# Portfolio Theory
# Diversification

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## *Objectives lecture on CAPM*

In previous lectures have discussed how to measure risk in form of variance or standard deviation for a single asset and found clear empirical evidence that there exists a trade-off between risk and return. In this lecture we discuss risk of a portfolio of several assets and show that variance does not measure risk the way it is compensated by the market.

- Effects of diversification.
- The Markovitz portfolio selection approach.

*Remark*: The Markovitz approach and the CAPM are both Nobel prize winning ideas.

## *Motivation*

We have seen that if we distinguish states ("scenarios") $s_1, \ldots, s_K$ then the expected value of the return of an asset is simply

$$\mathbf{E}[R] = \sum_{k=1}^{K} R(s_k)\mathbf{P}(s_k)$$

Similarly we have that the dispersion from the mean can be measured by

$$\mathbf{VAR}[R] = \sum_{k=1}^{K} (R(s_k) - \mathbf{E}[R])^2 \mathbf{P}(s_k)$$

But most investors do not just hold a single stock but a portfolio of stocks and bonds. $\Rightarrow$ How are portfolio returns determined ?

*Wealth Dynamics*

If we invest the amount of $\alpha_n^j$ in the asset $j$, which pays return $R_{n+1}^j$, the associated wealth is growing as follows:

$$\underbrace{W_{n+1} - W_n}_{\text{Wealth Increment}} = \sum_{j=1}^{d} \underbrace{\alpha_n^j}_{\text{Amount invested in risky asset j}} \underbrace{R_{n+1}^j}_{\text{Return of Asset j}}$$

where $\alpha_j$ is the the amount of money invested in asset $j$ in our portfolio of total $d$ assets.

## *Portfolio Return*

If we want to measure the return of our portfolio we divide both sides by $W_n$ and obtain

$$\underbrace{\frac{W_{n+1} - W_n}{W_n}}_{\text{Portfolio Return}} = \sum_{j=1}^{d} \underbrace{\delta_n^j}_{\text{Fraction of Wealth in asset j}} R_{n+1}^j$$

where $\delta_n^j$ now measures the fraction of wealth at the beginning of the investment period invested in the risky asset $j$.

$$\delta_n^j = \frac{\alpha_n^j}{W_n}$$

$\Rightarrow$ How do we measure *expected return* and *risk* of a portfolio ?

## *Expected Returns of a Portfolio*

We have seen that the return of a portfolio $R_{n+1}^{Portfolio} = \frac{W_{n+1} - W_n}{W_n}$ is given by

$$R_n^{Portfolio} = \sum_{j=1}^{d} \delta_n^j R_{n+1}^j$$

What is the expected return of this portfolio ?

$$\mathbf{E}[R_{n+1}^{Portfolio}] = \sum_{j=1}^{d} \delta_n^j \mathbf{E}[R_{n+1}^j]$$

*Remark*: The contribution of asset $j$ to the return of the portfolio is proportional to its weight in the portfolio.

## *Risk of a Portfolio*

We can consider the return of the portfolio as a new asset and measure the riskiness of the portfolio in the same way as we have measured the riskiness of a single asset

$$\mathbf{VAR}[R_{n+1}^{Portfolio}] = \mathbf{E}[(R_{n+1}^{Portfolio} - \mathbf{E}[R_{n+1}^{Portfolio}])^2]$$

This gives

$$\mathbf{VAR}[R_{n+1}^{Portfolio}] = \mathbf{E}[(R_{n+1}^{Portfolio} - \mathbf{E}[R_{n+1}^{Portfolio}])^2]$$

using the definition of $R_{n+1}^{Portfolio}$ this can be written

$$\mathbf{VAR}[R_{n+1}^{Portfolio}] = \sum_{i=1}^{d} \sum_{j=1}^{d} \delta_n^i \delta_n^j \mathbf{E}[(R_{n+1}^i - \mathbf{E}[R_{n+1}^i])(R_{n+1}^j - \mathbf{E}[R_{n+1}^j])]$$

## *Remark*

The double sum contains overall $d^2$ terms:

- $d$ *variances* (when $i = j$) $\mathbf{E}[(R_n^i - \mathbf{E}[R^i])^2] = \mathbf{VAR}[R_n^i]$
- $d(d-1)$ (when $i \neq j$ $\mathbf{E}[(R_n^i - \mathbf{E}[R^i])(R_n^j - \mathbf{E}[R^j])] = \mathbf{COV}[R_n^i, R_n^j]$ of which since $\mathbf{COV}[X,Y] = \mathbf{COV}[Y,X]$ only $\frac{d(d-1)}{2}$ are unknown.

⇒ To calculate the risk of a portfolio with $d = 100$ assets you have to calculate 100 variances and $\frac{1}{2} \times 100 \times 99 = 4590$ covariance terms ! We will see that often you don't need more than 10 assets in your portfolio.

⇒ The number of covariance terms on its own suggests that they may be important in the assessment of how the market compensates for risk. They capture the effects of *diversification*.

## *Diversification: The Covariance*

We have seen that the level of a risky prospect can be measured by the mean and the dispersion from the mean by the variance. Given several risky prospects you also want to measure the *co-movement* of the prospects

⇒ Does on average stock A has a low/high return when stock B has a low/high return ?

This can be measured by the covariance

*Definition*

$$\mathbf{COV}[X,Y] = \sum_{k=1}^{K} (X(s_k) - \mathbf{E}[X])(Y(s_k) - \mathbf{E}[Y])$$

## *Interpretation*

- If $COV[X,Y] < 0$ we say that $X, Y$ are *negatively correlated*: ⇒ On average "$X$ is *high* when $Y$ is *low* and $X$ is *low* when $Y$ is *high*".
- If $COV[X,Y] > 0$ we say that $X, Y$ are *positively correlated*: ⇒ On average "$X$ is *high* when $Y$ is *high* and $X$ is *low* when $Y$ is *high*". item If $COV[X,Y] = 0$ we say that $X, Y$ are *uncorrelated*: ⇒ On average "$X$ is *high* when $Y$ is *high* and $X$ is *low* when $Y$ is *high*".

## *Remark*

- Positive/negative correlation does not mean that there is necessarily a causal link between two random variables. (The number of flowers and biking accidents are positively correlated but there is not a real causal link ("except if you fall from your bike because you stared at the flowers").
- There can be a co-dependence between returns even if there are not correlated. Covariance measures co-movements of means but not for example of variances.

## *Example*

Consider a portfolio of a risk-free asset and two stocks (A,B) and three equally likely states.

| *state* <br> $s$ | *T-Bill* <br> *Return i %* | *Stock A* <br> *Return %* | *Stock B* <br> *Return in %* |
|---|---|---|---|
| $s_1$ "boom" | 5 % | 16 % | 3 % |
| $s_2$ "normal" | 5 % | 10 % | 9 % |
| $s_3$ "recession" | 5 % | 1 % | 16 % |

What is the expected return and variance of the portfolios formed by these three assets ?

## *Expected return*

Clearly the expected return is $\mathbf{E}[R^F] = 5\%$ then the expected return of stock A is

$$\mathbf{E}[R^A] = \frac{1}{3}[16 + 10 + 1]\% = 9\%$$

$$\mathbf{E}[R^B] = \frac{1}{3}[3 + 9 + 15]\% = 9\%$$

The Returns of the portfolio are

$$\mathbf{E}[R^{portfolio}] = \delta^{R^F} 5\% + (\delta^{R^A} + \delta^{R^A})9\%$$

## *Remark*

You may argue that since the expected return of stock A is equal to the expected return of the stock B you will hold equal shares of both. $\Rightarrow$ Does not take into account second moments. An equally weighted portfolio in all three assets has a return

$$\mathbf{E}[R^{ew}] = \frac{1}{3}[5\% + 2 \times 9\%] = \frac{23}{3} = 7.666\%$$

## *Variance of Portfolio*

Clearly since for the risk free asset the value in each state corresponds to the expected value their is no dispersion and therefore $\mathbf{VAR}[R^F] = 0$. For stock A we obtain

$$\mathbf{VAR}[R^A] = \frac{1}{3}[(16 - 9)^2 + (10 - 9)^2 + (1 - 9)^2] = \frac{114}{3}$$

whereas

$$\mathbf{VAR}[R^B] = \frac{1}{3}[(16 - 9)^2 + (9 - 9)^2 + (3 - 9)^2] = \frac{85}{3}$$

*Remark*: You may argue that since the variance of stock B is lower than the variance of stock A that you will never hold stock A since you obtain the same expected return with less risk. $\Rightarrow$ Does not take into account co-movement of stock A and B.

## *Diversification*

The covariance with the risky asset is equal to zero since the risk free asset does not vary for different states of nature. It remains to be calculated the covariance between stock A and stock B.

$$\mathbf{COV}[R^A, R^B] = \frac{1}{3}[(7)(-6) + (1)(0) + (-8)(7)]$$

which gives $\mathbf{COV}[R^A, R^B] = -\frac{98}{3}$

## *Portfolio Variance in Example*

The variance of the portfolio is given as follows:

$$\begin{aligned}
\mathbf{VAR}[R^{Portfolio}] &= \frac{1}{3}[((\delta^{R^F})^2 VAR[R^F] + (\delta^{R^A})^2 VAR[R^A] \\
&+ (\delta^{R^B})^2 VAR[R^B] + 2[\delta^{R^F} R^A \mathbf{COV}[R^F, R^A] \\
&+ \delta^{R^F} R^B \mathbf{COV}[R^F, R^B] + \delta^{R^A} R^B \mathbf{COV}[R^A, R^B]]
\end{aligned}$$

e

this gives

$$\mathbf{VAR}[R^{Portfolio}] = \frac{1}{9}[(\delta^{R^A})^2 114 + (\delta^{R^B})^2 85 - 2\delta^{R^A}\delta^{R^B} 98]$$

This gives for example for a portfolio with equal weights in all three assets

$$\mathbf{VAR}[R^{ew}] = \frac{1}{27}[114 + 85 - 2 \times 98] = \frac{3}{27} = \frac{1}{9}$$

## *Conclusion from example*

⇒ Risk free asset does not contribute to the variance of the portfolio only to the expected return.

⇒ The equally weighted portfolio has a considerably lower variance than the two risky assets on their own. (Its expected return is also lower but by not that much) But it is not clear whether or not the equally weighted portfolio gives the best trade-off between risk and return. *Markovitz portfolio selection approach* determines optimal weights by minimizing the variance for a given level of expected return.

⇒ The reduction in the risk of the portfolio is due to the negative correlation of stock A and stock B. Since return of stock A is high when stock B is low reduces the dispersion around the expected return of the portfolio. Benefits from *Diversification*. Should the market compensate you for risk that you can trade away by holding a portfolio rather than individual stock ? *CAPM* states that only *non-diversifiable* risk is compensated by the financial markets. Diversifiable risk is not priced.

⇒ Value of a stock has to be considered relative to its contribution to all possible portfolios that can be formed in combination with the other assets. A *"stock-picker"* in our example would have avoided to hold shares of stock A since it is riskier but does not give more return than stock B. But the equally weighted portfolio shows that stock B contributes considerably to reduce the riskiness of the return if we invest a positive amount.

## *The Effects of Diversification*

Theoretically the following analysis of the equally weighted portfolio with stock having the same expected value and variance $\sigma^2$ and a covariance of $\rho\sigma^2$ illustrates why diversification reduces the risk. Consider the following portfolio with return $R^{ew} = \frac{1}{d}\sum_{j=1}^{d} R^i$. We find that $\mathbf{E}[R^{ew}] = \frac{1}{d}\sum_{j=1}^{d} \mathbf{E}[R^j]$. Then the variance is given by

$$\begin{aligned}
\mathbf{VAR}[R^{ew}] &= \frac{1}{d^2}\sum_{i=1}^{d}\sum_{j=1}^{d}\delta^i\delta^j\mathbf{COV}[R^i, R^j] \\
&= \frac{1}{d^2}\sum_{i=1}^{d}\mathbf{VAR}[R^i] + 2\sum_{i=1}^{d}\sum_{j=i+1}^{d}\mathbf{COV}[R^i, R^j]
\end{aligned}$$

## *Effects of Diversification continued*

Using that $\mathbf{VAR}[R^i] = \sigma^2$ and $\mathbf{COV}[R^i, R^j] = \rho\sigma^2$ for all $i, j = 1, \ldots, d$ we obtain

$$\mathbf{VAR}[R^{ew}] = [\frac{1-\rho}{d} + \rho]\sigma^2$$

It follows that if $d$ increases the variance of the equally weighted portfolio decreases at the rate $\frac{1}{d}$.

## *Implications*

Our numerical and theoretical example show both that the combination of several risky assets can reduce the risk considerably without loosing to much expected return.

- If there is a strong common factors (inflation, monetary policy, technological growth etc.) driving the portfolio returns of single stocks $\Rightarrow$ stock returns are all related. There is a considerable amount of risk that is non-diversifiable. This risk is called the *market risk* or *systematic risk*.
- Empirical studies have also shown that the trade-off between risk and number of stocks is convex decreasing but to hold more than 20 stocks does not reduce risk much more than to hold 15 to 20 stocks.

## *Mean-Variance Portfolio Theory (Markovitz)*

We have asked in our example whether or not the equally weighted portfolio is optimal. If we think that risk is appropriately measured by the standard deviation the following objective function makes sense:

For a given return of the portfolio choose the weights such that the variance of the portfolio is minimal

*Remark*: The solution to this problem determines the optimal portfolio weights and therefore the minimal variance you can obtain for a level of return $c$.

## Mean-Variance Portfolio Selection

This can formally be written as

$$min_{\delta_1,\ldots,\delta_d}[\sum_{i=1}^{d}(\delta^i)^2\mathbf{VAR} + 2\sum_{i=1}^{d}\sum_{j=i+1}^{d}\delta^i\delta^j\mathbf{COV}[R^i, R^j]]$$

subject to the constraint

$$\mathbf{E}[R^{portfolio}] = \sum_{i=1}^{d}\delta^i\mathbf{E}[R^i] = c$$

and $\sum_{i=1}^{d}\delta^i = 1$.

## Efficient Frontier

Graphically the solution to the above problem is given by the *Efficient Frontier*: The efficient frontier describes for a given level of expected return the minimal variance we can obtain or equivalently for a given level of risk measured by the standard deviation the optimal level of expected return we can obtain.

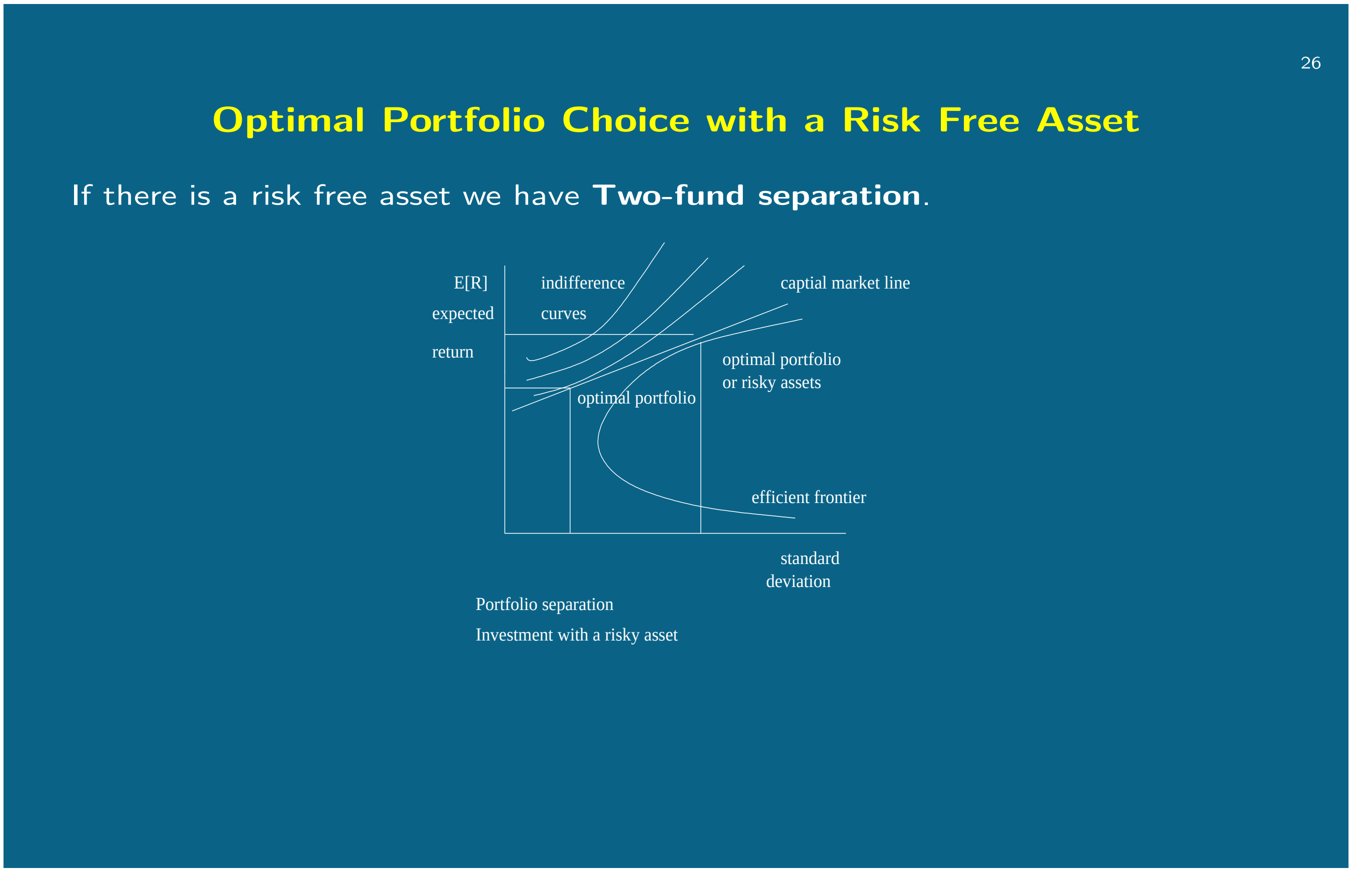
To find the optimal choice we then have to consider the *indifference* curves of the investor. How much of expected return are you willing to give up for a decrease in risk.

We have to distinguish two cases

- Optimal choice without a risk free portfolio
- Optimal choice with a risk free portfolio

## Optimal Choice without Risk Free Asset

The following graphic describes the optimal portfolio choice without a risk free asset

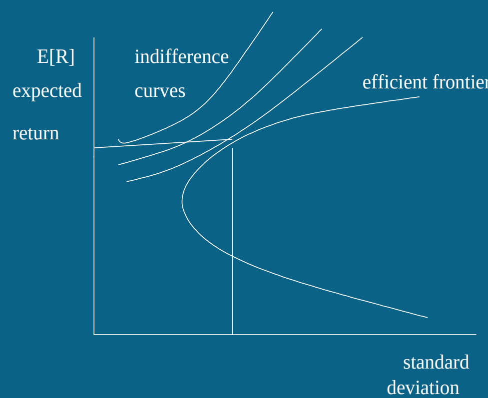



## *Optimal Portfolio Choice with a Risk Free Asset*

If there is a risk free asset we have *Two-fund separation*.

## Optimal Choice without risk free asset

The following graphic describes the optimal portfolio choice without a risk free asset




Investment without risk free asset

## *Interpretation*

The result has the following interpretations:

- If there is a risk free asset can first what is the optimal portfolio of risky assets by the tangency point with the capital market line. Its slope is given by $\frac{\mathbf{E}[R^{ra}]-R^f}{\sqrt{\mathbf{VAR}[R^{ra}]}}$. The optimal portfolio of risky assets does not depend on the preferences of the investor. It only depends on the risk free rate and the efficient frontier.
- The optimal combination of the risk free asset and the optimal portfolio of risky assets is then determined by the preferences.
- If the optimal combination of risk free asset is on the right of the optimal portfolio of risky assets the investor borrows money to invest in the stocks $\Rightarrow$ this may not be possible for all investors.
- If the optimal combination of is to the right the investor lends money at the risk free rate.

## *The Optimal Portfolio*

It can be shown that the vector of weights in the risky assets is given by

$$\begin{bmatrix} \delta_1 \\ \cdot \\ \cdot \\ \cdot \\ \delta_d \end{bmatrix} = \Sigma^{-1} \begin{bmatrix} \mathbf{E}[R^1] - R^f \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{E}[R^d] - R^f \end{bmatrix}.$$

where $\Sigma^{-1}$ is the inverse of the variance-covariance matrix of risky assets. That is $\Sigma = \mathbf{E}[[(R^i - \mathbf{E}[R^i]) \times (R^j - \mathbf{E}[R^j])]_{i,j=1,\ldots,d}]$ (Table with all variances on diagonal and covariances on off-diagonal).

## *Optimal Portfolio Policy in Example*

$$\begin{bmatrix} \delta^{R^A} \\ \delta^{R^B} \end{bmatrix} = \begin{bmatrix} 114 & -98 \\ -98 & 85 \end{bmatrix}^{-1} \begin{bmatrix} 9-5 \\ 9-5 \end{bmatrix} = \begin{bmatrix} 8.5116 \\ 9.8605 \end{bmatrix}$$

which gives that $\delta^{R^A} = 8.5116\%$ whereas $\delta^{R^B} = 9.8605$. The rest 81.628% will be invested in the risk-free asset.