, we need to find the roots of Eq.(50) numerically. However, we can obtain estimates that improve with increasing n by using the following asymptotic estimate of the Whittaker function with purely imaginary second index for $p > 0$, $\kappa \in \mathbb{R}$, and fixed $z > 0$ (Shrivastava, Vasil'ev, and Yakubovich (1998), Eq.(3.37) and Yakubovich (1996), Theorem 1.11):

$$W_{\kappa,\frac{ip}{2}}(z) = \sqrt{2z}e^{-\frac{\pi p}{4}}\left(\frac{p}{2}\right)^{\kappa-\frac{1}{2}}\cos\left(\frac{p}{2}\ln\left(\frac{z}{2p}\right)+\frac{p}{2}-\frac{\pi}{2}\left(\kappa-\frac{1}{2}\right)\right)\left[1+O\left(\frac{1}{p}\right)\right]. \quad (51)$$

With the view towards passing to the limit $b \to \infty$, we note the leading term in the large-b asymptotics of the $p_{n,b}$ for fixed n:

$$p_{n,b} = \frac{2\pi(n+\nu/4+1/2)}{\ln(b)} + o\left(\frac{1}{\ln(b)}\right), \quad (52)$$

and

$$p_{n+1,b} - p_{n,b} = \frac{2\pi}{\ln(b)} + o\left(\frac{1}{\ln(b)}\right). \quad (53)$$

As b increases, the $p_{n,b}$ are distributed denser and denser on the positive real line, and the corresponding eigenvalues are distributed denser and denser in the interval $(\nu^2/2, \infty)$. The corresponding eigenfunctions and eigenfunction norms are given in the following

Proposition 2 *For any $\nu \in \mathbb{R}$, the norms of the eigenfunctions corresponding to the eigenvalues (49),*

$$\phi(x, \lambda_{m_\nu(b)+n,b}) = x^{\frac{1-\nu}{2}} e^{\frac{1}{4x}} W_{\frac{1-\nu}{2},\frac{ip_{n,b}}{2}}\left(\frac{1}{2x}\right), \quad n = 0, 1, 2, ..., \quad (54)$$

are:

$$\frac{1}{\|\phi(x, \lambda_{m_\nu(b)+n,b})\|_b^2} = \frac{2p_{n,b}\Gamma(\frac{\nu+ip_{n,b}}{2})M_{\frac{1-\nu}{2},\frac{ip_{n,b}}{2}}\left(\frac{1}{2b}\right)}{\Gamma(1+ip_{n,b})\left(\frac{\partial W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2b}\right)}{\partial p}\right)\Bigg|_{p=p_{n,b}}}, \quad n = 0, 1, 2, ..., \quad (55)$$

where $\Gamma(z)$ is the Gamma function (Abramowitz and Stegun (1972, p.255)), and $M_{\kappa,\mu}(z)$ is the first Whittaker function (Abramowitz and Stegun (1972, p. 505), Slater (1960, p.9), and Buchholz (1969, p.9)). For $m_\nu(b) > 0$, the norms of the eigenfunctions corresponding to the $m_\nu(b)$ additional eigenvalues (45),

$$\phi(x, \lambda_{n,b}) = x^{\frac{1-\nu}{2}} e^{\frac{1}{4x}} W_{\frac{1-\nu}{2},\frac{q_{n,b}}{2}}\left(\frac{1}{2x}\right), \quad n = 0, ..., m_\nu(b) - 1, \quad (56)$$

are:

$$\frac{1}{\|\phi(x, \lambda_{n,b})\|_b^2} = \frac{2q_{n,b}\Gamma(\frac{\nu+q_{n,b}}{2})M_{\frac{1-\nu}{2},\frac{q_{n,b}}{2}}\left(\frac{1}{2b}\right)}{\Gamma(1+q_{n,b})\left(-\frac{\partial W_{\frac{1-\nu}{2},\frac{q}{2}}\left(\frac{1}{2b}\right)}{\partial q}\right)\Bigg|_{q=q_{n,b}}}, \quad n = 0, ..., m_\nu(b) - 1. \quad (57)$$

Proof. For any $\lambda \in \mathbb{C}$, let $\eta(x,\lambda)$ be the (unique) solution of the ODE (25), (16) with the initial conditions at b (prime denotes differentiation in x):

$$\eta(b,\lambda) = 0, \;\; \eta'(b,\lambda) = -1. \tag{58}$$

For fixed x, it is an entire function of λ. Proceeding similarly to the proof of Proposition 1, this solution is obtained in the form:

$$\eta(x,\lambda) = 2b^2 \left(\frac{x}{b}\right)^{\frac{1-\nu}{2}} e^{\frac{1}{4x}-\frac{1}{4b}} \Delta_{\frac{1-\nu}{2},\frac{1}{2}\sqrt{\nu^2-2\lambda}} \left(\frac{1}{2b}, \frac{1}{2x}\right), \tag{59}$$

where $\Delta_{\kappa,\mu}(y,z)$ is the solution of the Whittaker equation (42) with the initial conditions at $z = y$: $\Delta_{\kappa,\mu}(y,y) = 0$ and $\frac{d\Delta_{\kappa,\mu}(y,z)}{dz}\Big|_{z=y} = 1$. It can be taken in the form (see Davydov and Linetsky (2001b), Eq.(60)):

$$\Delta_{\kappa,\mu}(y,z) = W_{\kappa,\mu}(y) W_{-\kappa,\mu}(e^{i\pi}z) e^{i\kappa\pi} - W_{\kappa,\mu}(y) W_{-\kappa,\mu}(e^{i\pi}z) e^{i\kappa\pi}. \tag{60}$$

An equivalent representation using the first and second Whittaker functions $M_{\kappa,\mu}(z)$ and $W_{\kappa,\mu}(z)$ as the two linearly independent solutions of the Whittaker's equation is (Davydov and Linetsky (2001b), Eq.(69) and Buchholz (1969, p.20))

$$\Delta_{\kappa,\mu}(y,z) = \frac{\Gamma(\frac{1}{2}+\mu-\kappa)}{\Gamma(1+2\mu)} \left[W_{\kappa,\mu}(y) M_{\kappa,\mu}(z) - M_{\kappa,\mu}(y) W_{\kappa,\mu}(z) \right]. \tag{61}$$

For a generic value λ, the two solutions $\phi(x,\lambda)$ and $\eta(x,\lambda)$ are linearly independent, and their Wronskian is ($\mathfrak{s}(s)$ is the scale density (18)):

$$W_x(\eta(\cdot,\lambda), \phi(\cdot,\lambda)) = \mathfrak{s}(x) C(\lambda), \;\; C(\lambda) = b^{\frac{\nu+3}{2}} e^{-\frac{1}{4b}} W_{\frac{1-\nu}{2},\frac{1}{2}\sqrt{\nu^2-2\lambda}} \left(\frac{1}{2b}\right) = \frac{\phi(b,\lambda)}{\mathfrak{s}(b)}. \tag{62}$$

While $\eta(x,\lambda)$ satisfies the Dirichlet boundary condition at b for any complex λ, generally it does not satisfy the square-integrability condition. Only when λ is equal to an eigenvalue $\lambda_{n,b}$, the solutions $\eta(x,\lambda)$ and $\phi(b,\lambda)$ become linearly dependent (their Wronskian (62) vanishes), and $\eta(x,\lambda_{n,b}) = A_{n,b}\phi(x,\lambda_{n,b})$, where the (non-zero) constant $A_{n,b}$ is

$$A_{n,b} = -2b^{\frac{\nu+3}{2}} e^{-\frac{1}{4b}} \frac{\Gamma\left(\frac{\nu}{2} + \frac{1}{2}\sqrt{\nu^2 - 2\lambda_{n,b}}\right)}{\Gamma(1+\sqrt{\nu^2-2\lambda_{n,b}})} M_{\frac{1-\nu}{2},\frac{1}{2}\sqrt{\nu^2-2\lambda_{n,b}}} \left(\frac{1}{2b}\right), \tag{63}$$

and $\phi(x,\lambda_{n,b})$ is the corresponding (not normalized) eigenfunction. Finally, from Davydov and Linetsky (2001b) (Eq.(24) and discussion in Section 2.2) we have for the norm:

$$\|\phi(\cdot,\lambda_{n,b})\|_b^2 = -\frac{C'(\lambda_{n,b})}{A_{n,b}}, \;\; C'(\lambda_{n,b}) = \left(\frac{dC(\lambda)}{d\lambda}\right)\Bigg|_{\lambda=\lambda_{n,b}}. \tag{64}$$

Substituting Eqs.(62), (63), (45) and (49) into Eq.(64) gives the final results for the norms (55) and (57).□

Any payoff $f \in \mathcal{H}_b$ can now be developed into an eigenfunction expansion of the form (26), (27). For its price at time zero we have the following

Proposition 3 *For any $\tau \geq 0$ and $b > x > 0$,*

$$\mathbf{E}_x[\mathbf{1}_{\{\mathcal{T}_b > \tau\}} f(X_\tau)] = \sum_{n=0}^{\infty} c_{n,b}\, e^{-\lambda_{n,b}\tau} \frac{\phi(x, \lambda_{n,b})}{\|\phi(\cdot, \lambda_{n,b})\|_b^2}, \ \ c_{n,b} = \langle f, \phi(\cdot, \lambda_{n,b})\rangle_b. \tag{65}$$

Proof. The function $\phi(x, \lambda)$ is bounded on $(0, b]$. In fact, there is a limit (it follows from the asymptotics (43) for the Whittaker function)

$$\lim_{x \to 0} \phi(x, \lambda) = 2^{\frac{\nu - 1}{2}}. \tag{66}$$

For each eigenvalue $\lambda_{n,b}$, consider a process $\{e^{\lambda_{n,b}t}\phi(X_t, \lambda_{n,b}), t \in [0, \tau]\}$. Due to the boundedness of $\phi(x, \lambda_{n,b})$ on $(0, b]$, Ito's lemma, and the fact that $\phi(x, \lambda)$ is a solution of the ODE (25), (16), it is a martingale for $t \in [0, \tau]$. Furthermore, since $\phi(b, \lambda_{n,b}) = 0$, we have that ($x \wedge y := \min\{x, y\}$)

$$e^{\lambda_{n,b}(t \wedge \mathcal{T}_b)}\phi(X_{t \wedge \mathcal{T}_b}, \lambda_{n,b}) = e^{\lambda_{n,b}t}\phi(X_t, \lambda_{n,b})\mathbf{1}_{\{\mathcal{T}_b > t\}} \tag{67}$$

is also a martingale for $t \in [0, \tau]$ and, thus,

$$\mathbf{E}_x[\mathbf{1}_{\{\mathcal{T}_b > \tau\}}\phi(X_\tau, \lambda_{n,b})] = e^{-\lambda_{n,b}\tau}\phi(x, \lambda_{n,b}). \tag{68}$$

The expectation (65) of the eigenfunction expansion (26), (27) can now be obtained through an application of Fubini theorem.$\square$

We are now ready to come back to the calculation of $P_b^{(\nu)}(x, k, \tau)$. The payoff is $f(x) = (k - x)^+$, and we can readily calculate the coefficients of the eigenfunction expansion in closed form.

Proposition 4 *Define the function*

$$F_k^{(\nu)}(s) := \frac{1}{2}k^{\frac{\nu+3}{2}}e^{-\frac{1}{4k}}W_{-\frac{\nu+3}{2}, \frac{is}{2}}\left(\frac{1}{2k}\right), \ \ \nu \in \mathbb{R}, \ \ k > 0, \ \ s \in \mathbb{C}. \tag{69}$$

Then the coefficients of the expansion (65) of $f(x) = (k - x)^+$ corresponding to the eigenfunctions (54) and (56) are

$$c_{m_\nu(b)+n,b} = F_k^{(\nu)}(p_{n,b}), \ \ n = 0, 1, 2, ..., \tag{70}$$

and (for $m_\nu(b) > 0$)

$$c_{n,b} = F_k^{(\nu)}(-iq_{n,b}), \ \ n = 0, ..., m_\nu(b) - 1, \tag{71}$$

respectively.

Proof. The result follows from Eqs.(27), (54), (56), and the following integral identity for the Whittaker function ($\mu \in \mathbb{C}$, $\nu \in \mathbb{R}$, $k > 0$):

$$\int_0^k (k-x)x^{\frac{\nu-1}{2}} e^{-\frac{1}{4x}} W_{\frac{1-\nu}{2},\mu}\left(\frac{1}{2x}\right) dx = k^{\frac{\nu+3}{2}} e^{-\frac{1}{4k}} W_{-\frac{\nu+3}{2},\mu}\left(\frac{1}{2k}\right). \tag{72}$$

This identity follows from the integral given in Gradshteyn and Ryzhik (1994), p.867, Eq.(7) by changing the integration variable.□

Thus, we have explicitly calculated the function $P_b^{(\nu)}(x, k, \tau)$

$$P_b^{(\nu)}(x, k, \tau) = \sum_{n=0}^{\infty} c_{n,b}\, e^{-\lambda_{n,b}\tau} \frac{\phi(x, \lambda_{n,b})}{\|\phi(\cdot, \lambda_{n,b})\|_b^2} \tag{73}$$

with the eigenvalues, eigenfunctions, eigenfunction norms, and coefficients given above. The result was shown for each fixed $\tau \geq 0$, $\nu \in \mathbb{R}$, $x > 0$, $k > 0$, and $b > x \vee k$. This function enters the generalized Asian put valuation formula. To price the standard Asian put, we need to take the limit $x \to 0$ in Eq.(73) (recall that $x = \tau w/(1-w)$).

Proposition 5 *For each fixed $\tau > 0$, $\nu \in \mathbb{R}$, $0 < k < b$, the limit $P_b^{(\nu)}(k, \tau) = \lim_{x\to 0} P_b^{(\nu)}(x, k, \tau)$ is given by*

$$P_b^{(\nu)}(k, \tau) = 2^{\frac{\nu-1}{2}} \sum_{n=0}^{\infty} e^{-\lambda_{n,b}\tau} \frac{c_{n,b}}{\|\phi(\cdot, \lambda_{n,b})\|_b^2}. \tag{74}$$

The series is absolutely convergent.

Substituting in Eq.(74) the explicit expressions for the ingredients (the eigenvalues, eigenfunction norms and coefficients found previously), Eq.(74) yields the series formula (9).

Proof. The limit of each individual term in the sum follows from the limit (66). To show that we can interchange the order of taking the limit and the summation, we show that the series (74) is absolutely convergent for each fixed $\tau > 0$. For large $n > m_\nu(b)$, the individual terms in the sum can be estimated as

$$e^{-\lambda_{m_\nu(b)+n,b}\tau} \frac{c_{m_\nu(b)+n,b}}{\|\phi(\cdot, \lambda_{m_\nu(b)+n,b})\|_b^2} = 2e^{-\frac{(\nu^2+p_{n,b}^2)\tau}{2} + \frac{\pi p_{n,b}}{4}} \tag{75}$$

$$\times k^{\frac{\nu}{2}+1} e^{-\frac{1}{4k}} \frac{(p_{n,b}/2)^{\frac{\nu}{2}-2}}{\ln(4bp_{n,b})} \cos\left(\frac{p_{n,b}}{2}(1 - \ln(4kp_{n,b})) + \pi\left(\frac{\nu}{4}+1\right)\right) \left[1 + O\left(\frac{1}{p_{n,b}}\right)\right].$$

To show this estimate, use Eqs.(69), (70), and (51) to estimate the coefficient

$$c_{m_\nu(b)+n,b} = \frac{1}{2} k^{\frac{\nu}{2}+1} e^{-\frac{1}{4k}} e^{-\frac{\pi p_{n,b}}{4}} \left(\frac{p_{n,n}}{2}\right)^{-\frac{\nu}{2}-2} \tag{76}$$

$$\times \cos\left(\frac{p_{n,b}}{2}(1-\ln(4kp_{n,b}))+\pi\left(\frac{\nu}{4}+1\right)\right)\left[1+O\left(\frac{1}{p_{n,b}}\right)\right],$$

use the estimate of the eigenfunction norm (55)

$$\frac{1}{\|\phi(\cdot,\lambda_{m_\nu(b)+n,b})\|_b^2} = e^{\frac{\pi p_{n,b}}{2}}\frac{4(p_{n,b}/2)^\nu}{\ln(4bp_{n,b})}\left[1+O\left(\frac{1}{p_{n,b}}\right)\right], \tag{77}$$

and recall Eq.(49) for the eigenvalues. The series (74) is absolutely convergent for any $\tau>0$ due to the factor $e^{-\frac{\tau p_{n,b}^2}{2}}$ in Eq.(75).

To prove the estimate (77), we need to use the estimate (51), the estimate[9] (for $z>0$, $p>0$, $\kappa\in\mathbb{R}$)

$$M_{\kappa,\frac{ip}{2}}(z) = z^{\frac{1+ip}{2}}\left[1+O\left(\frac{1}{p}\right)\right] = \sqrt{z}\,e^{\frac{ip}{2}\ln z}\left[1+O\left(\frac{1}{p}\right)\right], \tag{78}$$

and a version of the Stirling formula (Yakubovich (1996, p.20)) (for $x\in\mathbb{R}$ and $y>0$)

$$\Gamma(x+iy) = \sqrt{2\pi}\,y^{x-\frac{1}{2}}e^{-\frac{\pi y}{2}}e^{i\left(\frac{\pi}{2}\left(x-\frac{1}{2}\right)+y\ln y-y\right)}\left[1+O\left(\frac{1}{y}\right)\right]. \tag{79}$$

Combining (78) and (79), we obtain the estimate

$$\frac{\Gamma\left(\frac{\nu+ip}{2}\right)M_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2b}\right)}{\Gamma(1+ip)} = b^{-\frac{1}{2}}2^{-\frac{\nu}{2}}p^{\frac{\nu}{2}-1}e^{\frac{\pi p}{4}}e^{i\left(\frac{p}{2}-\frac{p}{2}\ln(4bp)+\frac{\pi\nu}{4}-\frac{\pi}{2}\right)}\left[1+O\left(\frac{1}{p}\right)\right]. \tag{80}$$

Substituting $p=p_{n,b}$ and noting that, by Eqs.(50) and (51),

$$\cos\left(\frac{p_{n,b}}{2}(1-\ln(4bp_{n,b}))+\frac{\pi\nu}{4}\right) = O\left(\frac{1}{p_{n,b}}\right), \tag{81}$$

and

$$\sin\left(\frac{p_{n,b}}{2}(1-\ln(4bp_{n,b}))+\frac{\pi\nu}{4}\right) = (-1)^{n+1}+O\left(\frac{1}{p_{n,b}}\right), \tag{82}$$

we have

$$\frac{\Gamma\left(\frac{\nu+ip_{n,b}}{2}\right)M_{\frac{1-\nu}{2},\frac{ip_{n,b}}{2}}\left(\frac{1}{2b}\right)}{\Gamma(1+ip_{n,b})} = (-1)^{n+1}b^{-\frac{1}{2}}2^{-\frac{\nu}{2}}(p_{n,b})^{\frac{\nu}{2}-1}e^{\frac{\pi p_{n,b}}{4}}\left[1+O\left(\frac{1}{p_{n,b}}\right)\right]. \tag{83}$$

Next, for the derivative of the Whittaker function in the denominator of Eq.(55) we have the estimate (from (51) and (82)):

$$\left(\frac{\partial W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2b}\right)}{\partial p}\right)\Bigg|_{p=p_{n,b}} = (-1)^{n+1}b^{-\frac{1}{2}}\ln(4bp_{n,b})\left(\frac{p_{n,b}}{2}\right)^{-\frac{\nu}{2}}e^{-\frac{\pi p_{n,b}}{4}}\left[1+O\left(\frac{1}{p_{n,b}}\right)\right]. \tag{84}$$

[9]This estimate follows from the results in the proof of Theorem 1.11 in Yakubovich (1996, p.26).

Substituting the results (83) and (84) into Eq.(55), we arrive at the estimate (77).□

Remark 6.1. Note that while the series (73) is absolutely convergent for each fixed $x > 0$ and $\tau \geq 0$, the series (74) is absolutely convergent for each fixed $\tau > 0$. It is divergent for $\tau = 0$ (from (75), the terms grow as $e^{\frac{\pi p_{n,b}}{4}}$ as $n \to \infty$, as there is no factor $e^{-\frac{\tau p_{n,b}^2}{2}}$).

To see that for $x > 0$ the series (73) is absolutely convergent for each $\tau \geq 0$, estimate the terms for large $n > m_\nu(b)$:

$$c_{m_\nu(b)+n,b} e^{-\lambda_{m_\nu(b)+n,b}\tau} \frac{\phi(x, \lambda_{m_\nu(b)+n,b})}{\|\phi(\cdot, \lambda_{m_\nu(b)+n,b})\|_b^2} = 8k\, e^{-\frac{(\nu^2+p_{n,b}^2)\tau}{2}} \left(\frac{k}{x}\right)^{\frac{\nu}{2}} e^{\frac{1}{4x}-\frac{1}{4k}} \frac{1}{p_{n,b}^2 \ln(4bp_{n,b})} \tag{85}$$

$$\times \cos\left(\frac{p_{n,b}}{2}(1 - \ln(4kp_{n,b})) + \pi\left(\frac{\nu}{4} + 1\right)\right) \cos\left(\frac{p_{n,b}}{2}(1 - \ln(4xp_{n,b})) + \frac{\pi\nu}{4}\right) \left[1 + O\left(\frac{1}{p_{n,b}}\right)\right].$$

The series (73) is absolutely convergent even for $\tau = 0$ due to the factor $\frac{1}{p_{n,b}^2}$.

The estimate (85) follows from (75) and the estimate for the eigenfunction $\phi(x, \lambda_{m_\nu(b)+n,b})$ following from Eq.(51):

$$\phi(x, \lambda_{m_\nu(b)+n,b}) = x^{-\frac{\nu}{2}} e^{\frac{1}{4x}} e^{-\frac{\pi p_{n,b}}{4}} \left(\frac{p_{n,b}}{2}\right)^{-\frac{\nu}{2}} \tag{86}$$

$$\times \cos\left(\frac{p_{n,b}}{2}(1 - \ln(4xp_{n,b})) + \frac{\pi\nu}{4}\right) \left[1 + O\left(\frac{1}{p_{n,b}}\right)\right].$$

Remark 6.2. Computationally, from the estimate (75) we see that the numerical convergence of the series (74) (and (9)) is governed by the parameter $\tau = \frac{\sigma^2 T}{4}$. The larger the product of time to maturity and squared volatility, the faster the series converges.

7 Passing to the Limit $b \to \infty$: An Integral Transform Formula

We are now ready to pass to the limit $b \to \infty$ along the lines outlined in Section 5. Let $\mathcal{H} = L^2((0, \infty), \mathfrak{m})$ as in Section 5.

Proposition 6 *(i) For any $f \in \mathcal{H}$, $\tau \geq 0$, $\nu \in \mathbb{R}$, and $x > 0$,*

$$\mathbf{E}_x[f(X_\tau)] = \frac{1}{\pi^2} \int_0^\infty \mathcal{F}(p) e^{-\frac{(\nu^2+p^2)\tau}{2}} x^{\frac{1-\nu}{2}} e^{\frac{1}{4x}} W_{\frac{1-\nu}{2}, \frac{ip}{2}}\left(\frac{1}{2x}\right) \left|\Gamma\left(\frac{\nu + ip}{2}\right)\right|^2 \sinh(\pi p)\, p\, dp$$

$$+\mathbf{1}_{\{\nu<0\}} \sum_{n=0}^{m_\nu - 1} c_n\, e^{-2n(|\nu|-n)\tau} \frac{(-1)^n 2^{\frac{\nu+3}{2}}(|\nu| - 2n)}{\Gamma(1 + |\nu| - n)} (2x)^n L_n^{(|\nu|-2n)}\left(\frac{1}{2x}\right), \tag{87}$$

where

$$m_\nu = \begin{cases} [|\nu|/2]+1, & \nu<0 \\ 0, & \nu\geq 0 \end{cases}, \tag{88}$$

$$\mathcal{F}(p) = \frac{1}{2}\int_0^\infty f(x)x^{\frac{\nu-1}{2}}e^{-\frac{1}{4x}}W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2x}\right)dx, \tag{89}$$

$$c_n = (-1)^n 2^{n+\frac{\nu-3}{2}} n! \int_0^\infty f(x)x^{n+\nu-1}e^{-\frac{1}{2x}}L_n^{(|\nu|-2n)}\left(\frac{1}{2x}\right)dx, \tag{90}$$

and $L_n^{(\alpha)}(z)$ are the generalized Laguerre polynomials (Abramowitz and Stegun (1972, p.773)).

(ii) For the put payoff $f(x)=(k-x)^+$, $k>0$, the function $P^{(\nu)}(x,k,\tau)=\mathbf{E}_x[(k-X_\tau)^+]$ is given by Eq.(87) with the coefficients

$$\mathcal{F}(p) = F_k^{(\nu)}(p) = \frac{1}{2}k^{\frac{\nu+3}{2}}e^{-\frac{1}{4k}}W_{-\frac{\nu+3}{2},\frac{ip}{2}}\left(\frac{1}{2k}\right),\ \ p>0,\ \ \nu\in\mathbb{R}, \tag{91}$$

$$c_0 = 2^{-\frac{\nu+5}{2}}\{2k\Gamma(|\nu|,1/(2k))-\Gamma(|\nu|-1,1/(2k))\},\ \ \nu<0, \tag{92}$$

$$c_1 = 2^{-\frac{\nu+5}{2}}\Gamma(|\nu|-2,1/(2k)),\ \ \nu<-2, \tag{93}$$

$$c_n = (-1)^n 2^{-\frac{\nu+5}{2}}(n-2)!(2k)^{\nu+n+1}e^{-\frac{1}{2k}}L_{n-2}^{(|\nu|-2n)}\left(\frac{1}{2k}\right), \tag{94}$$

$$\nu<-4,\ \ n=2,\dots,m_\nu-1,$$

where $\Gamma(a,z)$ is the incomplete Gamma function (Abramowitz and Stegun (1972), p.260, Eq.(6.5.3)). With these coefficients the integral on the right-hand side of Eq.(87) is absolutely convergent for each fixed $x>0$ and $\tau\geq 0$.

(iii) For any $\nu\in\mathbb{R}$, $k>0$, $\tau>0$, the limit $P^{(\nu)}(k,\tau)=\lim_{x\to 0}P^{(\nu)}(x,k,\tau)$ is given by

$$\begin{aligned} P^{(\nu)}(k,\tau) = {} & \frac{1}{8\pi^2}\int_0^\infty e^{-\frac{(\nu^2+p^2)\tau}{2}}(2k)^{\frac{\nu+3}{2}}e^{-\frac{1}{4k}}W_{-\frac{\nu+3}{2},\frac{ip}{2}}\left(\frac{1}{2k}\right)\left|\Gamma\left(\frac{\nu+ip}{2}\right)\right|^2\sinh(\pi p)\,p\,dp \\ & +\mathbf{1}_{\{\nu<0\}}\frac{1}{2\Gamma(|\nu|)}\{2k\Gamma(|\nu|,1/(2k))-\Gamma(|\nu|-1,1/(2k))\} \\ & +\mathbf{1}_{\{\nu<-2\}}e^{-2(|\nu|-1)\tau}\frac{(|\nu|-2)}{\Gamma(|\nu|)}\Gamma(|\nu|-2,1/(2k)) \\ & +\mathbf{1}_{\{\nu<-4\}}\sum_{n=2}^{m_\nu-1}e^{-2n(|\nu|-n)\tau}\frac{(-1)^n(|\nu|-2n)}{2n(n-1)\Gamma(1+|\nu|-n)}(2k)^{\nu+n+1}e^{-\frac{1}{2k}}L_{n-2}^{(|\nu|-2n)}\left(\frac{1}{2k}\right). \end{aligned} \tag{95}$$

The integral on the right-hand side of (95) is absolutely convergent for each fixed $\tau>0$.

Proof. (i) We will show that the spectral function $\rho(\lambda)=\lim_{b\to\infty}\rho_b(\lambda)$ takes the form:

$$\rho(\lambda)=\mathbf{1}_{\{\nu<0\}}\sum_{n=0}^{m_\nu-1}\frac{4(|\nu|-2n)}{\Gamma(1+|\nu|-n)n!}\mathbf{1}_{\{\lambda\geq 2n(|\nu|-n)\}} \tag{96}$$

$$+\mathbf{1}_{\{\lambda>\frac{\nu^2}{2}\}}\frac{1}{\pi^2}\int_0^{\sqrt{2\lambda-\nu^2}}\left|\Gamma\left(\frac{\nu+ip}{2}\right)\right|^2\sinh(\pi p)\,p\,dp.$$

First, for $\nu<0$ consider the eigenvalues (45) and the corresponding eigenfunctions (56). From Eq.(48) we have:

$$\lambda_n=\lim_{b\to\infty}\lambda_{n,b}=2n(|\nu|-n),\ \ n=0,...,[|\nu|/2], \tag{97}$$

and

$$\phi(x,\lambda_n)=x^{\frac{1-\nu}{2}}e^{\frac{1}{4x}}W_{\frac{1-\nu}{2},-\frac{\nu}{2}-n}\left(\frac{1}{2x}\right)=(-1)^n2^{n+\frac{\nu-1}{2}}n!x^nL_n^{(|\nu|-2n)}\left(\frac{1}{2x}\right). \tag{98}$$

The last equality in (98) follows from the fact that when $\kappa=\mu+n+\frac{1}{2}$ (for an integer n) the Whittaker function $W_{\kappa,\mu}(z)$ reduces to the generalized Laguerre polynomial (Buchholz (1969, p.214)):

$$W_{\mu+n+\frac{1}{2},\mu}(z)=(-1)^nn!e^{-\frac{z}{2}}z^{\mu+\frac{1}{2}}L_n^{(2\mu)}(z). \tag{99}$$

The norm of $\phi(x,\lambda_n)$ can be computed directly using the following integral for $a>0$ (Prudnikov, Brychkov and Marichev (1986), p.479, Eq.(17); note that this equation in PBM has a typo — the factor $(-1)^n$ should not be included)

$$\int_0^\infty z^{a-1}e^{-z}(L_n^{(a)}(z))^2dz=\frac{\Gamma(1+a+n)}{n!a}. \tag{100}$$

The result is:

$$\frac{1}{\|\phi(\cdot,\lambda_n)\|^2}=\frac{4(|\nu|-2n)}{\Gamma(1+|\nu|-n)n!}. \tag{101}$$

For $\nu<0$ this gives the sum in Eq.(96).

Let b be large enough so that $m_\nu(b)=m_\nu$. Consider the part of the spectral function (34) with the eigenvalues (49) (to lighten notation we omit the subscript b in $\lambda_{n,b}$ and $p_{n,b}$)

$$\sum_{n=m_\nu}^\infty\frac{1}{\|\phi(\cdot,\lambda_n)\|_b^2}\mathbf{1}_{\{\lambda_n\leq\lambda\}}=\sum_{n=0}^\infty\frac{1}{(p_{n+1}-p_n)\|\phi(\cdot,\lambda_{m_\nu+n})\|_b^2}\mathbf{1}_{\{p_n\leq\sqrt{2\lambda-\nu^2}\}}(p_{n+1}-p_n), \tag{102}$$

where we divided and multiplied by $(p_{n+1}-p_n)$ and used the parameterization of eigenvalues (49). The norms in Eq.(102) are given by Eq.(55). We now show that for large b

$$\frac{1}{(p_{n+1}-p_n)\|\phi(\cdot,\lambda_{m_\nu+n})\|_b^2}=\frac{1}{\pi^2}\left|\Gamma\left(\frac{\nu+ip_n}{2}\right)\right|^2\sinh(\pi p_n)\,p_n+O\left(\frac{1}{\ln(b)}\right). \tag{103}$$

The Whittaker functions have the following asymptotics for small values of the argument (Slater (1960), Buchholz (1969)):

$$M_{\kappa,\mu}(z)=z^{\frac{1}{2}+\mu}e^{-\frac{z}{2}}[1+O(z)], \tag{104}$$

$$\begin{aligned} W_{\kappa,\mu}(z) &= \frac{\pi}{\sin(2\mu\pi)}\left\{\frac{M_{\kappa,-\mu}(z)}{\Gamma(1-2\mu)\Gamma(\frac{1}{2}+\mu-\kappa)}-\frac{M_{\kappa,\mu}(z)}{\Gamma(1+2\mu)\Gamma(\frac{1}{2}-\mu-\kappa)}\right\} \\ &= \frac{\pi}{\sin(2\mu\pi)}\left\{\frac{z^{\frac{1}{2}-\mu}e^{-\frac{z}{2}}[1+O(z)]}{\Gamma(1-2\mu)\Gamma(\frac{1}{2}+\mu-\kappa)}-\frac{z^{\frac{1}{2}+\mu}e^{-\frac{z}{2}}[1+O(z)]}{\Gamma(1+2\mu)\Gamma(\frac{1}{2}-\mu-\kappa)}\right\}. \end{aligned} \tag{105}$$

Thus, for the Whittaker function in the numerator of Eq.(55) we have the large-b asymptotics

$$M_{\frac{1-\nu}{2},\frac{ip_n}{2}}\left(\frac{1}{2b}\right)=\left(\frac{1}{2b}\right)^{\frac{1+ip_n}{2}}\left[1+O\left(\frac{1}{2b}\right)\right]. \tag{106}$$

We now obtain the large-b asymptotics of the denominator in Eq.(55). First, the Whittaker function has the large-b asymptotics

$$W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2b}\right)=\frac{-i\pi}{\sinh(\pi p)}\left\{\frac{\left(\frac{1}{2b}\right)^{\frac{1-ip}{2}}\left[1+O\left(\frac{1}{2b}\right)\right]}{\Gamma(1-ip)\Gamma(\frac{\nu+ip}{2})}-\frac{\left(\frac{1}{2b}\right)^{\frac{1+ip}{2}}\left[1+O\left(\frac{1}{2b}\right)\right]}{\Gamma(1+ip)\Gamma(\frac{\nu-ip}{2})}\right\}. \tag{107}$$

The large-b asymptotics for the derivative is

$$\begin{aligned} \frac{\partial W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2b}\right)}{\partial p} &= \left(\frac{1}{2b}\right)^{\frac{1-ip}{2}}\left\{\frac{\pi\ln(2b)}{2\sinh(\pi p)\Gamma(1-ip)\Gamma(\frac{\nu+ip}{2})}+O(1)\right\} \\ &+ \left(\frac{1}{2b}\right)^{\frac{1+ip}{2}}\left\{\frac{\pi\ln(2b)}{2\sinh(\pi p)\Gamma(1+ip)\Gamma(\frac{\nu-ip}{2})}+O(1)\right\}. \end{aligned} \tag{108}$$

We now evaluate it at $p=p_n$. Since at $p=p_n$ the Whittaker function vanishes, $W_{\frac{1-\nu}{2},\frac{ip_n}{2}}\left(\frac{1}{2b}\right)=0$, using Eq.(107) we have

$$\left[\frac{\partial W_{\frac{1-\nu}{2},\frac{ip}{2}}\left(\frac{1}{2b}\right)}{\partial p}\right]\Bigg|_{p=p_n}=\left(\frac{1}{2b}\right)^{\frac{1+ip_n}{2}}\left\{\frac{\pi\ln(2b)}{\sinh(\pi p_n)\Gamma(1+ip_n)\Gamma(\frac{\nu-ip_n}{2})}+O(1)\right\}. \tag{109}$$

Putting together (55), (106), (109), and (53), we arrive at (103). Substituting the result (103) into Eq.(102) and passing to the limit $b \to \infty$ (and, by Eq.(53), $(p_{n+1} - p_n) \to 0$), we obtain the integral term in Eq.(96). Finally, the eigenfunction expansion (87) for $\tau = 0$ is obtained on substituting the spectral function (96) into Eq.(37) and using Eqs.(40) and (98). For $\tau > 0$, there is an additional factor $e^{-\lambda_{n,b}\tau}$ in Eq.(65) that accelerates convergence of the series and results in the factors $e^{-\frac{(\nu^2+p^2)\tau}{2}}$ and $e^{-2n(|\nu|-n)\tau}$ in Eq.(87) in the limit $b \to \infty$.

(ii) Eq.(91) follows from Eqs.(89) and (72). To show Eq.(94), note that from Eqs.(69), (71), and (97) we have $c_n = F_k^{(\nu)}(-i(|\nu|-2n))$, and use Eq.(98) to express $W_{-\frac{\nu+3}{2},-\frac{\nu}{2}-n}(1/(2k))$, $n = 2, ..., m_\nu - 1$, in terms of the generalized Laguerre polynomials. To show Eq.(93), note that $c_1 = F_k^{(\nu)}(-i(|\nu| - 2))$ and use the identity (Buchholz (1969), p.209)

$$W_{\mu-\frac{1}{2},\mu}(z) = z^{\frac{1}{2}-\mu} e^{\frac{z}{2}} \Gamma(2\mu, z) \tag{110}$$

to express $W_{-\frac{\nu+3}{2},-\frac{\nu}{2}-1}(1/(2k))$ in terms of the incomplete Gamma function

$$\Gamma(a, z) = \int_z^\infty y^{a-1} e^{-y} dy. \tag{111}$$

To show Eq.(92), express the integral (90) for $n = 0$ and $f(x) = (k - x)^+$,

$$c_0 = 2^{\frac{\nu-3}{2}} \int_0^k (k - x) x^{\nu-1} e^{-\frac{1}{2x}} dx, \tag{112}$$

in terms of the incomplete Gamma function.

(iii) First, using Eq.(66), the integrand in the integral on the righ-hand side of Eq.(87) has the limit as $x \to 0$

$$\frac{1}{8\pi^2} e^{-\frac{(\nu^2+p^2)\tau}{2}} (2k)^{\frac{\nu+3}{2}} e^{-\frac{1}{4k}} W_{-\frac{\nu+3}{2},\frac{ip}{2}} \left(\frac{1}{2k}\right) \left|\Gamma\left(\frac{\nu + ip}{2}\right)\right|^2 \sinh(\pi p)\, p. \tag{113}$$

For fixed $k > 0$ and $\tau > 0$, this continuous function of p is absolutely integrable over $(0, \infty)$. To see this, use Eq.(51) to estimate the Whittaker function for large p and a version of the Stirling formula (79)

$$\left|\Gamma\left(\frac{\nu + ip}{2}\right)\right| = \sqrt{2\pi} e^{-\frac{\pi p}{4}} \left(\frac{p}{2}\right)^{\frac{\nu-1}{2}} \left[1 + O\left(\frac{1}{p}\right)\right] \tag{114}$$

to estimate the Gamma function. We thus have an estimate for the function (113):

$$\frac{2^{\frac{3}{2}}}{\pi} e^{-\frac{(\nu^2+p^2)\tau}{2} + \frac{\pi p}{4}} p^{\frac{\nu}{2}-2} k^{\frac{\nu}{2}+1} e^{-\frac{1}{4k}} \cos\left(\frac{p}{2}(1 - \ln(4pk)) + \pi\left(\frac{\nu}{4} + 1\right)\right) \left[1 + O\left(\frac{1}{p}\right)\right]. \tag{115}$$

This function is absolutely integrable over $(0, \infty)$ due to the factor $e^{-\frac{p^2\tau}{2}}$. We can now interchange the order of taking the limit and integration, and obtain the integral term in Eq.(95). The m_ν additional terms in (95) are obtained in the limit $x \to 0$ from the m_ν terms in Eq.(87) with the coefficients (92)-(94) using the result for the generalized Laguerre polynomials

$$\lim_{z\to\infty} \left(z^{-n} L_n^{(a)}(z) \right) = \frac{(-1)^n}{n!}. \ \Box \tag{116}$$

Proposition 6 gives closed-form analytical expressions for the functions $P^{(\nu)}(x,k,\tau)$ and $P^{(\nu)}(k,\tau)$ entering the Asian option pricing formulae.[10] As was discussed in Section 5, the spectrum of the problem on $(0,\infty)$ has a continuous portion on $(\nu^2/2,\infty)$ plus $[|\nu|/2]+1$ additional eigenvalues in $[0,\nu^2/2)$ for $\nu<0$ (there are no additional eigenvalues for $\nu \geq 0$ and the spectrum is purely continuous).

Remark 7.1. For $\nu \geq 0$ the eigenfunction expansion (87), (89) takes the form of an *integral transform* with the Whittaker function in the kernel. This integral transform first appeared in Wimp (1964) and was recently studied by Shrivastava, Vasil'ev, and Yakubovich (1998) and Yakubovich (1996) in the context of the theory of integral transforms and special functions, without any apparent connection to the spectral theory of Sturm-Liouville operators or Markov processes. In this literature, the integral transform is refereed to as a Wimp transform or the index transform with the Whittaker function in the kernel since the Whittaker function enters the kernel in (87) (see also Prudnikov, Brychkov, and Marichev (1992, pp.527-8)). From the mathematical standpoint, the mathematical contribution of this paper is two-fold. First, we solve the Sturm-Liouville problem for the self-adjoint operator (16) on the interval $(0,b)$ with the Dirichlet boundary condition at b and explicitly construct the associated eigenfunction expansion. Second, we pass to the limit $b\to\infty$ and obtain the integral transform as an eigenfunction expansion associated with the self-adjoint operator (16) on the infinite interval $(0,\infty)$. Moreover, in the literature on index transforms only the case $\nu \geq 0$ was considered. The calculations in this paper are done for $\nu\in\mathbb{R}$. For $n<0$ there are a finite number of additional terms in the eigenfunction expansion (87).

Remark 7.2. For $\nu<0$, Wong (1964) constructed a spectral representation of the transition probability density $p(t,x,y)$ for the diffusion process (15) in the form similar to our Eq.(87). Comtet, Monthus, and Yor (1998) have obtained expressions for the density similar to Wong (1964) in the context of disordered systems and exponential functionals (see also Monthus and Comtet (1994) and Comtet and Monthus (1996) for applications to disordered systems in physics). Our approach in this paper is different as we rely on the real-variable approach to the problems with continuous spectrum and take the limit of the problem on $(0,b]$. The benefit of our approach is that we end up with two formulae

[10] We consider an option pricing formula "closed-form" if it is expressed as a series or an integral where the terms of the series or the integrand are explicitly characterized in terms of known elementary or transcendental functions, including known special functions. For further discussions of the notion of "closed form" for option pricing formulae see Boyle, Tian and Guan (2001).

instead of one — the series formula and the integral formula. The series formula has the advantage that it does not require any numerical integrations, and as we will see in Section 8, is just as accurate. To the best of our knowledge, this is the first application of the real-variable approach to eigenfunction expansions in finance (both Lewis (1998) and Davydov and Linetsky (2001b) follow the complex-variable approach of Titchmarsh (1962) that relies on the Laplace transform inversion by means of the Cauchy Residue Theorem).

Remark 7.3. Note that while the integral in Eq.(87) with the function (91) is absolutely convergent for each fixed $x > 0$ and $\tau \geq 0$, the integral in Eq.(95) is absolutely convergent for each fixed $\tau > 0$. It is divergent for $\tau = 0$ (from (115), the integrand grows as $e^{\frac{\pi p}{4}}$ as $p \to \infty$, as there is no factor $e^{-\frac{\tau p^2}{2}}$).

To see that for $x > 0$ the integral in Eq.(87) with the function (91) is absolutely convergent for each fixed $\tau \geq 0$, estimate the integrand for large p:

$$\frac{4k}{\pi p^2} k\, e^{-\frac{(\nu^2+p^2)\tau}{2}} \left(\frac{k}{x}\right)^{\frac{\nu}{2}} e^{\frac{1}{4x}-\frac{1}{4k}} \tag{117}$$

$$\times \cos\left(\frac{p}{2}(1-\ln(4kp)) + \pi\left(\frac{\nu}{4}+1\right)\right) \cos\left(\frac{p}{2}(1-\ln(4xp)) + \frac{\pi\nu}{4}\right)\left[1 + O\left(\frac{1}{p}\right)\right].$$

The factor $\frac{1}{p^2}$ insures absolute convergence of the integral even for $\tau = 0$. The estimate (117) follows from (115) and the estimate for the function $\phi(x, \lambda)$ as in Eq.(86).

Remark 7.4. Computationally, from the estimate (115) we see that the numerical convergence of the integral formula (95) (and (10)) is governed by the parameter $\tau = \frac{\sigma^2 T}{4}$. The larger the product of time to maturity and squared volatility, the faster the integral converges.

Remark 7.5. Note that from the very beginning we chose to work with puts rather than calls. The reason is that the call payoff is *not* in the Hilbert space $L^2((0,\infty), \mathfrak{m})$, while the put payoff is. Thus, there is no eigenfunction expansion of the form of Proposition 6 for the call. To price calls, we price puts first and then use the call-put parity relationship to recover the call price.

8 Computational Results

In this Section we use the two analytical formulae to compute Asian option prices. We compute Asian put prices first, and then use the call-put parity for Asian options (4) to compute call prices. We consider seven combinations of parameters given in Table 1. These combinations of parameters have been used as test cases for various numerical methods by Geman and Eydeland (1995), Fu, Madan, and Wang (1997), Shaw (1997), Craddock, Heath, and Platen (2000), Dufresne (2000), and Vecer (2000). Table 1 gives the interest rate r, volatility σ, time to expiration T, and the initial asset price S_0. It is assumed that the underlying asset pays no dividends ($q = 0$), the strike price is $K = 2.0$ for all cases, and all options start at time zero, $t = 0$ (newly written contracts). The standardized parameters ν, τ, and k are computed using Eq.(5). Since in this Section we consider standard Asian options, $w = 0$, $x = 0$, and $k = \frac{\tau K}{S_0}$.

We now use the series formula (9) and the integral formula (10) to compute Asian option prices. We used *Mathematica 4.0* (Wolfram (1999)) running on a Pentium III PC for all computations in this paper. First, pick $b = 1$ and consider the steps needed to compute the Asian option price using the series formula (9) (note that $b = 1$ is much greater than the normalized strike k for all seven cases in Table 1). First, we need to determine the eigenvalues. The eigenvalues need to be determined only once for all options with different strikes, times to expiration, and current values of the underlying asset price. The eigenvalues depend only on the parameter ν, which is a characteristic of the underlying stochastic process, and are independent of the current value of the process or the terms of the option contract to be valued. To determine the eigenvalues, we need to numerically determine zeros of the Whittaker function (44), (50) with respect to the parameters q and p in the second index. The Whittaker functions are related to the Kumer and Tricomi confluent hypergeometric functions that are provided as built-in functions in *Mathematica*. The relationships are (Abramowitz and Stegun (1972, p.505)):

$$M_{\kappa,\mu}(z) = z^{\frac{1}{2}+\mu} e^{-\frac{z}{2}} M(1/2 + \mu - \kappa, 1 + 2\mu, z), \tag{118}$$

$$W_{\kappa,\mu}(z) = z^{\frac{1}{2}+\mu} e^{-\frac{z}{2}} U(1/2 + \mu - \kappa, 1 + 2\mu, z). \tag{119}$$

For the seven test cases in Table 1 we need to determine the eigenvalues twice for $\nu = 3$ and $\nu = -0.6$. For $b = 1$ and $\nu = 3$ and $\nu = -0.6$, the Whittaker function (44) has no zeros in the interval $0 \leq q < |\nu|$ (it is strictly positive in this interval). Next consider the Whittaker function (50). Using remarkable capabilities of *Mathematica* in handling special functions with arbitrary precision arithmetics, we are able to determine the first one thousand zeros of the Whittaker function (50) with the precision of two hundred significant digits in just over an hour on a PC. Thus, numerical determination of zeros of the Whittaker function does not present any computational problems and can be accomplished with arbitrary precision. Table 2 gives the first five zeros and the corresponding eigenvalues with the precision of ten significant digits. After the eigenvalues are determined, the Asian put prices are calculated using the series formula (9).

Alternatively, the integral formula (10) can be directly programmed in *Mathematica 4.0* using the built-in numerical integration routine. The advantage of the series formula is that no numerical integration is required. We have computed Asian option prices in the seven test cases of Table 1 using both the series formula and the integral formula. Both the series with $b = 1$ and the integral formula gave identical results in all seven cases at the precision level of ten decimals. The results for the Asian call options are reported in Table 3 in the column marked EE (Eigenfunction Expansion). The numbers in parenthesis next to the option prices give the number of terms in the series (9) needed to achieve the precision of ten significant digits. One can see that while for the case 7 with the largest

Case	r	σ	T	S_0	ν	τ	k
1	0.02	0.10	1	2.0	3	0.0025	0.0025
2	0.18	0.30	1	2.0	3	0.0225	0.0225
3	0.0125	0.25	2	2.0	-0.6	0.03125	0.03125
4	0.05	0.50	1	1.9	-0.6	0.0625	0.065789
5	0.05	0.50	1	2.0	-0.6	0.0625	0.0625
6	0.05	0.50	1	2.1	-0.6	0.0625	0.059524
7	0.05	0.50	2	2.0	-0.6	0.125	0.125

Table 1: **Asian Option Parameter Values.** For all seven cases $q = 0$ and $K = 2.0$.

value of $\tau = \frac{\sigma^2 T}{4}$ only thirteen terms are enough to achieve this level of precision, the case 1 with the smallest value of τ requires four hundred terms. The integral formula (10) using the built-in numerical integration routine gave identical option values in all seven cases. However, it was slower than the series due to numerical integration. As expected, in both cases the dimensionless time to expiration $\tau = \frac{\sigma^2 T}{4}$ is the crucial parameter that controls the speed of convergence for the series and the integral. The larger the value of this parameter, the faster the convergence.

For comparison, Table 3 gives Asian call option prices reported by authors using different numerical methods. Eydeland and Geman (1995) (EG) use the Fast Fourier Transform algorithm to numerically invert the Geman and Yor (1993) Laplace transform. Fu, Madan, and Wang (1997) (FMW) use the Euler algorithm of Abate and Whitt (1995) to invert the Geman and Yor (1993) Laplace transform. The column marked TW reports the values obtained using the Turnbull and Wakeman (1992) analytical approximation that approximates the distribution of the average price as lognormal with matched first and second moments (these numerical values are reported by Vecer (2000)). The column marked D reproduces the values obtained in Dufresne (2000) using the Laguerre series approach. The column marked MC reports the values and standard errors obtained by Monte Carlo simulation in Dufresne (2000). Vecer (2000) (V) develops numerical finite-difference PDE schemes. One can see from the Table that the prices reported by these authors are only accurate up to two to four decimals. It appears that the numbers reported by Dufresne (2000) are the most accurate, as all four digits agree with our results for the cases 2-7. However, Dufresne's method apparently fails to converge for the case 1 with the smallest τ.

9 Conclusion

We see the contribution of this paper as two-fold. Firstly, we derive two analytical formulae that allow exact pricing of Asian options. The first formula is an infinite series. The terms of the series are explicitly characterized in terms of special functions and do not involve any numerical integrations. The second formula is a limit of the series formula and is in the form of an integral transform. It involves a single real integral

	Zero	Eigenvalue
n	$\nu = 3$	
1	4.041095077	12.66522471
2	6.198998995	23.71379427
3	8.103928065	37.33682504
4	9.875120946	53.25900684
5	11.55802557	71.29397758
n	$\nu = -0.6$	
1	1.943531071	2.068656513
2	4.473537623	10.18626943
3	6.516559483	21.41277375
4	8.371763836	35.22321486
5	10.11419245	51.32844445

Table 2: **Eigenvalues for $b = 1$ and $\nu = 3$ and $\nu = -0.6$.**

Case	EE	EG	FMW	TW	D	MC	V
1	.0559860415 (400)	0.058	.0560	.0568	0.0197	.05602 (0.00017)	.0560
2	0.2183875466 (57)	0.227	0.219	0.220	0.2184	0.2185 (0.00059)	0.219
3	0.1722687410 (41)	.1722	0.172	0.173	0.1723	0.1725 (0.00063)	0.173
4	0.1931737903 (24)	0.195	0.194	0.195	0.1932	0.1933 (0.00084)	0.194
5	0.2464156905 (23)	0.248	0.247	0.250	0.2464	0.2465 (0.00095)	0.247
6	0.3062203648 (23)	0.308	0.307	0.311	0.3062	0.3064 (0.00106)	0.307
7	0.3500952190 (13)	0.351	0.352	0.359	0.3501	0.3503 (0.00146)	0.350

Table 3: **Asian Call Option Prices.** Parameters are as in Table 1.

that needs to be computed numerically. Our formulae can now serve as benchmarks for general-purpose numerical algorithms in option pricing, such as Monte Carlo simulation and finite-difference PDE schemes.

Secondly, we view the Asian option problem as an interesting case study in applying the powerful eigenfunction expansion methodology to problems in Financial Engineering, Applied Probability and Operations Research. We hope that this paper, along with our paper Davydov and Linetsky (2001b), will stimulate further interest in this methodology. It is a standard approach in many calculations dealing with transient behavior of Markov processes to first compute a Laplace transform in time, and then invert it numerically. A variety of transform inversion algorithms have been developed for this purpose (see, e.g., Abate and Whitt (1995) and references therein). The eigenfunction expansion approach offers an alternative. The idea is to find the eigenvalues and eigenfunctions of the infinitesimal generator of the Markov process (or, in other words, construct the spectral resolution of the Markov semigroup). If it can be done analytically, the result is an exact analytical characterization of the transient behavior of the process.

To conclude this paper, we point out an interesting problem for future research. In this paper we have focused on continuously averaged Asian options. Broadie, Glasserman, and Kou (1997), (1999) and Kou (2001) show how to estimate discretely sampled barrier and lookback option prices by using the analytical formulae for the continuously sampled versions of these contracts (the BGK continuity correction). It would be interesting to see if some continuity correction could be obtained for Asian options.

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