Pricing credit derivatives with uncertain default probabilities

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Abstract

One main problem of credit models, as in stochastic volatility models for instance, is that the range of arbitrage prices of risky bonds and credit derivatives is very wide. In this article, we present a model for pricing options on the spread in an environnement where the rating transition probabilities are uncertain parameters. The transition intensities are assumed to lie between two bounds which can be easily interpreted in the light of the rating agencies' transition matrices. These bounds are some kind of con…dence interval of the future values of the rating transition intensities. We show that the extremal arbitrage prices are solutions of a Black-Scholes-Barenblatt equation. In particular, when using realistic values for the rating transition (default) probabilities, the arbitrage range of credit derivatives prices is narrow.

1 INTRODUCTION

Credit derivatives are derivative securities whose payo¤ is contingent to the credit quality of a given issuer. This credit quality is measured by the credit rating of the issuer or by the spread of his bonds over the yield of a similar default free bond. In this article we focus on credit spread options.

From a theoretical point of view, a benchmark model as Black-Scholes' model for equities is still lacking. This is an obvious obstacle to the development of credit derivatives markets. Actually, the question is rather complex because, as we shall see, credit risk usually introduces incompleteness in the market since changes in the credit quality cannot be hedged away.

The way to tackle incomplete market problems is three-fold. We can choose a utility based method, as the one introduced by Davis ([9]) in order to price credit derivatives, but the main problem is that this method depends on the agent's preferences. Another approach is to select a criterion in order to choose one equivalent martingale measure out of the in…nite set of equivalent martingale measures available ; for instance, Follmer and Sondermann ([12]) have proposed the criterion of minimization of the quadratic risk in order to select an equivalent martingale measure. The last approach is to …nd the range of prices within the arbitrage bounds for credit derivatives and to keep all the equivalent martingale measures in the calculations. This method leads to solve the super-replication problem which consists in …nding the cheapest portfolio composed of the underlying asset and the riskless asset whose terminal value is almost surely superior to the payo¤ of the option. The key point with any of these approaches is the duality existing between the hedging problem an the set of equivalent martingale measures.

This last approach is of course the most satisfactory because it is not based on a choice of a utility function or risk measure ; however, it generally gives a trivial range for derivatives prices ([10][19][13]). For instance, in the case of credit risk models, the range of prices of a risky bond is simply determined by all the possible dates of occurrence of the default : the lower bound is the price of the bond if ever the default will occur immediatly, and the other bound is the price for a riskless bond. Thus, as in usual option models in incomplete markets, the problem of super-replication for credit derivatives often leads to trivial arbitrage prices.

Here, we propose a new methodology in order to get non trivial arbitrage bounds on credit derivatives prices. This model deals only with options on spread. As in Black-Scholes' or Vasicek's model ([4], [21]), we specify a continuous time dynamics for the underlying asset (here it is the instantaneous spread), and we consider a european option written on this asset. We also assume that the rating of the issuer can change, and the probabilities of such changes are given by the rating transition matrices. In our model we assume that the instantaneous spread follows an Ornstein-Uhlenbeck process with rating-dependent coe¢ cients. The incompleteness of the model comes from the rating transitions that cannot be hedged away by trading on the only asset available in the market. The main idea of our model is to deal with a subset of the equivalent martingale measures only as compared to bounded stochastic volatility models where we assume that the volatility lies between two extreme values. The bounded uncertain parameters are the intensities of rating transitions. For instance, it is excessively prudent to to hedge the risk that a AAA-rated …rm can default in the next minute : our model naturally avoids these non realistic cases that are possible when we keep all the equivalent martingale measures. As we shall see, the super-replication price of spread derivatives are solutions of a Partial Di¤erential Equation (PDE) that leads to non trivial arbitrage ranges for credit derivatives prices.

This article is organised as follows : in the second section, we make a short empirical study of the dynamics of the spreads according to the maturity of the contracts and to the rating of the issuer. In section 3 we describe the generating matrix formalism in order to model the rating transitions. Then, in section 4, we present the continuous time model and derive the non linear PDE that gives the arbitrage range of derivatives on the spread is decribed in section 5. These equations are solved numerically in section 6 in a simple three rating levels model. Section 7 concludes.

Figure 1: Mean value of credit spreads as a function of maturity and rating (left) and the corresponding 90% confidence interval (right).

Mean (bp)	2Y	5Y	10Y	20Y	30Y
AAA	-5.14	-8.66	-15.25	10.63	13.17
AA2	-0.38	-5.12	-9.12	16.62	20.41
AA3	4.97	2.23	-2.55	24.71	30.46
A1	12.76	12.62	6.95	36.79	42.04
A3	31.97	37.19	36.62	62.48	68.64
3B1	42.89	53.33	52.53	76.54	86.75
3B2	54.56	63.37	64.55	93.81	100.23
3B3	73.36	89.26	90.10	115.22	122.71

Confidence	2Y	5Y	10Y	20Y	30Y
AAA	38.80	33.37	32.14	66.72	86.47
AA2	39.39	35.45	41.95	66.44	106.13
AA3	51.97	56.99	43.38	63.75	56.28
A1	86.66	93.64	55.56	81.12	88.78
A3	214.08	169.37	88.96	189.98	295.73
3B1	286.89	273.72	177.30	238.13	382.88
3B2	339.38	276.79	238.93	376.96	532.22
3B3	561.95	497.76	296.40	528.94	690.69

Figure 2: Mean reverting speed of credit spreads as a function of maturity and rating (left) and the corresponding 90% confidence interval (right).

Speed	2Y	5Y	10Y	20Y	30Y
AAA	0.1572	0.1817	0.1958	0.0813	0.0555
AA2	0.1467	0.1708	0.1468	0.0784	0.0429
AA3	0.1219	0.1056	0.1404	0.0856	0.0824
A1	0.0697	0.0666	0.1109	0.0642	0.0483
A3	0.0321	0.0398	0.0728	0.0275	0.0159
3B1	0.0231	0.0248	0.0374	0.0239	0.0142
3B2	0.0204	0.0249	0.0301	0.0154	0.0101
3B3	0.0122	0.0149	0.0245	0.0113	0.0083

Confidence	2Y	5Y	10Y	20Y	30Y
AAA	0.8968	0.9559	0.9789	0.6597	0.5575
AA2	0.8660	0.9308	0.8747	0.6487	0.4941
AA3	0.7978	0.7481	0.8683	0.6747	0.6657
A1	0.6115	0.5974	0.7816	0.5887	0.5217
A3	0.4156	0.4598	0.6312	0.3763	0.3026
3B1	0.3537	0.3702	0.4626	0.3492	0.2728
3B2	0.3326	0.3673	0.4089	0.2780	0.2313
3B3	0.2646	0.2796	0.3583	0.2392	0.2058

2 SPREAD DYNAMICS

From an econometric point of view, the logarithm of the spreads are often approximated by an Ornstein-Uhlenbeck process ([18]). However, this implies a positive value for the spreads. This is not always the case, for instance when we consider the spread between high quality corporate rates and swap rates. Thus we assume that the spreads time series follow an AR(1) process of the form :

$$X_{t+1} - X_t = a(R;T)[b(R;T) - X_t] + s(R;T)\epsilon_t \quad (1)$$

These discrete dynamics involve three parameters : the parameter $a(R;T)$ is interpreted as the mean reverting speed, $b(R;T)$ is the long term equilibrium value of the spread and $s(R;T)$ is a volatility parameter of the spread. The variables R and T are the rating of the issuer and the maturity of the debt we are considering (we only consider here long term debts and credit qualities). The random variable ϵ_t is a gaussian white noise. Let us note that there is a term structure of the spreads and that the non arbitrage conditions would imply relations between these parameters in a continuous time model. Here, we only consider the relation (1) from an econometric point of view.

In order to estimate these three parameters, we have selected indexes of US industrial bonds built by Bloomberg. Each index corresponds to a given rating and Bloomberg has reconstructed a yield curve for each sector and rating. The datas are daily index yields from 02/28/98 to 12/01/99 ; the spreads we have calculated are the difference between these yields and the corresponding US swap rates of the same maturity. Our results are reproduced in tables 1-3. They provide the maximum likelihood estimations of the parameters $a(R;T)$; $b(R;T)$ and $s(R;T)$ and the 90% confidence intervals of these estimations.

These results provide intersting insights about the dynamics of credit spreads. First, the dynamics of credit spreads is mean reverting because of the positvity of the mean reverting speed of the process : for any value of the rating and of the maturity, the coefficient $a(R;T)$ is positive (see table 2). Moreover, as shown

Volatility	2Y	5Y	10Y	20Y	30Y
AAA	3.67%	3.65%	3.79%	3.26%	2.89%
AA2	3.48%	3.64%	3.71%	3.14%	2.73%
AA3	3.81%	3.62%	3.66%	3.28%	2.79%
A1	3.63%	3.74%	3.70%	3.13%	2.58%
A3	4.12%	4.03%	3.89%	3.12%	2.82%
3B1	3.94%	3.99%	3.95%	3.39%	3.25%
3B2	4.08%	4.05%	4.28%	3.42%	3.23%
3B3	3.82%	4.10%	4.23%	3.52%	3.40%

Confidence	2Y	5Y	10Y	20Y	30Y
AAA	3.06%	3.04%	3.15%	2.72%	2.40%
AA2	2.89%	3.03%	3.08%	2.61%	2.27%
AA3	3.17%	3.02%	3.05%	2.73%	2.32%
A1	3.02%	3.12%	3.08%	2.61%	2.15%
A3	3.44%	3.36%	3.24%	2.59%	2.35%
3B1	3.29%	3.33%	3.29%	2.81%	2.71%
3B2	3.38%	3.37%	3.57%	2.83%	2.69%
3B3	3.17%	3.41%	3.51%	2.93%	2.83%

Figure 3: Volatility parameter of credit spreads as a function of maturity and rating (left) and the corresponding 90% con…dence interval (right).

in Longsta¤ and Schwartz ([18]), the mean reverting speed of credit spreads decreases for lower-rated debts and also decreases with maturity.

The mean of the credit spreads is also a parameter of interest. In table 1, it is shown that $b(R;T)$ is clearly increasing with the rating. This behavior is of course intuitively correct : for a lower rated debt, we expect a higher return.

Another conclusion of our empirical study is that credit spread volatility parameters increase as the debt quality decreases (see Table 3). Here again, our results are in agreement with the results obtained by Longsta¤ and Schwartz in [18]. Table 3 details the volatility parameter of the spread as a function of the rating and the maturity.

Moreover, the average volatility parameter is quite high. Actually, as we can see directly in the time series themselves, the spread process has jumps, and the spread variations are not normally distributed. The con…dence intervals for the parameters con…rm this a¢rmation : in tables 1-3, we have computed the width of the 90% con…dence intervals for each parameter. We observe that for the mean reverting parameter and for the long term mean value of the spread, the width of the con…dence intervals are much larger than the parameters themselves. For the volatility parameter, the estimation is much better.

This analysis clearly shows that the spread process is far from a AR(1) process, even if the estimated parameters look friendly. However, in our continuous time model of section 4, we shall choose an Ornstein-Ulhenbeck process for the instantaneous spread dynamics in order to get a tracktable model. Before, this let us introduce the transition matrices formalism that models the rating changes.

3 TRANSITION MATRICES

In order to mathematically construct a coherent model for rating transitions, we consider the Markov chains formalism ([14], [17]). The rating process is a jump process that takes its values in a …nite set of integers. We assume that we have D levels of rating for the risky issuer, from 1=AAA to D=default and assume that the transition probability from level i to level j is proportional to the time interval

$$P\left[R_{t+\Delta t}=j|R_t=i\right]=b_{i;j}.\Delta t \qquad (2)$$

Rating agencies like Standard and Poors or Moody's give a one year matrix transition. Standard & Poor's ([20]) one year rating transition matrix (april 1996) is given in table 4.

Initial rating	Rating transition probability after one year(%)							
	AAA	AA	A	BBB	BB	B	C	D
AAA	90,81	8,33	0,68	0,06	0,12	0	0	0
AA	0,7	90,65	7,79	0,64	0,06	0,14	0,02	0
A	0,09	2,27	91,05	5,52	0,74	0,26	0,01	0,06
BBB	0,02	0,33	5,95	86,93	5,3	1,17	0,12	0,18
BB	0,03	0,14	0,67	7,73	80,53	8,84	1	1,06
B	0	0,11	0,24	0,43	6,48	83,46	4,07	5,2
C	0,22	0	0,22	1,3	2,38	11,24	64,86	19,79
D	0	0	0	0	0	0	0	100

Figure 4: One year rating transition probabilities (%).

Initial rating	Rating transition (%)							
	AAA	AA	A	BBB	BB	B	C	D
AAA	-9,68	9,18	0,35	0,02	0,14	0	0	0
AA	0,77	-9,96	8,57	0,45	0,01	0,14	0,02	0
A	0,09	2,49	-9,69	6,18	0,66	0,22	0,00	0,05
BBB	0,02	0,28	6,67	-14,50	6,29	1,03	0,09	0,12
BB	0,03	0,13	0,45	9,24	-22,40	10,70	1,07	0,77
B	0	0,12	0,24	0,10	7,86	-18,88	5,49	5,07
C	0,29	0	0,20	1,57	2,62	15,14	-43,77	24,00
D	0	0	0	0	0	0	0	0

Figure 5: Generating matrix (%).

Of course, the transition matrix over a given period depends on the period length. If we assume a stationarity property of the transition matrices (i.e. a matrix transition over a period does not depend on the date at which we consider the transition matrix), then we can easily build a transition matrix over any period from the one year matrices given by the rating institutes. Let us call $P(\Delta t)$ the transition matrix between t and $t + \Delta t$. We develop this matrix around the identity matrix up to order one in Δt :

$$P(\Delta t) \approx I + \Delta t.A \tag{3}$$

where I is the $D \times D$ identity matrix. Then, the transition matrix between time t and time $t + s$ writes :

$$P(s) = \exp(s.A) \tag{4}$$

The matrix A generates a semi-group structure for the transition matrices and is called the generating matrix whose properties are described in [1]. It is of course possible to compute the generating matrix from the matrix given by the rating agencies just by taking the logarithm of the transition matrix in a diagonal basis and by coming back in the original basis. The main property of the generating matrix is that the sum of the coefficients of a row of the matrix is equal to 0, and the only negative coefficients are the diagonal coefficients.

The main objection to this kind of model is that the markov property is not satisfied by historical datas : the 2 years historical matrix is not the square of the one year matrix. However, this assumption makes the model much simpler to handle.

4 A CONTINUOUS TIME MODEL

4.1 A rating driven spread dynamics

For the sake of clarity, we assume that the term structure of interest rates is flat and equal to r over time This assumption can be relaxed by specifying for instance a Vasicek like dynamics for the instantaneous rate, or a HJM model for the whole rate curve. Such a change would not change the generality of our purpose.

The market is supposed to be made of two kinds of assets : a risk free asset with a constant rate of return r, and risky zero-coupon bonds with any maturity issued by an only issuer. The price fluctuations of these bonds are driven by two sources. First, there are market fluctuations : because risk free rate is constant, they are interpretated as spread fluctuations. Second, credit events, such as a default, can induce price variations of the risky bond.

We propose here a one factor model in order to take into account the market risk. This factor is the instantaneous spread whose process $(X_t)_{t \geq 0}$ satisfies the following Stochastic Differential Equation (SDE) :

$$dX_t = a(R_t)[b(R_t) - X_t]dt + s(R_t)dW_t \tag{5}$$

where $(W_t)_{t \geq 0}$ is a standard brownian motion and $(R_t)_{t \geq 0}$ is the rating process of the issuer. We assume that there are D levels of rating ; the dynamics given in equation (5) is valid as long as the issuer has not defaulted ($R_t < D$) and the coefficients of the dynamics are rating dependent. If we forget rating changes, this model is exactly Vasicek's model. There is a unique probability change on the brownian part that makes the discounted price process of any risky bond a martingale. The price $B(t;T)$ at time t of the risky zero-coupon bond of maturity T writes as the expected value of the discounting factor under the risk neutral measure :

$$B(t;T) = E_t^Q \left[e^{-\int_t^T (r+X_s)ds} \right] \tag{6}$$

We can apply Vasicek's analysis to this model. The risky negotiable assets available in the market are risky bonds. The absence of arbitrage opportunities leads to construct portfolios of risky bonds with different maturities, and by fine-tuning the allocation we can get a riskless portfolio. This induces a term structure for the spreads of the same functional form as the interest rate term structure of Vasicek's model. Of course, in order to get a model able to take into account the whole spread curve as an input, it is necessary to introduce a multifactor spread dynamics. Nevertheless, this is beyond the scope of this article.

Before the time of default, the spread dynamics is a mean reverting stochastic dynamics with parameters depending only on the rating level of the debt and is solution of equation (5). After the default, the rating process is assumed to remain constant because, as we can see in table 4, the probability of coming back to a non defaulted situation is equal to zero. We assume that after the default, the risky bond turns into a riskless bond with the same maturity T but nominal $\omega < 1$ (which is the recovery rate of the risky debt). Let us call $B^0(t;T)$ the price of the riskless zero-coupon bond with maturity T. After the default, we deduce a theoretical value for the spread $s(t)$ at time t :

$$B(t,T) = B^0(t,T)e^{-s(t)(T-t)} = \omega B^0(t,T) \Longrightarrow s(t) = -\frac{1}{T-t}\ln \omega \qquad (7)$$

We assume that the observed spread after the default is no longer stochastic and reaches its equilibrium value $s(t)$ immediatly.

4.2 A dynamics for the rating

The specificity of this model is that it takes into account the possibility of rating transitions. It is easy to make a model of the rating process since we assume that it is a pure jump process with a finite number of possible values, and the intensities of the jumps are the coefficients of the generating matrix. The rating process $(R_t)_{t\geq 0}$ is solution of the SDE ($R_t \in \{1,\dots,D\}$):

$$dR_t = \sum_{i=1}^{D} (i - R_t)dN_t^i \qquad (8)$$

with initial value $R_0 \in \{1,\dots,D\}$. The $(N_t^i)_{i=1..D;\, t\geq 0}$ are independent Poisson point processes with intensity $\lambda(i, R_t) = a_{R_t,i}$ which are the coefficients of the generating matrix (see table 5). Each Poisson point process $(N_t^i)_{t\geq 0}$ represents the transition between the rating level R_t (the current rating level) to the rating level i. Between time t and time $t+dt$, the probability of this rating transition is equal to the coefficient (R_t, i) of the generating matrix times the time interval dt. The amplitude of the rating jump when the i-th Poisson's process $(N_t^i)_{t\geq 0}$ jumps is $(i - R_t)$.

This model is interesting from an empirical point of view. Indeed, we deal with a jump model, but there are no estimations of the jump parameters to be done since the amplitude are integer numbers and we have an estimation of the the intensities of the Poisson processes in the rating agencies' matrices.

4.3 Arbitrage range of option prices

In this subsection, we address the main goal of this article. We aim at computing the arbitrage range of prices of a contigent claim written on the spread. To this end, we calculate the super-replication (resp. under-replication) price of options on the spread. The option is written on the spread itself, but the negotiable underlying asset are the risky bonds. Let us consider the european option with maturity $H < T$ and pay-off $g(X_H)$. We consider any admissible strategies $(\pi_t)_{t\geq 0}$; we do not describe them in detail (see for instance [5]) but we just say that they are self-financing and that they satisfy some integrability conditions. An admissible hedging portfolio is then obtained from an admissible hedging strategy. The problem of super-replication is to calculate the cheapest portfolio over-hedging the contingent claim. At time t, this portfolio value is :

$$\Phi_t = \inf\left\{x \; : \; \exists \pi \text{ admissible}; \; V_H^{x,t,\pi} - g(X_H) \geq 0 \text{ a.s.}\right\}$$

where $V_H^{x,t,\pi}$ is the value at time H of the hedging portfolio knowing that it was x at time t and we have followed the strategy π. In its dual form (see Kramkov [16]), this optimization problem writes as the supremum over all equivalent martingale measures of the expected pay-off of the claim and the value of the portfolio is a function of t, X, and R :

$$V(t;X;R) = \sup_{Q} E_t^Q[g(X_H)|X_t = X; R_t = R] \tag{9}$$

Here, the supremum is taken over all the equivalent martingale measures ; in the next section, we will parametrise this set of measures and we discuss which measures should be kept in the analysis. In particular, we shall argue that, from a financial point of view, all equivalent martingale measures are not relevant.

5 THE RISK NEUTRAL DEFAULT PROBABILITIES

5.1 Constraining the risk-neutral probabilities

The market is incomplete because the rating is not a negotiable asset. As shown in the appendix, the set of martingale measures can be parametrized by D positive real numbers $(p^i)_{i=1..D} \in S$. These numbers are risk premiums associated to the D possible changes of rating The control set S is a subset of $\mathbb{R}_*^{+D}$. If we consider all the equivalent martingale measures, it is isomorphic to $\mathbb{R}_*^{+D}$. The function $V(t;X;R)$ is solution of the following non linear equation (see appendix and [15]) :

$$V_t + \frac{1}{2}s^2(R)V_{XX} + a(R)[b(R) - X]V_X + \sum_{i=1} \lambda(i;R) \sup_{p^i \in S} \left[p^i \Delta_i V \right] = rV \tag{10}$$

where $\Delta_i V = V(t;X;i) - V(t;X;R)$. Changing the probability measure is equivalent to changing the intensities of each Poisson process by a multiplicating factor p^i. This kind of transformation on the generating matrix leads to another generating matrix for any positive value of the parameters $(p^i)_{i=1..D}$. A risk neutral one year transition matrix can then be computed. Let us illustrate this by choosing one particular risk neutral measure characterized by a set of parameters $(p^i)_{i=1..D}$. We construct the diagonal matrix $D = \mathrm{diag}(p^1, ..., p^D)$. The matrix $D.A$ is still a generating matrix, and the risk neutral transition matrix between time t and time $t+s$ is :

$$P_{RN}(t;t+s) = e^{sD.A} \tag{11}$$

From a theoretical point of view, the $(p^i)_{i=1..D}$ are arbitrary strictly positive real numbers. Nevertheless, the question we address here, is to know whether they are all relevant from a financial point of view. Keeping all the equivalent martingale measures leads for instance to consider the case where the firm will default in the next minute with probability arbitrarily close to 1 (this corresponds to the limit $p^D \to \infty$). This scenario has a very important impact on the pricing procedure and affects deeply the numerical results : it is in particular responsible for the large ranges of arbitrage prices in credit models. In practice, it is not reasonable to hedge the risk that a AAA rated firm defaults in the next minute, more especially as the one year historical probability of default of a AAA rated firm is very close to zero : people are excessively prudent when they compute arbitrage prices. We argue that such a risk does not need to be hedged away in a realistic model of credit derivatives pricing (mathematically speaking, this means that the parameter p^D must not be sent to infinity). This is why we propose to bound upward the rating transition intensities instead of considering any positive real values. More precisely, our methodology is inspired from Avellaneda's model ([2]) : we assume that the risk neutral rating transition intensities are not uniquely defined, but they are uncertain parameters belonging to a fixed interval :

$$p^i \in]0; \bar{p}_i]; \qquad \forall i; \ \bar{p}_i \in \mathbb{R}^+_* \tag{12}$$

where the $(\bar{p}^i)_{i=1..D}$ are bayesian parameters of our model. The partial differential equation (10) thus writes, for $R < D$:

$$V_t + \frac{1}{2}\sigma^2(R)V_{XX} + a(R)[b(R) - X]V_X + \sum_{i=1} \lambda(i;R)\,\bar{p}^i(\Delta_i V)^+ = rV \tag{13}$$

This equation actually splits into $D - 1$ coupled equations because we have to solve the partial differential equations simultaneously for each value of the rating. In the next section, we will show this explicitely on a three rating levels model.

5.2 Intuitive meaning of the measure bounds

The parameters $(\bar{p}^i)_{i=1..D}$ that constrain the set of equivalent martingale measures have a very intuitive meaning in the special case $\bar{p}_i = \bar{p}$ (for all $i \in \{1..D\}$). Let us for instance consider the intensity $\lambda = a_{AAA;BBB}$ of the transition between the rating level AAA to the rating level BBB. Standard and Poor's value of this one year parameter is 0.02 %. When taking the supremum over all the equivalent martingale measures, we are led to examine the case where λ is multiplied by any positive real number $k \in]0; \bar{p}]$. Thus, as λ was a one year parameter, it is transformed into a k-years parameter and is bounded upward by the $\bar{p}$-years parameter. In terms of the one years transition probability matrix, we give an intuitive interpretation of the one year risk neutral transition matrices since they are of the form :

$$P_{RN(k)}(t; t + 1yr) = e^{k.A} = [P_{Hist}(t; t + 1yr)]^k = P_{Hist}(t; t + k\ yrs) \quad (k \in]0; \bar{p}]) \tag{14}$$

The risk neutral one year transition matrices are simply the k-years historical transition matrices with $k \in]0; \bar{p}]$: The standard and Poor's matrix is a kind of benchmark for rating transition probabilities ; in our model we keep any generating matrix equal to a k-years S&P's generating matrix. The parameter $\bar{p}$ tells how prudent I am when I hedge credit risk in the pricing procedure relative to using S&P's generating matrices. This remarkable property gives the model an intuitive understanding and a concrete criterion in order to choose the bounds on the risk neutral intensities we want to keep.

6 THREE LEVELS MODEL

In this section, we limit ourselves to a simple example of a three rating levels model. These levels are R1, R2 and D for default: For instance R1 stands for

Initial rating	Rating transition (%)		
	R1	**R2**	**D**
R1	-0.30	0.20	0.10
R2	0.30	-2.33	2.02
D	0.00	0.00	0.00

Figure 6: Generating matrix for the three levels model

Initial rating	Rating transition (%)		
	R1	**R2**	**D**
R1	99.70%	0.20%	0.10%
R2	0.30%	97.70%	2.00%
D	0.00%	0.00%	100.00%

Figure 7: One year transition matrix

the investment grade rating level and R2 stand for the speculative grade. The historical generating matrix for the transition probabilities is the one given in table 6.

The one year historical transition matrix associated to this generating matrix is obtained by exponentiation ; this leads to the matrix of table 7.

In order to write down the full system of partial differential equations we need the value of the derivative after the default. This goal is easily achieved since the spread after the default is at time t :

$$s(t) = -\frac{\ln \omega}{T - t}$$

After the default, the value of the derivative at time t is de terministic and is the discounted final payoff :

$$V^D = V(t; X; D) = e^{-r(H-t)} g(s(H)) \qquad (15)$$

From equation 13, the super-replication prices of the credit option for ratin levels R1 and R2 are solution of the following system of coupled PDE :

$$\begin{cases} V_t^{R1} + \frac{1}{2}\sigma^2(R1)V_{XX}^{R1} + \alpha(R1)[\beta(R1) - X]V_X^{R1} + \overline{\lambda}(R2; R1)(V^{R2} - V^{R1})^+ + \\ \qquad + \overline{\lambda}(D; R1)(V^D - V^{R1})^+ = rV^{R1} \\ V_t^{R2} + \frac{1}{2}\sigma^2(R2)V_{XX}^{R2} + \alpha(R2)[\beta(R2) - X]V_X^{R2} + \overline{\lambda}(R1; R2)(V^{R2} - V^{R1})^+ + \\ \qquad + \overline{\lambda}(D; R2)(V^D - V^{R2})^+ = rV^{R2} \end{cases} \qquad (16)$$

where $(A)^+ = \max(A; 0)$: This system of PDE is known to have a unique solution since it is a cooperative system ([7]). It also possesses important properties of numerical convergence. The lower bound of arbitrage prices is the solution of similar equations :

$$\begin{cases} V_t^{R1} + \frac{1}{2}\sigma^2(R1)V_{XX}^{R1} + \alpha(R1)[\beta(R1) - X]V_X^{R1} + \overline{\lambda}(R2; R1)(V^{R2} - V^{R1})^- + \\ \qquad + \overline{\lambda}(D; R1)(V^D - V^{R1})^- = rV^{R1} \\ V_t^{R2} + \frac{1}{2}\sigma^2(R2)V_{XX}^{R2} + \alpha(R2)[\beta(R2) - X]V_X^{R2} + \overline{\lambda}(R1; R2)(V^{R2} - V^{R1})^- + \\ \qquad + \overline{\lambda}(D; R2)(V^D - V^{R2})^- = rV^{R2} \end{cases} \qquad (17)$$

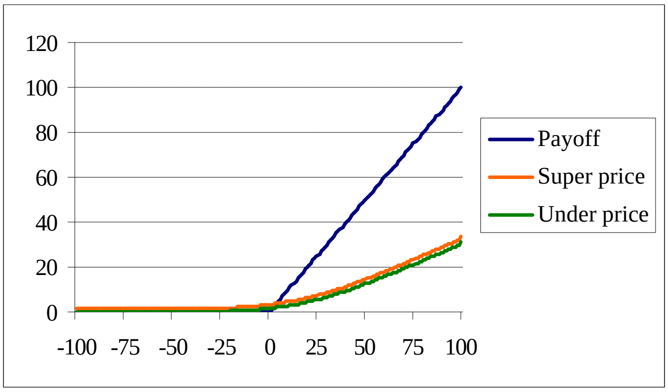

Figure 8: Arbitrage range of prices as a function of the instantaneous spread for a call option on the spread with strike 0 and maturity 1 month.

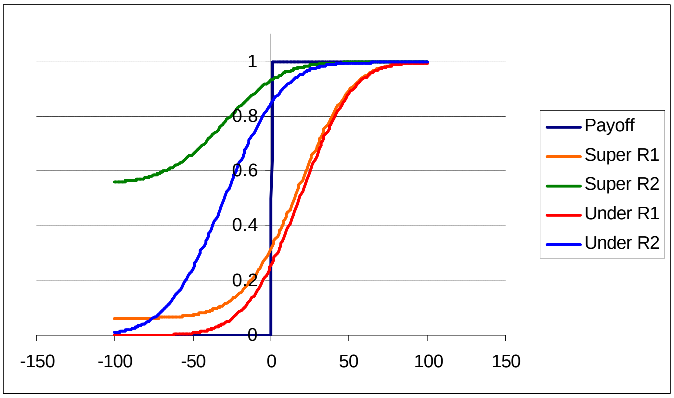

Figure 9: Arbitrage range of prices for a digital option of the spread with strike 0 and maturity 1 month

where $(A)^{-} = \min(A;0)$. We have computed the numerical solution for the call option on the spread and for the digital option. For instance when $\bar{p} = 1000$, we notice that the arbitrage range of prices is very narrow compared to the case treated in [6]. In figure 8, we give the numerical solution of equations (16a,b) and (17a,b), and we draw the curves for rating R1.

As we can see, the arbitrage range of prices is very narrow in this model in spite of a very large value of the parameter $\bar{p}$: The curves cross the payoff function contrary to Black-Scholes' model : the mean reverting structure of the spread dynamics makes the in-the-money prices of the european call options lower than the pay-off itself. For a digital option on the spread, we get the following curves for rating levels R1 and R2 :

7 CONCLUSION

We have discribed a simple model for the evolution of credit spreads based on credit ratings transitions. This model is appropriate for pricing european options on spread. The main interest of this article is not in the spread model itself, but lies in the choice of the risk-neutral probabilities. We assume that the default probabilities do not explain the whole spread, and on the reverse, the spread is not enough to choose the risk neutral transition probabilities. Our model is an uncertain probability model and leads to a range of arbitrage prices for credit derivatives. We have computed explicitely this range of prices and we have shown that it was very narrow.

This uncertain probability model is similar to Avellaneda's uncertain volatility model but leads to more complicated equations for the extreme bounds of the arbitrage prices. However, the interpretation of the probability interval is quite similar : we interpret it as a confidence interval for the rating transition probabilities.

8 APPENDIX : DERIVATION OF THE NON LINEAR PDE

The super-replication price of the contingent claim with maturity $H < T$; is :

$$V(t;X;R) = \sup_{Q} E_t^{Q}\left[g(X_H;X_H) | X_t = X;\ R_t = R\right] \qquad \text{(A1)} \qquad (18)$$

Changing the probability measure leads to introduce a risk premium for the brownian part (i.e. a process (μ_t)) and a risk premium for the jump part (one risk premium process (p_t^i) per jump process). The change of probability measure writes :

$$\begin{aligned} d\tilde{W}_t &= dW_t + \mu_t\, dt \\ d\tilde{M}_t^i &= dM_t^i + (1 - p_t^i)\lambda(i;R_t)dt \qquad (i = 1;\ldots;D) \end{aligned} \qquad (19)$$

Under an equivalent measure the risk premium (μ_t) linked to the brownian motion is zero because it is assumed that the brownian is risk-neutral. The risk premiums (p_t^i) associated to the jump processes are choosen to be markovian in order to deal only with markovian processes. We argue that the nature of the result remains unchanged whereas the calculations are simplified ([8]). The processes of spread, rate and rating write :

$$\begin{aligned} dX_t &= a(R_t)[b(R_t) - X_t]dt + s(R_t)dW_t \\ dR_t &= \sum_{i=1} (i - R_t)\lambda(i;R_t)p_t^i\, dt + \sum_{i=1} (i - R_t)d\tilde{M}_t^i \end{aligned} \qquad (20)$$

The processes (p_t^i) are free parameters of our model, and they parametrize the set of equivalent martingale measures ; we denote the equaivalent martingale measures as $Q^{(p^1)::(p^D)}$. Thanks to the markovian nature of all the processes under the equivalent martingale measures, we can apply Bellman's principle which leads to the equation

$$\sup_{(p_s^i);\ i=1::D;\ t \le s \le t+h} \frac{1}{h} E_{t;X;R}^{Q^{(p^1)::(p^D)}}\left[\int_t^{t+h} DV\, ds\right] = 0 \qquad (21)$$

where $DV(t;X;R)$ is the Dynkin operator applied to the function $V(t;X;R)$. In the limit $h \to 0$, the integral converges to the value of the integrand at time t, and the supremum over all the processes (p^i) is tranformed into a supremum over real parameters (value of the processes at time t). Lets us call S the euclidean sub-space of R^D in which the D-varied process $(p_t^1;\dots;p_t^D)$ takes its values. The Dynkin of the function $V(t;X;R)$ is obtained thanks to Itô's lemma extended to the case of jumping processes. Applying Itô's lemma to the function $V(t;X;R)$ leads :

$$dV = V_t\,dt + V_X\,dX + \frac{1}{2}V_{XX}\,d\langle X;X\rangle_t + \sum_{i=1}\Delta_i V\left[\lambda(i;R)p^i dt + dM_t^i\right]$$

where $\Delta_i V = V(t;X;r;i) - V(t;X;r;R)$. We obtain the partial differential equation satisfied by the super-replication price $V(t;X;R)$:

$$V_t + \sup_{p^i \in S}\left[\frac{1}{2}s^2(R)V_{XX} + a(R)[b(R) - X]V_X + \sum_{i=1}\lambda(i;R)p^i\Delta_i V\right] = rV \quad (22)$$

This is equation 10. At this stage, the control set S is not yet specified. In the article, we did not take the whole set of equivalent martingale measures, because, we have bounded the control set S to the rectangle $]0;\overline{p}_1]\times\dots\times]0;\overline{p}_D]$. If we keep the whole set of equivalent martingale measures, the control set S is $]0;+\infty[^D$. We notice that the jump term, and only this one, can be made arbitrarily large (in absolute value) by sending the parameters p^i towards infinity, whereas all the other terms remain finite. Thus, the jump term is shown to be arbitrarily close to zero. The way to do this correctly is for instance to send the parameters p^i to infinity but to keep constant ratios between them ; we put :

$$p^i = \chi q^i \quad (23)$$

Here, the parameters are positive and constrained, for instance by the relation $\sum_i q^i = 1$; the parameter χ is sent to infinity. This is enough to prove that the jump terms are equal to zero, and we can now explore the manifold of the parameters q^i. If we now send the parameter q^{i_0} close to 1, this means that the intensity of the transition between the rating R_{t-} to the rating i_0 will increase relative to the other intensities. As a consequence, exploring the manifold of the parameters q^i is equivalent to exploring all the possible values of the rating at date t. We then deduce the equation of the super-replication price:

$$V_t + \sup_R\left\{\frac{1}{2}s^2(R)V_{XX} + a(R)[b(R) - X]V_X\right\} = rV; \quad V(T;X;R) = g(X_H) \quad (24)$$

The super-replication price is the upper bound of the arbitrage range of prices of the contigent claim. The lower bound is solution of the same equation with the operator infimum instead of supremum. The problem of the case $S = R^D$ is that equation 24 is ill-defined because we assumed a jump in the spread process when a default occurs. A complete study of this case is provided in [6].

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