

Portfolio Selection Using Bayesian Analysis and Gibbs Sampling

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Abstract

This paper contributes to portfolio selection methodology using a Bayesian forecast of the distribution of returns by stochastic approximation. New hierarchical priors on the mean vector and covariance matrix of returns are derived and implemented. Comparison's between this approach and other Bayesian methods are studied with simulations on 20 years of historical data on global stock indices. It is demonstrated that a fully hierarchical Bayes procedure produces results superior to typical Bayesian formulations. In addition, an expected utility formulation increases performance significantly over methods that simply impute moment estimates into the Markowitz mean-variance model.

Portfolio theory is concerned with the allocation of an individual's wealth among various available assets. The basic Markowitz version of the portfolio selection problem is (Markowitz 1952):

$$\begin{aligned} & \text{Maximize } \mathbf{w}'\boldsymbol{\mu} - \frac{\lambda}{2} \mathbf{w}'\boldsymbol{\Sigma}\mathbf{w} \\ & \text{subject to } \mathbf{w}'\mathbf{e} = 1 \end{aligned} \quad (1)$$

where $\mathbf{w} = [w_1, w_2, \dots, w_m]'$ is a column vector of proportions representing a portfolio of assets, $\boldsymbol{\Sigma}$ and $\boldsymbol{\mu}$ are the covariance matrix and mean column vector of asset returns $\mathbf{y} = [y_1, y_2, \dots, y_m]'$, λ is the investor's risk-aversion parameter, and \mathbf{e} is a column unit vector. For no short sales restrictions, an additional constraint, $w_i \geq 0$, can be added. This portfolio selection approach is termed the Mean-Variance (*MV*) method because it ranks portfolio weights by their mean-variance pairs. The set of optimal portfolios obtained as the level of risk aversion, λ , varies is termed the *Markowitz efficient frontier*.

The Markowitz *MV* method can be viewed as maximizing expected utility. For example, if the investor's current wealth is W_0 , his terminal wealth is:

$$W = W_0 + \mathbf{w}'\mathbf{y} \quad (2)$$

According to Von Neumann and Morgenstern axioms, the investor determines \mathbf{w} by considering the expected value of a non-decreasing utility function of W . Using the exponential utility function,

$$U(W; \lambda) = -\exp[-\lambda W], \quad (3)$$

and assuming \mathbf{y} is distributed multivariate normal $N(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, the maximization of expected utility reduces to ranking *MV* portfolios using $(\boldsymbol{\mu}, \boldsymbol{\Sigma}; \lambda)$ in model (1).

Classical portfolio selection uses least-squares estimates of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ in model (1). However, *MV* portfolio selection based on estimates of population moments leads to a problem of estimation risk that arises from the difference between the estimates and the true parameter values. It has been well documented that the problem of estimation risk is significant (Dickinson 1974; Putnam and Quintana 1991; Pari and Chen 1985; Frankfurter, Phillips, and Seagle 1971; Jobson and Korkie 1980). Empirical studies of estimation risk associated with least-squares estimates appear in Levy and Sarnat (1970); Solnik (1982); Board and Sutcliffe, (1992); Chopra, Hensel, and Turner (1993); Chopra and Ziemba (1993). All of these studies conclude that resulting portfolios involve either extreme volatility or lack of diversification.

The use of Bayes and empirical Bayes estimators to estimate $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ have been advocated by several researchers (Brown 1976; Klein and Bawa 1976; Bawa, Brown and Klein 1979; Jorion 1986; Frost and Savarino 1986). Jorion (1986, 1991) employs Bayes modifications of James-Stein shrinking formulas (James and Stein 1960) to estimate $\boldsymbol{\mu}$, while Frost and Savarino (1986) employ empirical Bayes estimators of $\boldsymbol{\mu}$, assuming $\boldsymbol{\Sigma}$ has intraclass structure. They show through simulated and historical data that *MV* portfolios using their respective Bayes estimates in model (1) dominate *MV* portfolios using classical least squares estimates. See also Kadiyala and Karlsson (1997), Kandel, McCulloch and Stambaugh (1995), and Shaken (1987).

This paper examines a fully hierarchical Bayes model for $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$. These models are multivariate and thus can capture more complete information on the interde-

pendence between assets than previous models. Although these models are cross-sectional, one-step forward forecasts based on the posterior predictive distribution of returns are available for ranking portfolios. The posterior predictive distribution has been proposed for forecasting univariate ARMA models since Zellner (1971); also see West and Harrison (1989). This paper will empirically demonstrate that Bayesian forecasts are superior to moment estimates in portfolio ranking. Moreover, this approach applies to any utility function.

Marriott et al. (1994) show how to obtain the predictive distribution for a vector of future values via the Gibbs sampler and Monte Carlo integration. Kim, Shephard, and Chib (1998) exploit MCMC sampling methods to provide a practical likelihood based framework for the analysis of stochastic volatility models. These methods are used to compare the fit of stochastic volatility and GARCH models. Nakatsuma and Tsurumi (1996) compare small-sample properties of Bayes estimation and maximum likelihood estimation (MLE) of ARMA-GARCH models using MCMC sampling. McCulloch and Tsay (1994) use the Gibbs sampler for Bayesian analysis of AR models. This paper also exploits Monte Carlo Markov Chain (MCMC) sampling methods to obtain a practical stochastic approximation to the posterior predictive distribution and its moments.

This paper is structured as follows. The definition of posterior predictive distributions is given in section 1. Maximum expected utility is defined in section 2. In section 3, we describe data on eleven country-stock index funds provided by Morgan Stanley Capital International. In addition, designs for comparing the different Bayesian models are described. Bayesian data models including the fully

hierarchical prior are explained in section 4. Sections 5 and 6 describe the results and conclusion, respectively.

1 Bayes Posterior Predictive Distributions

Denote observed returns on m assets by \mathbf{y} and future, or unobserved, returns by $\tilde{\mathbf{y}}$.

Let $\theta \in \mathbb{R}^p$ and $\phi \in \mathbb{R}^q$ denote p parameters and q hyperparameters, respectively.

The parametric family of the joint likelihood of \mathbf{y} and $\tilde{\mathbf{y}}$ will be denoted by

$f(\mathbf{y}, \tilde{\mathbf{y}} | \theta, \phi) = f(\mathbf{y}, \tilde{\mathbf{y}} | \theta)$ and depends on the joint parameters only through the

low-level parameter θ . Denote the prior distribution of (ϕ, θ) by

$\pi(\phi, \theta) = \pi(\phi) \pi(\theta | \phi)$. Nonhierarchical models fix ϕ and compute posterior distributions using the prior $\pi(\theta | \phi)$, while hierarchical models compute posterior distributions using the joint prior $\pi(\phi, \theta)$.

In the portfolio selection problem, $\theta = (\mu, \Sigma)$ and ϕ will represent a vector of hyperparameters in the prior for (μ, Σ) . Portfolio selection using posterior predictive distributions addresses two unknown quantities, $\tilde{\mathbf{y}}$ and (ϕ, θ) , with the primary goal being to gain information about $\tilde{\mathbf{y}}$ with (ϕ, θ) as nuisance parameters. The advantage of the hierarchical model, with priors instead of point estimates of hyperparameters, is that the posterior distributions will reflect the appropriate uncertainty in the hyperparameters. The disadvantage is that the posterior predictive distribution will not be analytically tractable usually; however, the method based on the MCMC sampler provides a stochastic approximation of the posterior predictive distribution.

According to the likelihood principle all evidence about $(\tilde{y}, \phi, \theta)$ is contained in the joint likelihood function $f(\tilde{y}|\phi, \theta)$ (for an overview see Bjørnstad, 1990). Based on this likelihood, we wish to develop a posterior predictive distribution for \tilde{y} , $\pi(\tilde{y}|y)$, by eliminating (ϕ, θ) from the joint likelihood. The Bayes approach for this problem is to integrate out (ϕ, θ) using the joint prior. The resulting predictive distribution for \tilde{y} given the data, y , is the following:

$$\pi(\tilde{y}|y) = \int_{\phi \in \mathbb{R}^q} \int_{\theta \in \mathbb{R}^p} f(\tilde{y}|\theta) \pi(\theta, \phi|y) d\phi. \quad (4)$$

A stochastic approximation of the posterior predictive distribution is generated by simulation, using the MCMC sampler if necessary, using $\pi(\theta, \phi|y)$ to generate $\tilde{y} \sim f(\tilde{y}|\theta, \phi)$, and repeating these steps to obtain more simulated observations.

2 Maximum Expected Utility

For the reader's convenience, we repeat the basic notation. Let $w \in \mathbb{R}^m$ denote portfolio weights satisfying $w_i \geq 0, i = 1, \dots, m; \sum_{i=1, \dots, m} w_i = 1$. The inner product $w'\tilde{y}$ is the portfolio-return on future investment performance. An investor will choose a utility for wealth. This utility is denoted by a monotonically increasing, concave function $U(W; \lambda) \rightarrow \mathbb{R}$, where $\lambda \geq 0$ is a fixed parameter denoting risk aversion. The posterior expected utility of $W = W_0 + w'\tilde{y}$, where W_0 is initial wealth, given the data y is:

$$E U(\mathbf{w}'\tilde{\mathbf{y}} | \mathcal{V}_0; \lambda) = \int_{\tilde{\mathbf{y}} \in \mathbb{R}^m} U(\mathbf{w}'\tilde{\mathbf{y}} | \mathcal{V}_0; \lambda) \pi(\tilde{\mathbf{y}} | \mathcal{V}_0) d\tilde{\mathbf{y}}. \quad (5)$$

This expectation exists under standard regularity conditions.

The *direct utility (DU)* optimal portfolio model is a solution to the following model:

$$\begin{aligned} & \text{Maximize } E U(\mathbf{w}'\tilde{\mathbf{y}} | \mathcal{V}_0; \lambda) \\ & \text{subject to } \mathbf{w}'\mathbf{e} = 1 \end{aligned} \quad (6)$$

Typically, model (6) must be solved with a non-linear optimization algorithm. Many standard algorithms exist, such as sequential quadratic programming (see Gill, Murray, and Wright 1981, p. 237; Schittowski 1980, 1985) as implemented in *MATLAB*, enabling solutions for any utility function that is twice-continuously differentiable. In case the expected utility is not analytically tractable, it is necessary to contemplate samples from the posterior predictive distribution that can be used to approximate the expected utility. This is an advantage since portfolio selection can be carried out with a general utility function. Given a sample, $\tilde{\mathbf{y}}_1, \dots, \tilde{\mathbf{y}}_K$, from the predictive posterior distribution, the direct utility portfolio selection problem (6) is approximated by the model:

$$\begin{aligned} & \text{Maximize } K^{-1} \sum_{k=1}^K U(\mathbf{w}'\tilde{\mathbf{y}}_k | \mathcal{V}_0; \lambda) \\ & \text{subject to } \mathbf{w}'\mathbf{e} = 1 \end{aligned} \quad (7)$$

3 Empirical Data Analysis

The daily stock market indices for 11 different countries over the period 1975-1994 are used in this comparison of *DU* and *MV* using different data models. The countries include US, UK, Canada, Belgium, Australia, France, Japan, Austria,

Spain, Germany, and Hong Kong. The indices are compiled and provided by Morgan Stanley Capital International. Monthly returns were computed as the percentage changes in the index between consecutive last days of the month. Morgan Stanley Capital International provided two indices per country, one in local currency and one in \$US. Our study is based on the returns in \$US.

The data are partitioned into four periods of five consecutive years. This allows a comparison of the means, standard deviations, within-country serial correlations and between-country correlations. Table 1a displays the average-monthly asset-returns. Note the variability of the mean returns within countries across the four times periods. It appears that Spain exhibits the greatest change on mean return while the US is stable.

Table 1a also displays the standard deviation of the monthly returns, demonstrating considerable instability or risk over the four periods, especially for Hong Kong. Comparison of the means and standard deviations of returns reveals that the US has moderately stable returns with relatively low risk; while Hong Kong has consistently high return and risk.

Table 1b exhibits between-country covariance and correlation of monthly asset returns. When the covariance structure changes over time, there are important implications for the appropriate hierarchical model for the data. Although it is not shown in this table, there is much change over time in the covariance matrix for these data. For the extreme instance, the correlation between Spain and the other 10 countries appear to change this period; the correlation structure for Hong Kong appears to change as well.

3.1 Comparison Design

To examine the performance of different models, 180 overlapping, out-of-sample periods of one month each, covering the period Jan 1980 to Dec 1994, are used. That is, the first data set is from Jan 1975 through Dec 1979 (60 monthly observations) and the first out of sample observation is for Jan 1980. Our last out-of-sample portfolio is for Dec 1994. For a given model, we run the MCMC sampler independently on 180 data sets, and use the individual posterior distributions to form the portfolio. Our procedure can be summarized as follows:

1. Use 60 observations (initially those for months 1 to 60) to generate the joint posterior distributions of the means and covariances (via the MCMC sampler) and, in accordance with the decision theory rules, compute the posterior means of these distributions.
2. For a given λ , the risk aversion parameter, find the investment proportions w .
3. Apply these proportions to the actual returns observed in the next month to obtain the actual portfolio return for each model and value of λ .
4. Roll the sample forward by one month, e.g. months 2 to 61, and repeat steps (1) through (3).

This resulted in 180 sample periods being used. A common value, $\lambda = 0.02$, of the risk-aversion parameter was used. Computational time prevented expanding the procedure to a range of λ 's in this study. This is planned for future research.

4 Bayesian Data Models

We will examine two hierarchical data models, and apply MCMC sampling to obtain estimates of the posterior distribution of $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\tilde{\mathbf{y}}$. These estimates will then be used to solve models (1) and (6) for *MV* and *DU* portfolios respectively.

All models, even non-Bayesian, can be specified within a hierarchical Bayesian structure. In addition, the MCMC sampler can be used to solve even the simplest model (while a closed-form analytical solution may exist, it may still be easier to run the MCMC sampler to generate the posterior distributions). The following three data models will be tested empirically:

4.1 Classical Model

$$\begin{aligned}
 \mathbf{y} | \boldsymbol{\mu}, \sigma^2 &\sim N(\boldsymbol{\mu}, \sigma^2 \mathbf{I}) \\
 \mu_j &\sim iid \text{ Uniform}(b_j, \infty), g_j = 1, \dots, 11 \\
 \sigma^2 &\sim \text{Inv-Gamma}(b_1, \varepsilon_2)
 \end{aligned} \tag{8}$$

The values of ε_1 and ε_2 are equal to 0.0001 allowing for a proper, diffuse hyper-prior. The MCMC sampler with diffuse priors yields close approximations of the classical estimators.

4.2 James-Stein Model

$$\begin{aligned}
 \mathbf{y} | \boldsymbol{\mu}, \sigma^2 &\sim N(\boldsymbol{\mu}, \sigma^2 \mathbf{I}) \\
 \boldsymbol{\mu} | \boldsymbol{\mu}_0, \tau_0^2 &\sim N(\boldsymbol{\mu}_0, \tau_0^2 \mathbf{I}) \\
 \boldsymbol{\mu}_0 &\sim \text{Uniform}(b_1, \infty) \\
 \sigma^2 &\sim \text{Inv-Gamma}(b_1, \varepsilon_2) \\
 \tau_0^2 &\sim \text{Inv-Gamma}(b_3, \varepsilon_4)
 \end{aligned} \tag{9}$$

The values of $\varepsilon_1, \varepsilon_2, \varepsilon_3$, and ε_4 are equal to 0.0001 allowing for a proper, diffuse hyperprior.

4.3 Hierarchical Bayes Model

$$\begin{aligned}
 \mathbf{y} | \boldsymbol{\mu}, \boldsymbol{\Sigma} &\sim \text{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \\
 \boldsymbol{\mu} | \boldsymbol{\mu}_0, \boldsymbol{\Sigma} &\sim \text{N}(\boldsymbol{\mu}_0 + \boldsymbol{\Sigma}^{-1} \boldsymbol{\kappa}_0^{-1} \mathbf{g}, \boldsymbol{\Sigma} / \kappa_0) \\
 \boldsymbol{\Sigma} &\sim \text{Inv-Wishart}_{\nu_0}(\mathbf{G}_0^{-1} \mathbf{P}_0^{-1} \mathbf{r}) \\
 \boldsymbol{\mu}_0 &\sim \text{Uniform}(\mathbf{b}, \boldsymbol{\kappa}) \\
 \tau_0^2 &\sim \text{Inv-Gamma}(\varepsilon_1, \varepsilon_2)
 \end{aligned} \tag{10}$$

The degrees of freedom parameter, ν_0 , is unrestricted other than $\nu_0 \geq m$, where m is the number of asset, equal to 11 in this application. The values of ε_1 and ε_2 are equal to 0.0001 allowing for a proper, diffuse hyperprior; and κ_0 is equal to $0.10n$ where n represents the sample size. In addition, \mathbf{P}_0 is a known correlation matrix with structure:

$$\mathbf{P}_0 = \begin{pmatrix} 1 & \rho_0 & \cdots & \rho_0 \\ \rho_0 & 1 & \cdots & \rho_0 \\ \vdots & \vdots & \ddots & \vdots \\ \rho_0 & \cdots & \rho_0 & 1 \end{pmatrix} \tag{11}$$

In this application, the estimate of the correlation parameter ρ_0 is 0.5.

5 Results

We will employ a method of comparison based on portfolio performance. We use the posterior means of $\boldsymbol{\mu}_t$ and $\boldsymbol{\Sigma}_t$ as inputs to the *MV* framework (1); or the posterior marginal predictive distribution as inputs to the direct utility framework (6). In either framework, we obtain a vector of weights \mathbf{w}_t . The actual performance of the portfolio, $\mathbf{w}'_t \mathbf{y}_t$, is then tracked over time $t = 1, \dots, 180$. Consequently, we ob-

tain an actual simulation of how a model would have performed over the study period. This performance is compared for the various models: *Classical*, *James-Stein*, and *Hierarchical Bayes*. In addition to these three models, we include as a benchmark a heuristic portfolio-selection device that weighs each asset equally, denoted by the term *Weighted* in the ensuing figure.

Table 2 displays the portfolio performance comparisons summarized over the stream of 180 monthly returns. It is quite apparent that the *Classical Models* under-perform all the other models, including a naïve equal weight portfolio. The *James-Stein Model* portfolio produces superior results to the *Classical Model* portfolio. The additional edge in performance due to the *Hierarchical Bayes Model* is quite significant. In addition, the Direct Utility (*DU*) method increases performance significantly over methods that impute estimates into the *MV* model (1).

Figure 1 shows the equity performance of the various data models and their turnover rates (to be discussed shortly), where we show the growth of a hypothetical \$1000 portfolio. The top series display the *MV* portfolios, while the middle series display the *DU* factors. The actual performance of the *DU* portfolios are obtained as the *DU* factor times the performance of the *MV* performance. Note the *Hierarchical Bayes* factor remains above 1 after the mid-1980's.

An interesting point is to ask why the *James-Stein* and *Hierarchical Bayes Models* under-perform the *Classical Model* for the first 4 years and then significantly outperform the next 10 years (1985 and on). Closer inspection confirmed what we suspected from general knowledge of the history of these particular mar-

kets: some markets, such as the US, Japan, and Germany exhibited stable positive trends in the early 80's. The *Classical Model* interprets the data of the late 70's to invest in these markets. The shrinking characteristics of the Bayes models, however, hurt their performance during this period.

Portfolio return and the resulting commission costs are obviously of great interest in practical applications. We define turnover as

$$PT_i = \sum_{i=1}^{11} \mathbf{w}_{i,t} - w_{i,t-1} \mathbf{r}, \quad (12)$$

that is, the portfolio turnover in a given month is the sum of the changes in portfolio weights from the previous month to that month. We will analyze the portfolio turnover for each of our models.

Figure 1 (bottom series) compares the portfolio turnover rates of the three data models using the *DU* method. It is quite apparent that the *Classical Model* under-performs all the other models most of the time. The *James-Stein* portfolio produces superior results to the *Classical* portfolio. The *Hierarchical Bayes* model results in significantly lower turnover rates. Note that the *Classical* portfolio showed significant gains in the early 1980's but at the cost of high turnover rates.

6 Summary and Conclusion

A contribution of this paper was to employ practical hierarchical Bayesian models that incorporate a high degree of parameter uncertainty. A practical hierarchical Bayesian model accounting interclass covariance was applied to portfolio selection. The MCMC sampler was used to generate posterior prediction distribu-

tions and estimates of moments. The *James-Stein* model which has appeared previously in the finance literature is basically the Markowitz model using shrinking estimators in the mean, while the covariance matrix estimate is taken (independently of the mean) to be the sample covariance matrix. The *Hierarchical Bayes* model is a more general model, in both μ and Σ .

We carried out a numerical optimization procedure to maximize expected utility using the MCMC samples from the posterior predictive distribution. This model resulted in an extra 1.5 percentage points per year in additional portfolio performance (on top of the *Hierarchical Bayes* model to estimate μ and Σ and use the Markowitz model), which is quite a significant empirical result. This approach applies to a large class of utility functions and models for market returns.

References

- Bawa, V., Brown, S. and Klein, R., *Estimation Risk and Optimal Portfolio Choice*, New York: North-Holland, 1979.
- Bjørnstad, J.F., "Predictive Likelihood: A Review," *Statistical Science*, **5**, 1990, 242-265.
- Board, J. and Sutcliffe, C., "Estimation Methods in Portfolio Selection and the Effectiveness of Short Sales Restrictions: UK evidence," *Management Science*, **38**, 1992, 11-31.
- Brown, S., "Optimal Portfolio Choice under Uncertainty: A Bayesian Approach," Ph.D. dissertation, University of Chicago, 1976.

- Chopra, V. and Ziemba, W.T., "The Effect of Errors in Mean and Covariance Estimates on Optimal Portfolio Choice," *Journal of Portfolio Management*, **20**, 1993, 6-11.
- Chopra, V., Hensel, C. and Turner, A., "Massaging Mean-Variance Inputs: Returns from Alternative Global Investment Strategies," *Management Science*, **39**, 1993, 10-20.
- Dickinson, J., "The Reliability of Estimation Procedures in Portfolio Analysis," *Journal of Financial and Quantitative Analysis*, **9**, 1974, 447-462.
- Frankfurter, G., Phillips, H. and Seagle, J., "Portfolio Selection: the Effects of Uncertain Means, Variances and Covariances," *Journal of Financial and Quantitative Analysis*, **6**, 1971, 1251-1262.
- Frost A. and Savarino, J., "An Empirical Bayes Approach to Efficient Portfolio Selection," *Journal of Financial and Quantitative Analysis*, **21**, 1986, 293-305.
- Gelman, A., Carlin, J.B., Stern, H.S. and Rubin, D.B., *Bayesian Data Analysis*, New York: Chapman and Hall, 1995.
- Gill, P.E., Murray, W. and Wright, M.H., *Practical Optimization*, New York: Academic Press, 1981.
- James, W. and Stein, C., "Estimation with Quadratic Loss," *Proceedings of the Fourth Berkeley Symposium on Mathematical Statistics* (Vol. 1), ed. J. Neyman, Berkeley, CA: University of California Press, 1960, 361-380.

- Jobson, J. and Korkie, B., "Estimation for Markowitz Efficient Portfolios," *Journal of the American Statistical Association*, **75**, 1980, 544-554.
- Jorion, P., "Bayes-Stein Estimation for Portfolio Analysis," *Journal of Financial and Quantitative Analysis*, **21**, 1986, 279-291.
- Jorion, P., "Bayesian and CAPM Estimators of the Means: Implications for Portfolio Selection," *Journal of Banking and Finance*, **10**, 1991, 717-727.
- Kadiyala, K. R. and S. Karlsson "Numerical Methods for Estimation and Inference in Bayesian VAR-models," *Journal of Applied Econometrics*, **12**, 1997, 99-132.
- Kandel, S., McCulloch, R.E., and Stambaugh, R.F., "Bayesian Inference and Portfolio Efficiency," *Review of Financial Studies*, **8**, 1995, 1-53.
- Kim, S., Shephard, N., and Chib, S., "Stochastic Volatility: Likelihood Inference and Comparison with ARCH Models" *Review of Economic Studies*, **65**, 1998, 361-394.
- Levy, H. and Sarnat, M., "International Diversification of Investment Portfolios," *American Economic Review*, **60**, 1970, 668-675.
- Markowitz, H., "Portfolio Selection," *Journal of Finance*, **7**, 1952, 77-91.
- Marriott, J.; Ravishanker, N.; Gelfand, A. and Pai, J., "Bayesian Analysis of ARMA Processes: Complete Sampling Based Inference Under Exact Likelihoods", in: *Bayesian Statistics and Econometrics: Essays in Honor of Arnold Zellner*, D. Barry et.al (eds.), Amsterdam: North-Holland, 1993.
- MathWorks, Inc. *MATLAB*, Natick, MA: Author, 1996.

- McCulloch, R.E., and Rossi, P.E., "A Bayesian Approach to Testing the Arbitrage Pricing Theory," *Journal of Econometrics*, **49**, 1991, 141-168.
- McCulloch, R.E., and Tsay, R.S., "Bayesian Analysis of Autoregressive Time Series Via the Gibbs Sampler," *Journal of Time Series Analysis*, **15**, 1994, 235-250.
- Nakatsuma, T., and Tsurumi, H., "ARMA-GARCH Models: Bayes Estimation Versus MLE, and Bayes Non-Stationarity Test," Working Paper No. 9619, Department of Economics, Rutgers University, New Jersey, 1996.
- Pari, A. and Chen, S., "Estimation Risk and Optimal Portfolios," *Journal of Portfolio Management*, **12**, 1985, 120-130.
- Putnam, B. and Quintana, J., "Mean-Variance Optimal Portfolio Models and the Inappropriateness of the Assumption of a Time-Stable Variance-Covariance Matrix", *Review of Financial Economics*, **1**, 1991.
- Schittowski, K., *Nonlinear Programming Codes*, (Vol. 183), New York: Springer-Verlag, 1980.
- Schittowski, K., "NLQPL: A FORTRAN Subroutine Solving Constrained Nonlinear Programming Problems," *Operations Research*, **5**, 1985, 458-500.
- Shaken, J., "A Bayesian Approach to Testing Portfolio Efficiency," *Journal of Financial Economics*, **19**, 1987, 195-215.
- Solnik, B., "Optimal International Asset Allocation," *Journal of Portfolio Management*, **9**, 1982, 11-21.

West, M. and J. Harrison, *Bayesian Forecasting and Dynamic Models*, New York: Springer-Verlag, 1989.

Zellner, A., *An Introduction to Bayesian Inference in Econometrics*, New York: John Wiley and Sons, Inc., 1971.

Tables

Table 1a. Monthly Asset Returns Mean (Standard Deviation)

	75-94	75-79	80-84	85-89	90-94
Hong Kong	2.1 (9.1)	3.0 (9.1)	1.2 (10.5)	2.1 (9.0)	2.0 (8.0)
United Kingdom	1.5 (6.6)	2.4 (9.7)	1.6 (4.5)	1.4 (6.0)	0.5 (4.8)
Australia	1.2 (6.2)	1.6 (5.4)	0.7 (7.2)	1.9 (7.4)	0.4 (4.6)
France	1.1 (6.3)	1.2 (6.9)	1.1 (5.8)	2.0 (7.0)	0.2 (5.2)
United States	0.9 (4.2)	0.7 (4.1)	0.8 (4.0)	1.3 (5.1)	0.6 (3.4)
Japan	0.9 (5.3)	0.9 (3.6)	1.3 (4.2)	2.1 (5.3)	-0.7 (7.0)
Belgium	0.8 (5.2)	0.3 (3.7)	0.8 (5.4)	2.0 (6.5)	0.1 (4.5)
Germany	0.8 (5.1)	0.4 (3.7)	0.9 (4.0)	1.5 (7.0)	0.2 (5.0)
Canada	0.7 (4.9)	1.4 (4.8)	0.6 (6.3)	0.9 (4.6)	0.1 (3.5)
Austria	0.6 (5.9)	0.0 (1.5)	-0.2 (2.5)	2.6 (8.0)	0.0 (7.9)
Spain	0.4 (6.3)	-1.4 (5.4)	0.9 (4.7)	2.1 (7.7)	0.2 (6.4)

Table 1b. Monthly Asset Returns Covariance and Correlation

	75-94	HK	UK	AU	FR	US	JA	BE	GE	CA	AUS	SP
Hong Kong	84	0.5	0.4	0.3	0.4	0.2	0.3	0.3	0.3	0.4	0.2	0.3
United Kingdom	26	43	0.4	0.5	0.5	0.4	0.5	0.4	0.4	0.5	0.3	0.4
Australia	24	18	39	0.4	0.4	0.3	0.3	0.3	0.3	0.6	0.2	0.3
France	17	22	14	39	0.4	0.4	0.6	0.6	0.6	0.4	0.4	0.4
United States	14	14	11	11	17	0.3	0.4	0.4	0.4	0.7	0.2	0.3
Japan	11	13	9	14	6	28	0.4	0.4	0.3	0.3	0.2	0.4
Belgium	14	17	9	20	9	12	27	0.6	0.4	0.4	0.4	0.4
Germany	15	14	8	18	8	10	16	26	0.3	0.6	0.6	0.4
Canada	17	16	18	12	14	7	9	7	24	0.2	0.3	0.3
Austria	13	10	7	15	4	6	12	17	6	35	0.3	0.3
Spain	16	15	12	16	8	13	12	12	9	11	39	0.3

Table 2. Portfolio Yearly Return Performance Comparisons

Data Model	Method			
	Mean Variance (<i>MV</i>)		Direct Utility (<i>DU</i>)	
	Average	SD	Average	SD
Weighted	12.0	.58	12.0	.58
Classical	10.9	.46	11.2	.48
James-Stein	12.5	.53	12.7	.57
Hierarchical Bayes	14.5	.65	16.0	.70

Figure

Figure 1. Equity Performance Mean Variance (Top), Direct Utility Factors (Mid) and Turnover Rates (Bottom).

