Appendix to
A Multivariate Model of Strategic Asset Allocation

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A Appendix A: Derivation of Main Equations in Text

We first summarize three results on matrix algebra that will be convenient in deriving the expressions given in the text.

Result 1.

$$\begin{aligned}
\mathbf{z}_{t+1}\mathbf{z}_{t+1}' - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right) &= \left(\Phi_0 + \Phi_1\mathbf{z}_t + \mathbf{v}_{t+1}\right)\left(\Phi_0 + \Phi_1 z_t + \mathbf{v}_{t+1}\right)' - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right) \\
&= \Phi_0\Phi_0' + \Phi_1\mathbf{z}_t\Phi_0' + \mathbf{v}_{t+1}\Phi_0' + \Phi_0\mathbf{z}_t'\Phi_1' + \Phi_1\mathbf{z}_t\mathbf{z}_t'\Phi_1' \\
&\quad + \mathbf{v}_{t+1}\mathbf{z}_t'\Phi_1' + \Phi_0\mathbf{v}_{t+1}' + \Phi_1\mathbf{z}_t\mathbf{v}_{t+1}' + \mathbf{v}_{t+1}\mathbf{v}_{t+1}' - \mathrm{E}_t\left(\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right) \\
&= \mathbf{v}_{t+1}\Phi_0' + \mathbf{v}_{t+1}\mathbf{z}_t'\Phi_1' + \Phi_0\mathbf{v}_{t+1}' + \Phi_1\mathbf{z}_t\mathbf{v}_{t+1}' + \mathbf{v}_{t+1}\mathbf{v}_{t+1}' - \Sigma_v.
\end{aligned}$$

■

Result 2.

$$\begin{aligned}
r_{i,t+1} - \mathrm{E}_t\left(r_{i,t+1}\right) &= \mathbf{x}_{t+1}^{(i-1)} + r_{1,t+1} - \mathrm{E}_t\left(\mathbf{x}_{t+1}^{(i-1)} + r_{1,t+1}\right) \\
&= \mathbf{v}_{t+1}^{(i)} + \mathbf{v}_{t+1}^{(1)}
\end{aligned}$$

where $\mathbf{x}_{t+1}^{(i-1)}$ denotes the $(i-1)$th element of the excess return vector $\mathbf{x}_{t+1}$ and likewise with $\mathbf{v}_{t+1}$.■

Result 3. Any quadratic form $\mathbf{z}_{t+1}'\mathbf{M}\mathbf{z}_{t+1}$ admits the following vector form representation:

$$\mathbf{z}_{t+1}'\mathbf{M}\mathbf{z}_{t+1} = \mathrm{vec}\left(\mathbf{M}\right)'\mathrm{vec}\left(\mathbf{z}_{t+1}\mathbf{z}_{t+1}'\right),$$

where vec$(\cdot)$ is the vectorization operator.■

Result 4. (Muirhead, 1982, pp.518)

$$\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right) = \left(\mathbf{I}_{m^2} + \sum_{i,j}^{m}\left(\mathbf{Q}_{ij}\otimes\mathbf{Q}_{ij}'\right)\right)\left(\Sigma_v\otimes\Sigma_v\right),$$

where $\mathbf{Q}_{ij}$ is a $m\times m$ zero matrix except for the (i,j)th element which is equal to 1, and $\otimes$ is the kronecker product operator.■

Unconditional distribution of the state vector $\mathbf{z}_t$.

The linearity of the VAR system (4) implies that the state vector $\mathbf{z}_t$ inherits the normality of the shocks $\mathbf{v}_{t+1}$. It has unconditional mean $\boldsymbol{\mu}_z$ and variance-covariance matrix $\boldsymbol{\Sigma}_{zz}$ given by

$$\begin{aligned}
\boldsymbol{\mu}_z &= \left(\mathbf{I}_m - \boldsymbol{\Phi}_1\right)^{-1}\boldsymbol{\Phi}_0, \\
\mathrm{vec}(\boldsymbol{\Sigma}_{zz}) &= \left(\mathbf{I}_{m^2} - \boldsymbol{\Phi}_1\otimes\boldsymbol{\Phi}_1\right)^{-1}\mathrm{vec}\left(\boldsymbol{\Sigma}_v\right).
\end{aligned} \tag{30}$$

$\blacksquare$

Derivation of Equation (10)

The log return on the portfolio $r_{p,t+1}$ is a discrete-time approximation to its continuous-time counterpart. We begin by specifying the return processes for the short-term instrument B_t and other risky assets $\mathbf{P}_t$ in continuous time:

$$\begin{aligned} \frac{dB_t}{B_t} &= \mu_{b,t}dt + \boldsymbol{\sigma}_b d\mathbf{W}_t, & (31)\\ \frac{d\mathbf{P}_t}{\mathbf{P}_t} &= \boldsymbol{\mu}_t dt + \boldsymbol{\sigma} d\mathbf{W}_t, & (32) \end{aligned}$$

where $\mu_{b,t}$ and $\boldsymbol{\mu}_t$ are the drifts, $\boldsymbol{\sigma}_b$ and $\boldsymbol{\sigma}$ are the diffusion, and $\mathbf{W}_t$ is a m-dimensional standard Brownian motion.[15] We allow the drifts to depend on other state variables, but for notational simplicity, we suppress this dependency and simply use the time subscript. Moreover, note that the same $\mathbf{W}_t$ appears in the two equations.

We can obtain the log return on each asset using Ito's Lemma:

$$\begin{aligned} d\log B_t &= \left(\frac{dB_t}{B_t}\right) - \frac{1}{2}\left(\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\right)dt, & (33)\\ d\log P_{i,t} &= \left(\frac{dP_{i,t}}{P_{i,t}}\right) - \frac{1}{2}\left(\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right)dt, & (34) \end{aligned}$$

where $\boldsymbol{\sigma}_i$ is the ith row of the diffusion matrix $\boldsymbol{\sigma}$, and $i = 1, .., n-1$.

Let V_t be the value of the portfolio at time t. We will use $d\log V_t$ to approximate $r_{p,t+1}$. By Ito's Lemma,

$$d\log V_t = \left(\frac{dV_t}{V_t}\right) - \frac{1}{2}\left(\frac{dV_t}{V_t}\right)^2. \qquad (35)$$

We will now derive these two terms in order:

$$\begin{aligned} \frac{dV_t}{V_t} &= \boldsymbol{\alpha}_t'\left(\frac{d\mathbf{P}_t}{\mathbf{P}_t}\right) + \left(1-\boldsymbol{\alpha}_t'\boldsymbol{\iota}\right)\frac{dB_t}{B_t}\\ &= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t + \frac{1}{2}\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right]dt\right) + \left(1-\boldsymbol{\alpha}_t'\boldsymbol{\iota}\right)\left(d\log B_t + \frac{1}{2}\left(\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\right)dt\right)\\ &= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right) + d\log B_t\\ &\quad +\frac{1}{2}\boldsymbol{\alpha}_t'\left(\left[\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'\right] - \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\cdot\boldsymbol{\iota}\right)dt + \frac{1}{2}\boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'dt, \end{aligned}$$

where $\boldsymbol{\iota}$ is a $n\times 1$ vector of ones and the bracket $[\cdot]$ denotes a vector with $\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'$ the ith entry. Next,

$$\begin{aligned} \left(\frac{dV_t}{V_t}\right)^2 &= \boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)'\boldsymbol{\alpha}_t + \left(d\log B_t\right)^2\\ &\quad +2\boldsymbol{\alpha}_t'\left(d\log\mathbf{P}_t - d\log B_t\cdot\boldsymbol{\iota}\right)\left(d\log B_t\right) + o\left(dt\right), \end{aligned}$$

[15] The dimensions of $\mu_b, \boldsymbol{\mu}, \boldsymbol{\sigma}_b, \boldsymbol{\sigma}$ are $1\times 1, (n-1)\times 1, 1\times m, (n-1)\times m$, respectively.

where the $o(dt)$ terms vanish because they involve either $(dt)^2$ or $(dt)(d\mathbf{W}_t)$.

Now, from equation (31)–(33) and ignoring dt terms,

$$d\log \mathbf{P}_t - d\log B_t \cdot \boldsymbol{\iota} = (\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)\, d\mathbf{W}_t.$$

Thus,

$$\begin{aligned}(d\log \mathbf{P}_t - d\log B_t \cdot \boldsymbol{\iota})(d\log \mathbf{P}_t - d\log B_t \cdot \boldsymbol{\iota})' &= (\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)', \\ (d\log \mathbf{P}_t - d\log B_t \cdot \boldsymbol{\iota})(d\log B_t) &= (\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_1)\cdot\boldsymbol{\sigma}_b'.\end{aligned}$$

Collecting these results and using our notation for excess returns: $\mathbf{x}_{t+1} = d\log \mathbf{P}_t - d\log B_t \cdot \boldsymbol{\iota}$, $r_{1,t+1} = d\log(B_t)$ and $dt = 1$,

$$\begin{aligned}r_{p,t+1} &= d\log V_t \\ &= \boldsymbol{\alpha}_t'\mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t'\left([\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'] - \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b'\cdot\boldsymbol{\iota}\right) \\ &\quad -\frac{1}{2}\left[\boldsymbol{\alpha}_t'(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)'\boldsymbol{\alpha}_t + 2\boldsymbol{\alpha}_t'(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)\boldsymbol{\sigma}_b'\right]\end{aligned}$$

Using the notation in the VAR system with the Cholesky decomposition for $\Sigma_v = \mathbf{G}\mathbf{G}'$, $\boldsymbol{\sigma}_i - \boldsymbol{\sigma}_b$ is equal to the ith row of $\mathbf{G}, \mathbf{G}_i$. Hence,

$$\begin{aligned}(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)(\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)' &= \mathbf{G}_{2:n}\mathbf{G}_{2:n}' = \boldsymbol{\Sigma}_{xx}, \\ \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b' &= \mathbf{G}_1\mathbf{G}_1' = \sigma_1^2, \\ \boldsymbol{\sigma}_i\boldsymbol{\sigma}_i' &= \mathbf{G}_i\mathbf{G}_i' + \boldsymbol{\sigma}_b\mathbf{G}_i' + \mathbf{G}_i\boldsymbol{\sigma}_b' + \boldsymbol{\sigma}_b\boldsymbol{\sigma}_b', \\ [\boldsymbol{\sigma}_i\boldsymbol{\sigma}_i'] &= \sigma_x^2 + 2\boldsymbol{\sigma}_{1x} + \sigma_1^2\boldsymbol{\iota}, \\ (\boldsymbol{\sigma} - \boldsymbol{\iota}\cdot\boldsymbol{\sigma}_b)\boldsymbol{\sigma}_b' &= \mathbf{G}_{2:n}\mathbf{G}_1' = \boldsymbol{\sigma}_{1x},\end{aligned}$$

where $\mathbf{G}_{2:n}$ denotes the submatrix formed by taking the 2nd to nth rows of $\mathbf{G}$.

With these terms, the return on the portfolio is

$$\begin{aligned}r_{p,t+1} &= \boldsymbol{\alpha}_t'\mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t'\left(\sigma_x^2 + 2\boldsymbol{\sigma}_{1x}\right) - \frac{1}{2}\boldsymbol{\alpha}_t'\boldsymbol{\Sigma}_{xx}\boldsymbol{\alpha}_t - \boldsymbol{\alpha}_t'\boldsymbol{\sigma}_{1x}, \\ &= \boldsymbol{\alpha}_t'\mathbf{x}_{t+1} + r_{1,t+1} + \frac{1}{2}\boldsymbol{\alpha}_t'\left(\sigma_x^2 - \boldsymbol{\Sigma}_{xx}\boldsymbol{\alpha}_t\right).\end{aligned}$$

■

Solving for the Optimal Portfolio Rule.

Subtracting the log Euler equation (12) with $i = 1$ from (12), we obtain

$$\begin{aligned}\mathrm{E}_t(r_{i,t+1} - r_{1,t+1}) + \frac{1}{2}\mathrm{Var}_t(r_{i,t+1} - r_{1,t+1}) &= \mathrm{Cov}_t\left(\frac{\theta}{\psi}\Delta c_{t+1} + (1-\theta)\, r_{p,t+1}, r_{i,t+1}\right) \\ &\quad -\mathrm{Cov}_t\left(\frac{\theta}{\psi}\Delta c_{t+1} + (1-\theta)\, r_{p,t+1}, r_{1,t+1}\right) \\ &\quad -\frac{1}{2}\left(\mathrm{Var}_t(r_{i,t+1}) - \mathrm{Var}_t(r_{1,t+1}) - \mathrm{Var}_t(r_{i,t+1} - r_{1,t+1})\right).\end{aligned} \tag{36}$$

Using the budget constraint (11) and the trivial identity $\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}$,

$$\frac{\theta}{\psi}\Delta c_{t+1} + (1-\theta)\, r_{p,t+1} = \frac{\theta}{\psi}\left(c_{t+1} - w_{t+1}\right) + \gamma r_{p,t+1} + \text{ time } t \text{ terms and constants.}$$

Thus, equation (36) can be written as

$$\begin{aligned}
\mathrm{E}_t\left(r_{i,t+1} - r_{1,t+1}\right) + \frac{1}{2}\mathrm{Var}_t\left(r_{i,t+1} - r_{1,t+1}\right) &= \frac{\theta}{\psi}\left[\sigma_{i,c-w,t} - \sigma_{1,c-w,t}\right] + \gamma\left[\sigma_{i,p,t} - \sigma_{1,p,t}\right] \\
&\quad -\frac{1}{2}\left(\mathrm{Var}_t\left(r_{i,t+1}\right) - \mathrm{Var}_t\left(r_{1,t+1}\right) - \mathrm{Var}_t(r_{i,t+1} - r_{1,t+1})\right).
\end{aligned}$$

We will derive these terms now.

Using the equation for log return on the portfolio and ignoring time t terms and constants,

$$\begin{aligned}
\sigma_{i,p,t} &= \mathrm{Cov}_t\left(\boldsymbol{\alpha}_t'\mathbf{x}_{t+1} + r_{1,t+1}, r_{i,t+1}\right) \\
&= \boldsymbol{\alpha}_t'\left(\Sigma_{xx}^{(i-1)} + \boldsymbol{\sigma}_{1x}\right) + \boldsymbol{\sigma}_{1x}^{(i-1)} + \sigma_1^2,
\end{aligned}$$

$$\begin{aligned}
\sigma_{1,p,t} &= \mathrm{Cov}_t\left(\boldsymbol{\alpha}_t'\mathbf{x}_{t+1} + r_{1,t+1}, r_{1,t+1}\right) \\
&= \boldsymbol{\alpha}_t'\boldsymbol{\sigma}_{1x} + \sigma_1^2.
\end{aligned}$$

To evaluate the conditional covariances $\sigma_{i,c-w,t}$ and $\sigma_{1,c-w,t}$, we use the conjectured policy rule for the consumption-wealth ratio.

$$\begin{aligned}
&\sigma_{i,c-w,t} \\
=\;& \mathrm{Cov}_t\left(c_{t+1} - w_{t+1} - \mathrm{E}_t\left(c_{t+1} - w_{t+1}\right), r_{i,t+1} - \mathrm{E}_t(r_{i,t+1})\right) \\
=\;& \mathrm{Cov}_t\left(\mathbf{B}_1'\mathbf{v}_{t+1} + \left(\mathbf{\Phi}_0 + \mathbf{\Phi}_1\mathbf{z}_t + \mathbf{v}_{t+1}\right)'\mathbf{B}_2\left(\mathbf{\Phi}_0 + \mathbf{\Phi}_1\mathbf{z}_t + \mathbf{v}_{t+1}\right), \mathbf{v}_{t+1}^{(i)} + \mathbf{v}_{t+1}^{(1)}\right) \\
=\;& \mathrm{Cov}_t\left(\mathbf{B}_1'\mathbf{v}_{t+1} + \mathbf{\Phi}_0'\mathbf{B}_2\mathbf{v}_{t+1} + \mathbf{z}_t'\mathbf{\Phi}_1'\mathbf{B}_2\mathbf{v}_{t+1} + \mathbf{v}_{t+1}'\mathbf{B}_2\mathbf{\Phi}_0 + \mathbf{v}_{t+1}'\mathbf{B}_2\mathbf{\Phi}_1\mathbf{z}_t, \mathbf{v}_{t+1}^{(i)} + \mathbf{v}_{t+1}^{(1)}\right) \\
=\;& \mathbf{B}_1'\left(\Sigma_v^{(i)} + \Sigma_v^{(1)}\right) + \mathbf{\Phi}_0'\mathbf{B}_2\left(\Sigma_v^{(i)} + \Sigma_v^{(1)}\right) + \left(\Sigma_v^{(i)} + \Sigma_v^{(1)}\right)'\mathbf{B}_2\mathbf{\Phi}_0 \\
&+\mathbf{z}_t'\mathbf{\Phi}_1'\mathbf{B}_2\left(\Sigma_v^{(i)} + \Sigma_v^{(1)}\right) + \left(\Sigma_v^{(i)} + \Sigma_v^{(1)}\right)'\mathbf{B}_2\mathbf{\Phi}_1\mathbf{z}_t,
\end{aligned}$$

where the second equality follows from using Result 1 and 2, and $\Sigma_v^{(i)}$ denotes the ith column of Σ_v. Note that $\mathbf{B}_2$ is not necessarily symmetric, so that we cannot combine any of the terms in the expression above. Similarly, for the return on the short-term bond we have

$$\sigma_{1,c-w,t} = \mathbf{B}_1'\Sigma_v^{(1)} + \mathbf{\Phi}_0'\mathbf{B}_2\Sigma_v^{(1)} + \left(\Sigma_v^{(1)}\right)'\mathbf{B}_2\mathbf{\Phi}_0 + \mathbf{z}_t'\mathbf{\Phi}_1'\mathbf{B}_2\Sigma_v^{(1)} + \left(\Sigma_v^{(1)}\right)'\mathbf{B}_2\mathbf{\Phi}_1\mathbf{z}_t.$$

Therefore

$$\begin{aligned}
\sigma_{i,c-w,t} - \sigma_{1,c-w,t} &= \mathbf{B}_1'\Sigma_v^{(i)} + \left(\mathbf{\Phi}_0'\mathbf{B}_2 + \mathbf{z}_t'\mathbf{\Phi}_1'\mathbf{B}_2\right)\Sigma_v^{(i)} + \left(\Sigma_v^{(i)}\right)'\left(\mathbf{B}_2\mathbf{\Phi}_0 + \mathbf{B}_2\mathbf{\Phi}_1\mathbf{z}_t\right) \\
&= \mathbf{B}_1'\Sigma_v^{(i)} + \left(\mathbf{\Phi}_0' + \mathbf{z}_t'\mathbf{\Phi}_1'\right)\left(\mathbf{B}_2 + \mathbf{B}_2'\right)\Sigma_v^{(i)} \\
&= \left(\Sigma_v^{(i)}\right)'\mathbf{B}_1 + \left(\Sigma_v^{(i)}\right)'\left(\mathbf{B}_2 + \mathbf{B}_2'\right)\left(\mathbf{\Phi}_0 + \mathbf{\Phi}_1\mathbf{z}_t\right)
\end{aligned}$$

Stacked, this equations give

$$
\begin{aligned}
&\boldsymbol{\sigma}_{c-w,t} - \sigma_{1,c-w,t}\boldsymbol{\iota} \\
= &\left(\begin{bmatrix} \Sigma_v^{(2)\prime} \\ \vdots \\ \Sigma_v^{(n)\prime} \end{bmatrix} \mathbf{B}_1 + 2 \begin{bmatrix} \Sigma_v^{(2)\prime} \\ \vdots \\ \Sigma_v^{(n)\prime} \end{bmatrix} \left(\mathbf{B}_2 + \mathbf{B}_2'\right) \boldsymbol{\Phi}_0 \right) + \left(\begin{bmatrix} \Sigma_v^{(2)\prime} \\ \vdots \\ \Sigma_v^{(n)\prime} \end{bmatrix} \left(\mathbf{B}_2 + \mathbf{B}_2'\right) \boldsymbol{\Phi}_1 \right) \mathbf{z}_t \\
= &\left[\left(\Sigma_v \mathbf{H}_x'\right)' \mathbf{B}_1 + \left(\Sigma_v \mathbf{H}_x'\right) \left(\mathbf{B}_2 + \mathbf{B}_2'\right) \boldsymbol{\Phi}_0 \right] + \left[\left(\Sigma_v \mathbf{H}_x'\right)' \left(\mathbf{B}_2 + \mathbf{B}_2'\right) \boldsymbol{\Phi}_1 \right] \mathbf{z}_t \\
= &\Lambda_0 + \Lambda_1 \mathbf{z}_t, \qquad (37)
\end{aligned}
$$

as claimed in equation (17).■

Solving for the Optimal Consumption Rule.

We derive first equation (26). To derive this equation, note that log consumption growth verifies the following trivial identity: $\Delta c_{t+1} = (c_{t+1} - w_{t+1}) - (c_t - w_t) + \Delta w_{t+1}$. Substituting the log-linearized budget constraint (11) into this equation and taking expectations we obtain

$$
\begin{aligned}
\mathrm{E}_t(\Delta c_{t+1}) &= \mathrm{E}_t\left(c_{t+1} - w_{t+1}\right) - \left(c_t - w_t\right) + \mathrm{E}_t\left(\Delta w_{t+1}\right) \qquad (38) \\
&= \mathrm{E}_t\left(c_{t+1} - w_{t+1}\right) - \left(c_t - w_t\right) + \mathrm{E}_t(r_{p,t+1}) + \left(1 - \frac{1}{\rho}\right)\left(c_t - w_t\right) + k.
\end{aligned}
$$

Combining the two equations (24) and (38), we obtain a difference equation in $c_t - w_t$, given in (26).

Next we show that both the expected log return on the wealth portfolio $\mathrm{E}_t r_{p,t+1}$ and the variance term $\chi_{p,t}$ in equation (24) for expected log consumption growth are quadratic functions of the vector of state variables.

Taking conditional expectations of equation (10) and substituting the portfolio policy rule $\boldsymbol{\alpha}_t = \mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t$,

$$
\begin{aligned}
\mathrm{E}_t\left(r_{p,t+1}\right) &= \boldsymbol{\alpha}_t' \mathrm{E}_t\left(\mathbf{x}_{t+1}\right) + \mathrm{E}_t\left(r_{1,t+1}\right) + \frac{1}{2}\boldsymbol{\alpha}_t'\left(\sigma_x^2 - \Sigma_{xx}\boldsymbol{\alpha}_t\right) \\
&= \left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\mathbf{H}_x\left(\Phi_0 + \Phi_1\mathbf{z}_t\right) + \mathbf{H}_1\left(\Phi_0 + \Phi_1\mathbf{z}_t\right) \\
&\quad + \frac{1}{2}\left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\sigma_x^2 - \frac{1}{2}\left(\mathbf{A}_0' + \mathbf{z}_t'\mathbf{A}_1'\right)\Sigma_{xx}\left(\mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t\right) \\
&= \Gamma_0 + \Gamma_1\mathbf{z}_t + \Gamma_2 \mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right),
\end{aligned}
$$

where

$$
\begin{aligned}
\Gamma_0 &\equiv \mathbf{A}_0'\mathbf{H}_x\Phi_0 + \mathbf{H}_1\Phi_0 + \frac{1}{2}\mathbf{A}_0'\sigma_x^2 - \frac{1}{2}\mathbf{A}_0'\Sigma_{xx}\mathbf{A}_0, \\
\Gamma_1 &\equiv \Phi_0'\mathbf{H}_x'\mathbf{A}_1 + \mathbf{A}_0'\mathbf{H}_x\Phi_1 + \mathbf{H}_1\Phi_1 + \frac{1}{2}\sigma_x^2\mathbf{A}_1 - \mathbf{A}_0'\Sigma_{xx}\mathbf{A}_1, \\
\Gamma_2 &\equiv \mathrm{vec}\left(\mathbf{A}_1'\mathbf{H}_x\Phi_1\right)' - \frac{1}{2}\mathrm{vec}\left(\mathbf{A}_1'\Sigma_{xx}\mathbf{A}_1\right)',
\end{aligned}
$$

and $\mathbf{H}_1$ and $\mathbf{H}_x$ are selection matrices that select the short-term real interest rate and the vector of excess returns from the full state vector.

We now evaluate the variance term

$$\chi_{p,t} = \frac{1}{2}\left(\frac{\theta}{\psi}\right)\mathrm{Var}_t\left(\Delta c_{t+1} - \psi r_{p,t+1}\right).$$

Using the trivial identity for Δc_{t+1} and the budget constraint (11), substituting the conjecture for the consumption rule

$$\begin{aligned} c_t - w_t &= b_0 + \mathbf{B}_1'\mathbf{z}_t + \mathbf{z}_t'\mathbf{B}_2\mathbf{z}_t \\ &= b_0 + \mathbf{B}_1'\mathbf{z}_t + \mathrm{vec}\left(\mathbf{B}_2\right)'\mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right) \end{aligned}$$

and $\boldsymbol{\alpha}_t = \mathbf{A}_0 + \mathbf{A}_1\mathbf{z}_t$, and ignoring time t terms and constants, we can write the argument of the variance as:

$$\begin{aligned} & \Delta c_{t+1} - \psi r_{p,t+1} \\ = \; & \left[\mathbf{B}_1' + \mathbf{\Phi}_0'\left(\mathbf{B}_2 + \mathbf{B}_2'\right) + (1-\psi)\mathbf{A}_0'\mathbf{H}_x + (1-\psi)\mathbf{H}_1\right]\mathbf{v}_{t+1} \\ & + \mathbf{z}_t'\left[\mathbf{\Phi}_1'\left(\mathbf{B}_2 + \mathbf{B}_2'\right) + (1-\psi)\mathbf{A}_1'\mathbf{H}_x\right]\mathbf{v}_{t+1} \\ & + \mathrm{vec}\left(\mathbf{B}_2\right)'\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right) \\ = \; & \left[\Pi_1 + \mathbf{z}_t'\Pi_2\right]\mathbf{v}_{t+1} + \mathrm{vec}\left(\mathbf{B}_2\right)'\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right), \end{aligned}$$

where

$$\begin{aligned} \Pi_1 &\equiv \mathbf{B}_1' + \mathbf{\Phi}_0'\left(\mathbf{B}_2 + \mathbf{B}_2'\right) + (1-\psi)\mathbf{A}_0'\mathbf{H}_x + (1-\psi)\mathbf{H}_1, \\ \Pi_2 &\equiv \mathbf{\Phi}_1'\left(\mathbf{B}_2 + \mathbf{B}_2'\right) + (1-\psi)\mathbf{A}_1'\mathbf{H}_x. \end{aligned}$$

Since $\mathbf{v}_{t+1}$ is conditionally normally distributed, all third moments are zero. Thus,

$$\begin{aligned} & \mathrm{Var}_t\left(\Delta c_{t+1} - \psi r_{p,t+1}\right) \\ = \; & \Pi_1\Sigma_v\Pi_1' + \left[2\Pi_1\Sigma_v\Pi_2'\right]\mathbf{z}_t + \mathrm{vec}\left(\Pi_2\Sigma_v\Pi_2'\right)'\mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right) \\ & + \mathrm{vec}\left(\mathbf{B}_2\right)'\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)\mathrm{vec}\left(\mathbf{B}_2\right), \end{aligned}$$

and $\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)$ is given by the expression in Result 4 above. Putting these pieces together, we have

$$\chi_{p,t} = V_0 + \mathbf{V}_1\mathbf{z}_t + \mathbf{V}_2\mathrm{vec}\left(\mathbf{z}_t\mathbf{z}_t'\right),$$

where

$$\begin{aligned} V_0 &\equiv \frac{\theta}{2\psi}\left[\Pi_1\Sigma_v\Pi_1' + \mathrm{vec}\left(\mathbf{B}_2\right)'\mathrm{Var}_t\left(\mathrm{vec}\left(\mathbf{v}_{t+1}\mathbf{v}_{t+1}'\right)\right)\mathrm{vec}\left(\mathbf{B}_2\right)\right], \\ \mathbf{V}_1 &\equiv \frac{\theta}{2\psi}\left[2\Pi_1\Sigma_v\Pi_2'\right], \\ \mathbf{V}_2 &\equiv \frac{\theta}{2\psi}\left[\mathrm{vec}\left(\Pi_2\Sigma_v\Pi_2'\right)'\right]. \end{aligned}$$

We can now solve for the coefficients of the optimal consumption rule. Simple substitution of the expressions for $\mathrm{E}_t r_{p,t+1}$ and $\chi_{p,t}$, and the expression for the conditional expectation of $(c_{t+1}-w_{t+1})$ into the RHS of (26) yields

$$c_t - w_t = \Xi_0 + \Xi_1 \mathbf{z}_t + \Xi_2 \mathrm{vec}\left(\mathbf{z}_t \mathbf{z}_t'\right), \tag{39}$$

where

$$\begin{aligned}
\Xi_0 &\equiv \rho[-\psi \log\delta + k - V_0 + (1-\psi)\,\Gamma_0 + b_0 + \mathbf{B}_1'\Phi_0 \\
&\quad + \mathrm{vec}\left(\mathbf{B}_2\right)' \mathrm{vec}(\mathbf{\Phi}_0\mathbf{\Phi}_0') + \mathrm{vec}\left(\mathbf{B}_2\right)' \mathrm{vec}(\Sigma_v)], \\
\Xi_1 &\equiv \rho\left[-\mathbf{V}_1 + (1-\psi)\,\Gamma_1 + \mathbf{B}_1'\mathbf{\Phi}_1 + 2\mathbf{\Phi}_0'\left(\mathbf{B}_2' + \mathbf{B}_2\right)\mathbf{\Phi}_1\right], \\
\Xi_2 &\equiv \rho\left[-\mathbf{V}_2 + (1-\psi)\,\Gamma_2 + \mathrm{vec}\left(\mathbf{\Phi}_1'\mathbf{B}_2\mathbf{\Phi}_1\right)'\right].
\end{aligned}$$

Equation (39) confirms our initial conjecture on the form of the consumption-wealth ratio. Notice that Ξ_0, Ξ_1, Ξ_2 depend on $b_0, \mathbf{B}_1$ and $\mathbf{B}_2$. Therefore, for the solution to be consistent, $\{b_0, \mathbf{B}_1, \mathbf{B}_2\}$ must solve the following set of equations:

$$\begin{aligned}
b_0 &= \Xi_0, \\
\mathbf{B}_1 &= \Xi_1', \\
\mathrm{vec}\left(\mathbf{B}_2\right) &= \Xi_2'.
\end{aligned} \tag{40}$$

The resulting set of values for $b_0, \mathbf{B}_1$ and $\mathrm{vec}\left(\mathbf{B}_2\right)$ determines the optimal consumption rule.■

Verification that the optimal portfolio rule is independent of ψ given ρ.

>From equations (22) and (23) in text, and equation (37) in the Appendix, we can write $\mathbf{A}_0$ and $\mathbf{A}_1$ as:

$$\begin{aligned}
\mathbf{A}_0 &= \left(\frac{1}{\gamma}\right)\mathbf{\Sigma}_{xx}^{-1}\left(\mathbf{H}_x\Phi_0 + \frac{1}{2}\boldsymbol{\sigma}_x^2 + (1-\gamma)\boldsymbol{\sigma}_{1x}\right) + \left(1-\frac{1}{\gamma}\right)\mathbf{\Sigma}_{xx}^{-1}\frac{-\mathbf{\Lambda}_0}{1-\psi} \\
&= \left(\frac{1}{\gamma}\right)\Sigma_{xx}^{-1}\left(\mathbf{H}_x\Phi_0 + \frac{1}{2}\boldsymbol{\sigma}_x^2 + (1-\gamma)\Sigma\boldsymbol{\sigma}_{1x}\right) - \left(1-\frac{1}{\gamma}\right)\Sigma_{xx}^{-1}\left[\left(\Sigma_v\mathbf{H}_x'\right)'\frac{\mathbf{B}_1}{1-\psi} + \left(\Sigma_v\mathbf{H}_x'\right)\left(\frac{\mathbf{B}_2+\mathbf{B}_2'}{1-\psi}\right)\mathbf{\Phi}_0\right] \\
\mathbf{A}_1 &= \left(\frac{1}{\gamma}\right)\mathbf{\Sigma}_{xx}^{-1}\mathbf{H}_x\Phi_1 + \left(1-\frac{1}{\gamma}\right)\mathbf{\Sigma}_{xx}^{-1}\frac{-\mathbf{\Lambda}_1}{1-\psi} \\
&= \frac{1}{\gamma}\Sigma_{xx}^{-1}\left(\mathbf{H}_x\Phi_1\right) - \left(1-\frac{1}{\gamma}\right)\Sigma_{xx}^{-1}\left[\left(\Sigma_v\mathbf{H}_x'\right)'\left(\frac{\mathbf{B}_2+\mathbf{B}_2'}{1-\psi}\right)\mathbf{\Phi}_1\right].
\end{aligned}$$

Thus, showing that the optimal portfolio rule is independent of ψ given ρ is equivalent to showing that $\mathcal{B}_1 \equiv \mathbf{B}_1/(1-\psi)$ and $\mathcal{B}_2 \equiv \mathbf{B}_2/(1-\psi)$ are independent of ψ given ρ.

First, consider $\mathcal{B}_2$. From (40), we have

$$(1-\psi)\,\mathrm{vec}\left(\mathcal{B}_2\right) = \rho\left[-\mathbf{V}_2' + (1-\psi)\Gamma_2' + (1-\psi)\,\mathrm{vec}\left(\mathbf{\Phi}_1'\mathcal{B}_2\mathbf{\Phi}_1\right)\right]. \tag{41}$$

Using the definition of $\mathbf{V}_2$, we have

$$\begin{aligned}
-\mathbf{V}_2' &= \frac{1-\gamma}{2(1-\psi)} \operatorname{vec}\left[\left(\mathbf{\Phi}_1' \left(\mathbf{B}_2 + \mathbf{B}_2'\right) + (1-\psi)\,\mathbf{A}_1'\mathbf{H}_x\right)\Sigma_v \left(\mathbf{\Phi}_1' \left(\mathbf{B}_2 + \mathbf{B}_2'\right) + (1-\psi)\,\mathbf{A}_1'\mathbf{H}_x\right)'\right] \\
&= \frac{1-\gamma}{2(1-\psi)} \operatorname{vec}\left[(1-\psi)^2 \left(\mathbf{\Phi}_1' \left(\mathcal{B}_2 + \mathcal{B}_2'\right) + \mathbf{A}_1'\mathbf{H}_x\right)\Sigma_v \left(\mathbf{\Phi}_1' \left(\mathcal{B}_2 + \mathcal{B}_2'\right) + \mathbf{A}_1'\mathbf{H}_x\right)'\right] \\
&= \frac{1-\gamma}{2}(1-\psi) \operatorname{vec}\left[\left(\mathbf{\Phi}_1' \left(\mathcal{B}_2 + \mathcal{B}_2'\right) + \mathbf{A}_1'\mathbf{H}_x\right)\Sigma_v \left(\mathbf{\Phi}_1' \left(\mathcal{B}_2 + \mathcal{B}_2'\right) + \mathbf{A}_1'\mathbf{H}_x\right)'\right] \\
&\equiv (1-\psi)\overline{\mathbf{V}}_2',
\end{aligned}$$

which is independent of ψ, since $\mathbf{A}_1$ does not depend on ψ, given $\mathcal{B}_2$.

Similarly, using the definition of Γ_2, we have:

$$(1-\psi)\Gamma_2' = (1-\psi)\left[\operatorname{vec}\left(\mathbf{A}_1'\mathbf{H}_x\Phi_1\right) - \frac{1}{2}\operatorname{vec}\left(\mathbf{A}_1'\Sigma_{xx}\mathbf{A}_1\right)\right],$$

which is also independent of ψ, since $\mathbf{A}_1$ does not depend on ψ, given $\mathcal{B}_2$.

Thus, (41) reduces to

$$\operatorname{vec}\left(\mathcal{B}_2\right) = \rho\left[\overline{\mathbf{V}}_2' + \Gamma_2' + \operatorname{vec}\left(\mathbf{\Phi}_1'\mathcal{B}_2\mathbf{\Phi}_1\right)\right].$$

This is a quadratic equation in $\mathcal{B}_2$, whose coefficients do not depend on ψ, except for ρ. The loglinearization coefficient $\rho \equiv 1 - \exp(\mathrm{E}_t[c_t - w_t])$ does depend onψ indirectly, through the dependence of $\mathrm{E}_t[c_t - w_t]$ on b_0, $\mathbf{B}_1$ and $\mathbf{B}_2$, which are functions of ψ. Consequently, the solution for $\mathcal{B}_2$ will also be independent of ψ given ρ.

Using the same logic, we can show that $\mathcal{B}_1$ is independent of ψ. From (40) we have

$$(1-\psi)\mathcal{B}_1 = \rho\left[-\mathbf{V}_1' + (1-\psi)\,\Gamma_1' + (1-\psi)\mathbf{\Phi}_1'\mathcal{B}_1 + (1-\psi)\mathbf{\Phi}_1'\left(\mathcal{B}_2 + \mathcal{B}_2'\right)\mathbf{\Phi}_0\right]. \tag{42}$$

Now,

$$\begin{aligned}
-\mathbf{V}_1' &= \frac{1-\gamma}{(1-\psi)}\left[\left((1-\psi)\overline{\Pi}_1\right)\Sigma_v'\left(\mathbf{\Phi}_1'\left(\mathcal{B}_2 + \mathcal{B}_2'\right) + \mathbf{A}_1'\mathbf{H}_x\right)\right]' \\
&= (1-\gamma)\,(1-\psi)\left[\overline{\Pi}_1\Sigma_v'\left(\mathbf{\Phi}_1'\left(\mathcal{B}_2 + \mathcal{B}_2'\right) + \mathbf{A}_1'\mathbf{H}_x\right)\right]' \\
&\equiv (1-\psi)\overline{V}_1'
\end{aligned}$$

where

$$\overline{\Pi}_1 \equiv \mathcal{B}_1' + \mathbf{\Phi}_0'\left(\mathcal{B}_2 + \mathcal{B}_2'\right) + \mathbf{A}_0'\mathbf{H}_x + \mathbf{H}_1.$$

Both $\overline{V}_1$ and $\overline{\Pi}_1$ are also independent of ψ, since $\mathbf{A}_0$ and $\mathbf{A}_1$ do not depend on ψ, given $\mathcal{B}_1$ and $\mathcal{B}_2$.

Also, Γ_1 is only a function of $\mathcal{B}_1$ and $\mathcal{B}_2$ via its dependence on $\mathbf{A}_0$ and $\mathbf{A}_1$, not of ψ. Therefore, (42) becomes

$$\mathcal{B}_1 = \rho\left[\overline{\mathbf{V}}_1' + \Gamma_1' + \mathbf{\Phi}_1'\mathcal{B}_1 + \mathbf{\Phi}_1'\left(\mathcal{B}_2 + \mathcal{B}_2'\right)\mathbf{\Phi}_0\right],$$

which again implies that the solution for $\mathcal{B}_1$ does not depend on ψ given ρ This completes our proof.■

Value function when $\psi = 1$.

First, note that we have just proved that $\mathcal{B}_1$ and $\mathcal{B}_2$ do not depend on ψ given ρ. We now derive an expression for the value function when $\psi = 1$. The value function is given by

$$\begin{aligned} V_t &= (1-\delta)^{-\frac{\psi}{1-\psi}} \left(\frac{C_t}{W_t}\right)^{\frac{1}{1-\psi}} \\ &= \exp\left\{-\frac{\psi}{1-\psi}\log(1-\delta) + \frac{b_0}{1-\psi} + \frac{\mathbf{B}_1'}{1-\psi}\mathbf{z}_t + \frac{\operatorname{vec}(\mathbf{B}_2)'}{1-\psi}\operatorname{vec}(\mathbf{z}_t\mathbf{z}_t')\right\} \\ &= \exp\left\{\mathcal{B}_0 + \mathcal{B}_1'\mathbf{z}_t + \operatorname{vec}(\mathcal{B}_2)'\operatorname{vec}(\mathbf{z}_t\mathbf{z}_t')\right\}, \end{aligned}$$

where $\mathcal{B}_1'$, and $\operatorname{vec}\mathcal{B}_2$ are independent of ψ given ρ, but $\mathcal{B}_0$ does depend on ψ.

We now find the limiting expression for $\mathcal{B}_0$ when $\psi = 1$:

$$\begin{aligned} \mathcal{B}_0 &= -\frac{\psi}{1-\psi}\log(1-\delta) + \frac{b_0}{1-\psi} \\ &= -\frac{\psi}{1-\psi}\log(1-\delta) \\ &\quad + \frac{\rho}{1-\rho}\left[-\frac{\psi}{1-\psi}\log\delta + \frac{k}{1-\psi} - \frac{V_0}{1-\psi} + \Gamma_0 + \frac{\mathbf{B}_1'}{1-\psi}\Phi_0 + \frac{\operatorname{vec}(\mathbf{B}_2)'}{1-\psi}\operatorname{vec}(\Phi_0\Phi_0') + \frac{\operatorname{vec}(\mathbf{B}_2)'}{1-\psi}\operatorname{vec}(\Sigma_v)\right] \\ &= \frac{1}{1-\psi}\frac{\rho}{1-\rho}\left[-\frac{1-\rho}{\rho}\psi\log(1-\delta) - \psi\log\delta + k\right] \\ &\quad + \frac{\rho}{1-\rho}\left[-\frac{V_0}{1-\psi} + \Gamma_0 + \frac{\mathbf{B}_1'}{1-\psi}\Phi_0 + \frac{\operatorname{vec}(\mathbf{B}_2)'}{1-\psi}\operatorname{vec}(\Phi_0\Phi_0') + \frac{\operatorname{vec}(\mathbf{B}_2)'}{1-\psi}\operatorname{vec}(\Sigma_v)\right]. \end{aligned}$$

Substituting $k = \log\rho + ((1-\rho)/\rho)\log(1-\rho)$ into the first term of the last equality, and noting that $\rho = \delta$ when $\psi = 1$, we have

$$\begin{aligned} &\frac{1}{1-\psi}\frac{\rho}{1-\rho}\left[-\frac{1-\rho}{\rho}\psi\log(1-\delta) - \psi\log\delta + k\right] \\ &= \frac{1}{1-\psi}\frac{\rho}{1-\rho}\left[-\frac{1-\rho}{\rho}\psi\log(1-\delta) - \psi\log\delta + \log\rho + \frac{1-\rho}{\rho}\log(1-\rho)\right] \\ &= \frac{1}{1-\psi}\frac{\rho}{1-\rho}\left[\frac{1-\rho}{\rho}(1-\psi)\log(1-\delta) + (1-\psi)\log\delta\right] \\ &= \log(1-\delta) + \frac{\delta}{1-\delta}\log\delta, \end{aligned}$$

which is independent of ψ.

>From previous results, we know that all the terms in the second term of the last equality are independent of ψ given ρ, except for $V_0/(1-\psi)$. We now verify that this term is also independent of ψ given ρ:

$$
\begin{aligned}
\mathcal{V}_0 &\equiv \frac{V_0}{1-\psi} \\
&= -\frac{1-\gamma}{2}\left[\frac{\Pi_1\Sigma_v\Pi_1'}{(1-\psi)^2} + \frac{\operatorname{vec}(\mathbf{B}_2)'}{1-\psi}\operatorname{Var}_t(.)\,\frac{\operatorname{vec}(\mathbf{B}_2)'}{1-\psi}\right] \\
&= -\frac{1-\gamma}{2}\left[\frac{\Pi_1\Sigma_v\Pi_1'}{(1-\psi)^2} + \operatorname{vec}(\mathcal{B}_2)'\operatorname{Var}_t(.)\operatorname{vec}(\mathcal{B}_2)\right],
\end{aligned}
$$

where the second term is independent of ψ given ρ, and the first one is also verifiably independent of ψ:

$$
\begin{aligned}
\frac{\Pi_1\Sigma_v\Pi_1'}{(1-\psi)^2} &= \left(\frac{\mathbf{B}_1'}{(1-\psi)} + \mathbf{A}_0'\mathbf{H}_x + \mathbf{H}_1 + \Phi_0'\frac{\mathbf{B}_2+\mathbf{B}_2'}{1-\psi}\right)\Sigma_v\left(\frac{\mathbf{B}_1}{(1-\psi)} + \mathbf{H}_x'\mathbf{A}_0 + \mathbf{H}_1' + \frac{\mathbf{B}_2+\mathbf{B}_2{}'}{(1-\psi)}\Phi_0\right) \\
&= \left(\mathcal{B}_1' + \mathbf{A}_0'\mathbf{H}_x + \mathbf{H}_1 + \Phi_0'\left(\mathcal{B}_2+\mathcal{B}_2'\right)\right)\Sigma_v\left(\mathcal{B}_1 + \mathbf{H}_x'\mathbf{A}_0 + \mathbf{H}_1' + \left(\mathcal{B}_2+\mathcal{B}_2'\right)\Phi_0\right),
\end{aligned}
$$

which does not depend on ψ given ρ.

Thus when $\psi = 1$ we have:

$$
\begin{aligned}
\mathcal{B}_0 &= \log(1-\delta) + \frac{\delta}{1-\delta}\log\delta \\
&\quad + \frac{\delta}{1-\delta}\left[-\mathcal{V}_0 + \Gamma_0 + \mathcal{B}_1'\Phi_0 + \operatorname{vec}(\mathcal{B}_2)'\operatorname{vec}(\Phi_0\Phi_0') + \operatorname{vec}(\mathcal{B}_2)'\operatorname{vec}(\Sigma_v)\right].
\end{aligned}
$$

Now, since ρ is independent of ψ in the special case $\psi = 1$—because $\rho = \delta$—, we conclude that $\mathcal{B}_0$, $\mathcal{B}_1$ and $\mathcal{B}_2$ are independent of ψ when $\psi = 1$.

Derivation of $\mathrm{E}[V_t]$.

>From the previous section, we learn how to obtain the coefficient matrices $\mathcal{B}_0, \mathcal{B}_1$ and $\mathcal{B}_2$. Now we want to evaluate explicitly

$$
\mathrm{E}(V_t) = \mathrm{E}\left[\exp\left(\mathcal{B}_0 + \mathcal{B}_1'\mathbf{z}_t + \mathbf{z}_t'\mathcal{B}_2\mathbf{z}_t\right)\right]
$$

where $\mathbf{z}_t$ has a multivariate normal distribution

$$
\mathbf{z}_t \sim \mathcal{N}\left(\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_{zz}\right),
$$

with

$$
\begin{aligned}
\boldsymbol{\mu}_z &= \left(\mathbf{I}_m - \boldsymbol{\Phi}_1\right)^{-1}\boldsymbol{\Phi}_0, \\
\operatorname{vec}\left(\boldsymbol{\Sigma}_{zz}\right) &= \left(\mathbf{I}_{m^2} - \boldsymbol{\Phi}_1\otimes\boldsymbol{\Phi}_1\right)^{-1}\operatorname{vec}\left(\boldsymbol{\Sigma}_v\right).
\end{aligned}
$$

First, consider a change of variable. Define

$$\mathbf{h}_t = \mathbf{z}_t - \boldsymbol{\mu}_z.$$

Then,

$$\mathcal{B}_0 + \mathcal{B}_1'\mathbf{z}_t + \mathbf{z}_t'\mathcal{B}_2\mathbf{z}_t = \mathcal{C}_0 + \mathcal{C}_1\mathbf{h}_t + \mathbf{h}_t'\mathcal{C}_2\mathbf{h}_t,$$

where

$$\mathcal{C}_0 \equiv \mathcal{B}_0 + \mathcal{B}_1'\boldsymbol{\mu}_z + \boldsymbol{\mu}_z'\mathcal{B}_2\boldsymbol{\mu}_z, \tag{43}$$

$$\mathcal{C}_1 \equiv \mathcal{B}_1' + 2\boldsymbol{\mu}_z'\mathcal{B}_2, \tag{44}$$

$$\mathcal{C}_2 \equiv \mathcal{B}_2. \tag{45}$$

To calculate $\mathrm{E}(V_t)$,

$$\begin{aligned}
&\mathrm{E}(V_t) \\
=\ & \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(\mathcal{B}_0 + \mathcal{B}_1'\mathbf{z}_t + \mathbf{z}_t'\mathcal{B}_2\mathbf{z}_t\right) \\
& \times \left(\frac{1}{\sqrt{2\pi}}\left|\boldsymbol{\Sigma}_{zz}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\mathbf{z}_t - \boldsymbol{\mu}_z)'\boldsymbol{\Sigma}_{zz}^{-1}(\mathbf{z}_t - \boldsymbol{\mu}_z)\right)\right) dz_{1t}...dz_{mt} \\
=\ & \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(\mathcal{C}_0 + \mathcal{C}_1\mathbf{h}_t + \mathbf{h}_t'\mathcal{C}_2\mathbf{h}_t\right)\cdot\left(\frac{1}{\sqrt{2\pi}}\left|\boldsymbol{\Sigma}_{zz}\right|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\mathbf{h}_t'\boldsymbol{\Sigma}_{zz}^{-1}\mathbf{h}_t\right)\right) dh_{1t}...dh_{mt} \\
=\ & \frac{1}{\sqrt{2\pi}}\left|\boldsymbol{\Sigma}_{zz}\right|^{-\frac{1}{2}}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(\mathcal{C}_0 + \mathcal{C}_1\mathbf{h}_t - \frac{1}{2}\mathbf{h}_t'\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)\mathbf{h}_t\right) dh_{1t}...dh_{mt} \\
=\ & \frac{\left|\boldsymbol{\Sigma}_{zz}\right|^{-\frac{1}{2}}}{\sqrt{2\pi}} \frac{\left|\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)^{-1}\right|^{-\frac{1}{2}}}{\left|\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)^{-1}\right|^{-\frac{1}{2}}}\cdot\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(\mathcal{C}_0 + \mathcal{C}_1\mathbf{h}_t - \frac{1}{2}\mathbf{h}_t'\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)\mathbf{h}_t\right) dh_{1t}...dh_{mt} \\
=\ & \frac{\left|\boldsymbol{\Sigma}_{zz}\right|^{-\frac{1}{2}}}{\left|\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)\right|^{\frac{1}{2}}} \\
& \times\frac{1}{\sqrt{2\pi}}\left|\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)^{-1}\right|^{-\frac{1}{2}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \exp\left(\mathcal{C}_0 + \mathcal{C}_1\mathbf{h}_t - \frac{1}{2}\mathbf{h}_t'\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)\mathbf{h}_t\right) dh_{1t}...dh_{mt} \\
=\ & \frac{\left|\boldsymbol{\Sigma}_{zz}\right|^{-\frac{1}{2}}}{\left|\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)\right|^{\frac{1}{2}}}\cdot\widetilde{\mathrm{E}}\left(\exp\left(\mathcal{C}_0 + \mathcal{C}_1\mathbf{h}_t\right)\right),
\end{aligned}$$

where the expectation $\widetilde{\mathrm{E}}$ is taken as if $\mathbf{h}_t$ is normally distributed with mean zero and covariance matrix $\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)^{-1}$. Thus, we immediately have

$$\mathrm{E}(V_t) = \frac{\left|\boldsymbol{\Sigma}_{zz}\right|^{-\frac{1}{2}}}{\left|\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)\right|^{\frac{1}{2}}}\cdot\exp\left(\mathcal{C}_0 + \frac{1}{2}\mathcal{C}_1\left(\boldsymbol{\Sigma}_{zz}^{-1} - 2\mathcal{C}_2\right)^{-1}\mathcal{C}_1'\right). \tag{46}$$

■

B Appendix B: Numerical Procedure

Equations (21) and (40) show that the coefficients $\{\mathbf{A}_0, \mathbf{A}_1\}, \{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2)\}$ in the optimal policy rules are functions of the underlying parameters. When there is one state variable as in Campbell and Viceira (1999), solving explicitly for these coefficients is manageable. However, with multiple state variables, such an exercise is practically impossible. Therefore, we employ a simple numerical procedure to find these coefficients instead.

To find the coefficients of the optimal portfolio rule for each value of γ, we use the fact that they are independent of ψ given ρ. Thus, for each γ, we fix a value for ρ, choose an arbitrary value for ψ, and start with some initial values for $\{\mathbf{B}_1, \text{vec}(\mathbf{B}_2)\}$—denote these by $\{\mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2)^{(1)}\}$. Through equation (21), this implies a set of values for $\{\mathbf{A}_0, \mathbf{A}_1\}$—denote these by $\{\mathbf{A}_0^{(1)}, \mathbf{A}_1^{(1)}\}$.

With $\rho, \{\mathbf{A}_0^{(1)}, \mathbf{A}_1^{(1)}\}, \{\mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2)^{(1)}\}$, we can compute the coefficients $\{\Xi_1, \Xi_2\}$ in the $c - w$ difference equation (39). By equating these coefficients with the $\{\mathbf{B}_1, \text{vec}(\mathbf{B}_2)\}$ in the conjectured policy function, we have a new set of values for $\{\mathbf{B}_1, \text{vec}(\mathbf{B}_2)\}$—call them $\{\mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2)^{(2)}\}$. Since the initial values are arbitrary, $\{\mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2)^{(2)}\}$ will be different from $\{\mathbf{B}_1^{(1)}, \text{vec}(\mathbf{B}_2)^{(1)}\}$ in general. Thus, we recompute $\{\mathbf{A}_0, \mathbf{A}_1\}$ using ρ,and $\{\mathbf{B}_1^{(2)}, \text{vec}(\mathbf{B}_2)^{(2)}\}$ to get $\{\mathbf{A}_0^{(2)}, \mathbf{A}_1^{(2)}\}$. We obtain then a new set of values for $\{\Xi_1, \Xi_2\}$. We continue until values of $\{\mathbf{B}_1, \text{vec}(\mathbf{B}_2)\}$, and hence $\{\mathbf{A}_0, \mathbf{A}_1\}$, converge.

The convergence criterion for $\{\mathbf{B}_1, \text{vec}(\mathbf{B}_2)\}$ is rather stringent. We first calculate the maximum of the squared deviations of all elements from 2 consecutive iterations. We then require for parameter convergence that the sum of 20 such consecutive maxima be less than 0.00001.

Note once again that coefficients $\{\mathbf{A}_0, \mathbf{A}_1\}$ are the same for all values of ψ given the loglinearization parameter ρ. In the special case $\psi = 1$, we also have that $\rho = \delta$, so that the solution is exact, and choosing a value for ρ is equivalent to choosing a value for δ. When ψ is not equal to one, we implement a recursive procedure similar to the one described in Campbell and Viceira (1999, 2001). Given an initial value of ρ, we compute coefficients $\{\mathbf{A}_0, \mathbf{A}_1\}, \{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2)\}$ using the procedure described above. From $\{b_0, \mathbf{B}_1, \text{vec}(\mathbf{B}_2)\}$, we can compute $\text{E}_t[c_t - w_t]$, and a new value of ρ. We iterate until convergence.

C Appendix C: Construction of Hypothetical Real Bonds

Recall that the first element of our VAR system is the ex post real bill return. Therefore, the ex ante log real bill return at time $t+1$ is the first element of $\mathrm{E}_t(\mathbf{z}_{t+1}) = \Phi_0 + \Phi_1 \mathbf{z}_t$. In other words, the log real yield at time t is given by

$$\widehat{y}_{1t} = \mathbf{H}_1 \cdot \mathrm{E}_t(\mathbf{z}_{t+1}) \equiv \mathbf{H}_1 \cdot \widehat{\mathbf{z}}_{t,t+1},$$

where $\mathbf{H}_1 \equiv (1, 0, ..., 0)$ and $\widehat{\mathbf{z}}_{t,t+1} \equiv \mathrm{E}_t(\mathbf{z}_{t+1})$.

The next step is to assume that the log expectations hypothesis holds for the real term structure; that is,

$$y_{n,t} = \frac{1}{n} \sum_{i=0}^{n-1} \mathrm{E}_t(y_{1,t+i}),$$

where $y_{n,t}$ is the log yield on a real bond with maturity n. Note that we have implicitly assume that inflation risk premium is zero. An estimate of $y_{n,t}$ can be easily constructed as follows:

$$\widehat{y}_{n,t} = \frac{1}{n} \sum_{i=0}^{n-1} \widehat{y}_{1,t+i} = \frac{1}{n} \sum_{i=0}^{n-1} \mathbf{H}_1 \cdot \widehat{\mathbf{z}}_{t,t+i+1}.$$

To compute $\widehat{\mathbf{z}}_{t,t+i+1}$, we can iterate the VAR(1) system forward to get

$$\widehat{\mathbf{z}}_{t,t+k} = \left(\sum_{j=0}^{k-1} \Phi_1^j \right) \Phi_0 + \Phi_1^k \mathbf{z}_t.$$

Using this result, log yield can be expressed as a function of current state variables:

$$\begin{aligned} \widehat{y}_{n,t} &= \frac{1}{n} \mathbf{H}_1 \sum_{i=1}^{n} \widehat{\mathbf{z}}_{t,t+i} \\ &= \frac{1}{n} \mathbf{H}_1 \sum_{i=1}^{n} \left[\left(\sum_{j=0}^{i-1} \Phi_1^j \right) \Phi_0 + \Phi_1^i \mathbf{z}_t \right] \\ &\equiv \frac{1}{n} \mathbf{H}_1 (\mathbf{Q}_c + \mathbf{Q}_n \mathbf{z}_t) \end{aligned}$$

where

$$\begin{aligned} \mathbf{Q}_n &\equiv \sum_{i=1}^{n} \Phi_1^i = \Phi_1 (\mathbf{I}_m - \Phi_1)^{-1} (\mathbf{I}_m - \Phi_1^n), \\ \mathbf{Q}_c &\equiv (\mathbf{I}_m - \Phi_1)^{-1} (\mathbf{I}_m - \mathbf{Q}_n) \Phi_0, \end{aligned}$$

and $\mathbf{I}_m$ is the identity matrix, $m = \dim(\mathbf{z}_t)$.

Finally, the 1-period return on a hypothetical real n-period bond is calculated as

$$\begin{aligned} r_{n,t+1} &= n\widehat{y}_{n,t} - (n-1)\,\widehat{y}_{n-1,t+1} \\ &\approx n\widehat{y}_{n,t} - (n-1)\,\widehat{y}_{n,t+1} \end{aligned}$$

And the excess return on the hypothetical real n-period bond is

$$\begin{aligned} & r_{n,t+1} - r_{1,t+1} \\ = & \left(n\widehat{y}_{n,t} - (n-1)\,\widehat{y}_{n,t+1}\right) - \widehat{y}_{1t} \\ = & \left\{\mathbf{H}_1\left(\mathbf{Q}_c + \mathbf{Q}_n\mathbf{z}_t\right) - \frac{n-1}{n}\mathbf{H}_1\left(\mathbf{Q}_c + \mathbf{Q}_n\mathbf{z}_{t+1}\right)\right\} - \mathbf{H}_1\left(\Phi_0 + \Phi_1\mathbf{z}_t\right). \end{aligned}$$

The next step is to construct a real perpetuity from these zero-coupon bonds. Campbell, Lo and MacKinlay (1997) show how to use a loglinearization framework to construct real perpetuity returns. Specifically,their equations (10.1.16) and (10.1.17) show that the log yield on a real perpetuity or "consol" $y_{c,\infty,t}$ is given by

$$y_{c,\infty,t} = (1-\rho_c)\sum_{i=0}^{\infty}\rho_c^i r_{c,\infty,t+1+i},$$

where $r_{c,\infty,t+i}$ is the one-period log return on a perpetuity at time $t+i$ and $\rho_c = 1 - \exp\left(\mathrm{E}[-p_{c,t}]\right)$, where $p_{c,t}$ is the log "cum-dividend" price of the perpetuity including its current coupon payout.

Taking conditional expectations at time t and imposing the expectations hypothesis,

$$\begin{aligned} y_{c,\infty,t} &= (1-\rho_c)\sum_{i=0}^{\infty}\rho_c^i\mathbf{H}_1\widehat{\mathbf{z}}_{t,t+i+1} \\ &= \mathbf{H}_1(1-\rho_c)\left(\sum_{i=0}^{\infty}\rho_c^i\sum_{j=0}^{i}\Phi_1^j\right)\Phi_0 + \mathbf{H}_1(1-\rho_c)\left(\sum_{i=0}^{\infty}\rho_c^i\Phi_1^{i+1}\right)\mathbf{z}_t. \end{aligned}$$

It is straightforward to show that

$$\begin{aligned} \sum_{i=0}^{\infty}\rho_c^i\sum_{j=0}^{i}\Phi_1^j &= \frac{1}{1-\rho_c}\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}, \\ \sum_{i=0}^{\infty}\rho_c^i\Phi_1^{i+1} &= \left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_1. \end{aligned}$$

Thus, the log yield can be expressed as function of the VAR parameters, current state variables and the loglinearization constant ρ_c:

$$y_{c,\infty,t} = \mathbf{H}_1\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_0 + \mathbf{H}_1(1-\rho_c)\left(\mathbf{I}_m - \rho_c\Phi_1\right)^{-1}\Phi_1\mathbf{z}_t.$$

D Appendix D: Tables

TABLE A

VAR Estimation Results
Nominal Bills, Stocks and Real Consol Bond

A: Quarterly Sample (1952.Q2 - 1999.Q4)

Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
VAR Estimation Results							
rtb_{t+1}	0.435	0.005	0.015	0.270	-0.001	0.428	0.338
	(6.154)	(0.775)	(0.359)	(3.478)	(-1.173)	(2.261)	(0.000)
	(6.154)	(0.775)	(0.359)	(3.478)	(-1.173)	(2.261)	(0.000)
xr_{t+1}	1.866	0.079	0.919	-2.341	0.050	0.201	0.084
	(1.559)	(0.953)	(1.580)	(-2.627)	(2.404)	(0.077)	(0.008)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
y_{t+1}	-0.001	0.004	-0.000	0.948	0.000	0.120	0.868
	(-0.024)	(1.505)	(-0.009)	(18.671)	(0.002)	(1.151)	(0.000)
$(d-p)_{t+1}$	-1.972	-0.072	-0.819	1.640	0.959	-0.955	0.932
	(-1.615)	(-0.803)	(-1.333)	(1.765)	(44.168)	(-0.355)	(0.000)
spr_{t+1}	0.009	-0.000	0.009	0.026	-0.000	0.743	0.540
	(0.304)	(-0.045)	(0.586)	(0.815)	(-0.232)	(10.996)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	$xrcb$	y	$(d-p)$	spr
rtb	0.551	0.228	-0.465	-0.390	-0.228	0.183
xr	-	7.764	-0.351	-0.164	-0.981	0.023
$xrcb$	-	-	1.236	-0.408	0.333	0.111
y	-	-	-	0.256	0.196	-0.776
$(d-p)$	-	-	-	-	7.946	-0.056
spr	-	-	-	-	-	0.172

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset.

TABLE A (Ctd.)

VAR Estimation Results
Nominal Bills, Stocks and Real Consol Bond

B: Annual Sample (1890 - 1998)

Dependent Variable	rtb_t (t)	xr_t (t)	$xrcb_t$ (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
VAR Estimation Results							
rtb_{t+1}	0.309	-0.056	0.000	0.604	-0.009	-0.548	0.235
	(2.299)	(-1.467)	(0.004)	(2.546)	(-0.320)	(-1.081)	(0.000)
xr_{t+1}	-0.096	0.078	-0.211	0.052	0.136	1.682	0.058
	(-0.270)	(0.650)	(-1.100)	(0.080)	(2.385)	(1.217)	(0.314)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
y_{t+1}	-0.058	-0.013	-0.017	0.897	-0.006	0.098	0.774
	(-2.142)	(-1.928)	(-1.035)	(11.157)	(-1.343)	(0.956)	(0.000)
$(d-p)_{t+1}$	-0.362	-0.136	0.192	-0.928	0.826	-1.505	0.719
	(-1.202)	(-1.310)	(1.047)	(-1.498)	(12.505)	(-1.113)	(0.000)
spr_{t+1}	0.030	0.003	0.011	0.093	0.004	0.767	0.541
	(1.565)	(0.500)	(0.962)	(1.672)	(1.245)	(9.785)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	$xrcb$	y	$(d-p)$	spr
rtb	7.619	-0.169	-0.839	0.126	0.109	-0.160
xr	-	17.429	0.166	-0.149	-0.721	0.196
$xrcb$	-	-	12.416	-0.579	-0.026	0.617
y	-	-	-	1.235	0.205	-0.892
$(d-p)$	-	-	-	-	16.104	-0.185
spr	-	-	-	-	-	0.977

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset.

TABLE B

VAR Estimation Results
Nominal Bills, Stocks, Real Consol Bond, and Nominal Bond

A: Quarterly Sample (1952.Q2 - 1999.Q4)

Dependent Variable	rtb_t (t)	xr_t (t)	xnb_t (t)	$xrcb_t$ (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
VAR Estimation Results								
rtb_{t+1}	0.593	0.012	-0.058	0.128	0.236	-0.001	0.409	0.352
	(5.344)	(1.984)	(-1.929)	(1.967)	(3.129)	(-0.721)	(2.178)	(0.000)
xr_{t+1}	0.962	0.038	0.329	0.276	-2.148	0.047	0.310	0.087
	(0.558)	(0.394)	(0.729)	(0.261)	(-2.286)	(2.277)	(0.119)	(0.006)
xnb_{t+1}	0.375	-0.038	-0.189	0.280	0.314	0.003	2.968	0.099
	(0.727)	(-1.473)	(-1.128)	(0.866)	(0.731)	(0.441)	(2.700)	(0.002)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
y_{t+1}	-0.042	0.002	0.015	-0.029	0.957	-0.000	0.125	0.869
	(-0.769)	(0.816)	(0.778)	(-0.935)	(19.692)	(-0.260)	(1.204)	(0.000)
$(d-p)_{t+1}$	-0.980	-0.027	-0.362	-0.112	1.428	0.962	-1.074	0.932
	(-0.539)	(-0.262)	(-0.757)	(-0.102)	(1.457)	(44.524)	(-0.401)	(0.000)
spr_{t+1}	0.019	0.000	-0.004	0.017	0.024	-0.000	0.741	0.540
	(0.506)	(0.175)	(-0.298)	(0.802)	(0.779)	(-0.138)	(10.975)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	xnb	$xrcb$	y	$(d-p)$	spr
rtb	0.545	0.239	0.389	-0.462	-0.384	-0.240	0.181
xr	-	7.751	0.228	-0.355	-0.170	-0.981	0.025
xnb	-	-	2.670	0.543	-0.764	-0.245	0.195
$xrcb$	-	-	-	1.236	-0.414	0.337	0.112
y	-	-	-	-	0.255	0.202	-0.776
$(d-p)$	-	-	-	-	-	7.932	-0.058
spr	-	-	-	-	-	-	0.172

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, xnb_t = excess nominal long bond return, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset.

TABLE B (Ctd.)

VAR Estimation Results

Nominal Bills, Stocks, Real Consol Bond, and Nominal Bond

B: Annual Sample (1890 - 1998)

Dependent Variable	rtb_t (t)	xr_t (t)	xnb_t (t)	$xrcb_t$ (t)	y_t (t)	$(d-p)_t$ (t)	spr_t (t)	R^2 (p)
VAR Estimation Results								
rtb_{t+1}	0.305	-0.052	0.122	0.002	0.700	-0.004	-0.781	0.240
	(2.258)	(-1.314)	(0.902)	(0.026)	(2.380)	(-0.147)	(-1.177)	(0.000)
xr_{t+1}	-0.093	0.074	-0.098	-0.212	-0.025	0.132	1.869	0.059
	(-0.262)	(0.616)	(-0.332)	(-1.114)	(-0.036)	(2.371)	(1.251)	(0.305)
xnb_{t+1}	0.223	0.106	-0.196	0.023	-0.117	0.012	2.566	0.393
	(1.890)	(2.954)	(-1.485)	(0.274)	(-0.333)	(0.612)	(5.118)	(0.000)
$xrcb_{t+1}$	0.000	0.000	0.000	0.000	0.000	0.000	0.000	0.000
	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(0.000)	(1.000)
y_{t+1}	-0.059	-0.012	0.036	-0.017	0.925	-0.005	0.029	0.779
	(-2.239)	(-1.746)	(1.280)	(-1.037)	(12.568)	(-1.115)	(0.249)	(0.000)
$(d-p)_{t+1}$	-0.373	-0.124	0.363	0.197	-0.642	0.840	-2.198	0.723
	(-1.260)	(-1.174)	(1.163)	(1.103)	(-0.992)	(13.412)	(-1.459)	(0.000)
spr_{t+1}	0.030	0.002	-0.013	0.011	0.083	0.004	0.791	0.543
	(1.593)	(0.415)	(-0.649)	(0.953)	(1.557)	(1.142)	(8.432)	(0.000)

Cross-Correlation of Residuals

	rtb	xr	xnb	$xrcb$	y	$(d-p)$	spr
rtb	7.592	-0.168	-0.020	-0.842	0.115	0.101	-0.155
xr	-	17.422	-0.017	0.166	-0.146	-0.723	0.195
xnb	-	-	5.099	0.257	-0.651	-0.059	0.262
$xrcb$	-	-	-	12.416	-0.585	-0.026	0.618
y	-	-	-	-	1.221	0.191	-0.894
$(d-p)$	-	-	-	-	-	15.996	-0.179
spr	-	-	-	-	-	-	0.975

Note: rtb_t = ex post real T-Bill rate, xr_t = excess stock return, $xrcb_t$ = excess real consol bond return, $(d-p)_t$ = log dividend-price ratio, y_t = nominal T-bill yield, xnb_t = excess nominal long bond return, spr_t = yield spread. The bond is a 5-year nominal bond in the quarterly dataset and a 20-year for the annual dataset.

TABLE C

Mean Asset Demands with Hypothetical Real Bonds (Annual Sample: 1890 - 1998)

A: Nominal Bills, Stocks, and Real Consol Bond

State Variables:	Constant	Full VAR
$\gamma = 1, \psi = 1, \rho = 0.92$		
Stocks	200.45	226.91
Real Consol Bond	-54.43	-64.46
Cash	-46.02	-62.45
$\gamma = 2, \psi = 1, \rho = 0.92$		
Stocks	100.58	140.21
Real Consol Bond	-1.02	-10.05
Cash	0.43	-30.16
$\gamma = 5, \psi = 1, \rho = 0.92$		
Stocks	40.67	65.69
Real Consol Bond	31.03	44.67
Cash	28.30	-10.36
$\gamma = 20, \psi = 1, \rho = 0.92$		
Stocks	10.71	17.40
Real Consol Bond	47.06	82.94
Cash	42.24	-0.34
$\gamma = 2000, \psi = 1, \rho = 0.92$		
Stocks	0.82	-0.84
Real Consol Bond	52.34	97.86
Cash	46.84	2.98

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. "Full VAR" column reports mean asset demands when the VAR system includes all state variables. The nominal bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE C (ctd.)

Mean Asset Demands with Hypothetical Real Bonds

(Annual Sample: 1890 - 1998)

B: Nominal Bills, Stocks, Real Consol Bond and Nominal Bond		
State Variables:	Constant	Full VAR
$\gamma = 1, \psi = 1, \rho = 0.92$		
Stocks	198.39	232.42
Real Consol Bond	-68.22	-97.56
Nominal Bond	143.96	301.23
Cash	-174.13	-336.09
$\gamma = 2, \psi = 1, \rho = 0.92$		
Stocks	99.61	137.51
Real Consol Bond	-7.54	-30.05
Nominal Bond	68.06	163.36
Cash	-60.13	-170.81
$\gamma = 5, \psi = 1, \rho = 0.92$		
Stocks	40.34	60.01
Real Consol Bond	28.87	39.15
Nominal Bond	22.51	64.02
Cash	8.27	-63.18
$\gamma = 20, \psi = 1, \rho = 0.92$		
Stocks	10.71	15.95
Real Consol Bond	47.08	81.05
Nominal Bond	-0.26	19.78
Cash	42.47	-16.79
$\gamma = 2000, \psi = 1, \rho = 0.92$		
Stocks	0.93	0.61
Real Consol Bond	53.09	95.75
Nominal Bond	-7.77	7.58
Cash	53.75	-3.94

Note: "Constant" column reports mean asset demands when the VAR system only has a constant in each regression, corresponding to the case in which risk premia are constant and realized returns on all assets, including the short-term real interest rate, are i.i.d. "Full VAR" column reports mean asset demands when the VAR system includes all state variables. The nominal bond is a 5-year nominal bond in the quarterly dataset and a 20-year in the annual dataset.

TABLE D

Mean Value Function ($\psi = 1$ Case)
(Annual Sample: 1890 - 1998)

γ	$E[V_t]$
Nominal Bills and Stocks	
1	0.164
2	0.088
5	0.050
20	0.015
2000	0.000
Nominal Bills, Stocks, and Nominal Bond	
1	14.911
2	0.446
5	0.086
20	0.018
2000	0.000
Nominal Bills, Stocks, and Real Consol Bond	
1	0.175
2	0.090
5	0.053
20	0.038
2000	0.014
Nominal Bills, Stocks, Nominal Bond, and Real Consol Bond	
1	37.37
2	0.526
5	0.095
20	0.023
2000	0.014