# ‘Maximal’ Affine Model of Convenience Yields Implied from Interest Rates and Commodity Futures *

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**Abstract**

We develop a three-factor Gaussian model of commodity spot prices, convenience yields and interest rates, which extends previous work in two ways. First, the model nests several (Brennan (1991), Gibson and Schwartz (1990), Schwartz (1997), Ross (1997), Schwartz and Smith (2000)) proposed specifications. Second, it allows for time-varying risk-premia. It appears that previous models implicitly restrict the unconditional correlation structure of commodity prices, convenience yields and interest rates. In particular, our model allows convenience yields to be a function of the spot price and interest rates. Using data on crude oil, copper, gold and silver commodity futures, we empirically estimate the model using maximum likelihood. We find both features of the model to be economically and empirically significant. In particular, for crude oil and copper we find strong evidence for spot-price level dependence in convenience yields, which implies mean-reversion in spot prices under the risk-neutral measure, and is consistent with the "theory of storage." We also find evidence for time-varying risk-premia, which implies mean-reversion of commodity prices under the physical measure albeit with different strength and long-term mean. The model thus disentangles the different sources of mean-reversion in spot commodity prices. The results suggest that the relative contribution of both effects (level dependent convenience yield vs. time-varying risk-premia) to mean reversion depends on the nature of the commodity. We find that for store-of-value commodities like gold and silver, the negative correlation between risk-premia and spot prices explains most of the mean reversion. For commodities that serve as inputs to production technologies like crude oil, the mean-reversion in spot prices is attributable also to the convenience yield. The results are robust to the inclusion of jumps in the spot dynamics. We illustrate the economic significance of the two sources of mean reversion with two simple examples. The spot-price level dependence in convenience yields has a substantial impact for option prices, while the time-varying risk-premia affect risk management decisions.



**Key words.** Commodity prices, Futures Prices, Convenience Yields, Risk-Premia, Term Structure of Interest rates, Affine Jump-Diffusion Models

**JEL Classification Numbers.** G10, G11, G13, D81, E43.

## 1 Introduction

Commodity derivatives markets have witnessed a tremendous growth in recent years. A variety of models have been proposed for pricing commodity derivatives such as futures and options.[1] In his presidential address, Schwartz (1997) selects and empirically compares three models. His empirical results suggest that three factors, driving spot prices, interest rates and convenience yields are necessary to capture the dynamics of futures prices. Further, models accommodating mean-reversion in spot prices under the risk-neutral measure seem desirable, although in that case, Schwartz argues commodities cannot be seen as "an asset in the usual sense,"[2] because they do not satisfy the standard no-arbitrage condition for traded assets.

Below, we develop a three factor Gaussian model of commodity futures prices which nests the three specifications analyzed by Schwartz (1997) as well as Brennan (1991), Gibson and Schwartz (1990), Ross (1997) and Smith and Schwartz (1998). Instead of modeling separately the dynamics of spot, interest rates and convenience yield process, we start by directly specifying the most general identifiable three (latent) factor Gaussian model of futures prices.[3] Assuming the term structure of risk-free interest rates is driven by a single factor and imposing a restriction on the drift of spot prices which amounts to the standard no-arbitrage condition, we identify the convenience yield implied by our general model of futures prices. We show that it allows for a richer unconditional covariance structure of convenience yields, commodity prices and interest rates than previous models. In particular, the convenience yield may depend both on the spot price and the risk-free rate itself.

One simple insight of our framework is that the models by Ross (1997) and Schwartz (1997), which allow for mean-reversion under the risk-neutral measure of spot prices, can simply be interpreted as arbitrage-free models of commodity spot prices, where the convenience yield is a function of the spot price. Spot price level dependence in convenience yields leads to mean-reversion of spot prices under the risk-neutral measure. The latter feature seems to be empirically desirable to fit the cross-section of futures prices.

Several papers (Working (1949), Brennan (1958), Deaton and Laroque (1992), Routledge, Seppi and Spatt (2000)) have shown that convenience yields arise endogenously as a result of the interaction between supply, demand and storage decisions. In particular, Routledge, Seppi and Spatt (RSS 2000) show that, in a competitive rational expectations model of storage, when storage in the economy is driven to its lower bound, e.g. in periods of relative scarcity of the commodity

[1] Early description of the market can be found in the collection of papers "Managing Energy Price Risk" (1995). Seppi (2003) offers a more up to date survey.

[2] See Schwartz (1997) p. 926. See also Ross (1997).

[3] In Dai and Singleton's (2000) terminology we use the 'maximal' $A_0(3)$ model.

available for trading, convenience yields should be high. This provides some economic rational for allowing the convenience yield to depend on spot prices as in our model. Indeed, assuming that periods of scarcity, e.g., low inventory, correspond to high spot prices, this theory predicts a positive relation between the convenience yield and spot prices.

Further, RSS 2000 note that the correlation structure between spot prices and convenience yields should be time-varying, in contrast to the prediction of standard commodity derivatives pricing models such as Brennan (1991), Gibson and Schwartz (1990), Amin, Ng and Pirrong (1995), Schwartz (1997) and Hilliard and Reis (1998). While the model we develop has a constant instantaneous correlation structure (since it is Gaussian) it allows for a more general unconditional correlation structure of spot price and convenience yields than previous papers.

Most theoretical models of convenience yields (such as RSS 2000), assume that interest rates are zero, and thus do not deliver predictions about how interest rates should affect the convenience yield. However, to the extent that inventory and interest rates are correlated, it seems consistent with the theory to find a relation between interest rates and convenience yields. Further, interest rates in general proxy (at least partially) for economic activity, which in turn may affect convenience yields.

To empirically implement the model and estimate the significance of the previously imposed over-identifying restrictions, we need a specification of risk-premia. Following Duffee (2002), we allow risk-premia to be affine in the state variables. This specification nests the constant risk-premium assumption made in previous empirical analysis of commodity futures (e.g., Schwartz (1997)). Existing theoretical models of commodity prices based on the theory of storage (such as RSS 2000) assume risk-neutrality, and thus make no prediction about risk-premia. However, allowing for time-varying risk-premia is important since, as argued by Fama and French (1987, 1988), negative correlation between risk-premia and spot prices may generate mean-reversion in spot prices. In the context of our affine model, allowing for risk-premia to be level dependent implies that state variables have different strength of mean-reversion under the historical and risk-neutral measures. Mean-reversion under the risk-neutral measure is due to convenience yields, whereas mean-reversion under the historical measure results from both the convenience yield and the time-variation in risk-premia. The former is important to capture the cross-section of futures prices, whereas the latter affects the time series properties of spot and futures prices.

We use weekly data on crude oil, copper, gold and silver futures contracts and U.S. treasury bills, from 8/1/1995 to 3/25/02. We estimate the model using maximum-likelihood, since it takes full advantage of the Gaussian-affine structure of our model.[4] Using standard pricing results on

[4]The same approach has been widely used in the literature: Chen and Scott (1993), Pearson and Sun (1994), Duffie and Singleton (1997).

Gaussian-affine models (Langetieg (1980), Duffie, Pan and Singleton (2000)) we obtain closed-form solutions for futures, zero-coupon bond prices and the transition density of the state vector. Results indicate that the maximal convenience yield model improves over all (nested) specifications previously investigated. Three factors are needed to capture the dynamics of futures prices. Allowing convenience yields and risk-premia to be a function of the level of spot commodity prices as well as interest rates is an important feature of the data. For crude oil and copper we find convenience yields are increasing in spot commodity prices, in line with predictions of the theory of storage. For gold and silver this dependence is negative, but the level of convenience yield is much lower and not very variable. For all commodities the sign of the dependence of convenience yields on interest rates is positive and significant. It is higher for gold and silver. For all commodities we find statistically and economically significant negative correlation between risk-premia and spot prices. The point estimates further suggest that the contribution of time variation in risk-premia to the total mean-reversion strength under the historical measure is increasing in the degree to which an asset serves as a store of value, e.g. a financial asset (lowest for oil, highest for gold). Related, the level of convenience yields is increasing in the degree to which an asset serves for production purposes (high for oil and copper, and low for gold and silver).

These results are robust to the inclusion of jumps in the spot dynamics. Specifically, we decompose the jump component of spot commodity prices into three parts. We find evidence for a high-intensity jump with stochastic jump size with approximately zero mean, and two lower intensity jumps with constant jump sizes (one positive and one negative). The estimates of risk-neutral drift parameters of the state vector are almost unchanged. Including jumps mainly affects the estimates of the volatility coefficients and the risk-premia parameters. Indeed, we show that jumps in the spot price have little impact on the predicted cross-section of futures prices.[5] However, accounting for jumps helps better capture the historical measure dynamics of futures prices.

Bessembinder et al. (1995) also find evidence for mean-reversion in commodity prices by comparing the sensitivity of long-maturity futures prices to changes in spot prices (or, effectively, short maturity futures prices). Since their test uses only information from the cross-section of futures prices, it cannot detect mean-reversion resulting from "movements in the risk-premium component" (see their discussion p.362). Consequently, their test cannot determine whether historical time series of commodity prices actually exhibit mean-reversion.[6] In contrast to their paper,

[5] As we show in an appendix, jumps impact futures prices only when the convenience yield depends on the spot price. The intuition is that futures prices are martingales under the risk-neutral measure. Combined with the martingale restriction on the drift of the spot price process, this implies that jumps in the spot price can only 'matter' if there is a common jump in the convenience yield (or the interest rate). Hilliard and Reis (1998) for example, find that, in their model, jumps have no impact on futures prices. Their convenience yield model is not maximal however.

[6] Indeed, the risk-premia could, in principle, be time-varying in a way to offset the 'risk-neutral' mean-reversion induced by convenience yields.

our model allows to disentangle the various sources of mean-reversion: level dependence in convenience yield vs. time-variation in risk-premia. Fama and French (1988) study the importance of time-variation in risk-premia for mean-reversion in commodity prices using simple univariate linear-regressions of changes in spot prices and forward premium on the basis (similar to Fama (1984)). Their results are inconclusive for most commodities (and in particular for the metals studied here), mainly, they argue, because the basis exhibits too little volatility for regressions to reliably identify time-variation in risk-premia. In contrast, viewing commodity futures through the 'filter' of affine models potentially allows us to obtain more reliable estimates of time-variation in risk-premia.[7]

Finally, we document the economic importance of disentangling the two sources of mean-reversion, by studying two applications: option pricing and value at risk computations. Ignoring spot price dependence of convenience yields results in a mis-specification of the risk-neutral dynamics of the spot price and can result in gross mis-valuation of options. Mean-reversion under the risk-neutral measure effectively reduces the term volatility of the spot price and the expected convenience yield (which acts as a stochastic dividend) which tend to reduce option values. This is especially true for oil and copper, where an important fraction of the total mean-reversion is due to the positive relation between spot prices and convenience yields. Comparing option prices using our parameter estimates with those obtained using a restricted model (with parameters estimated imposing that convenience yield be linearly independent of the spot price) results in sizable errors of about 20% for in and at the money options. An implication is that for crude oil and copper investments the naive model will predict much higher real-option values and tend to differ investment more than the more realistic 'maximal' model.

Similarly, ignoring time-variation in risk-premia may lead to severe over-estimation of the value at risk of real, commodity-related investments. Comparing the value at risk of an investment in one unit of the asset obtained when estimating the naive restricted model vs. the 'maximal' model we find that the tails of the distribution of the naive model tend to be fatter the longer the maturity of the investment considered. For copper, gold and silver, we find that for a five year horizon investment the loss implied by a 5% value at risk more than doubles when computed with models which ignore time-variation in risk-premia.

These examples illustrate that disentangling the sources of mean-reversion in risk-premia can have a substantial impact on valuation, investment decision[8] and risk-management.

[7] Piazzesi (2002) offers further discussion of the advantages of the affine framework, which explicitly imposes cross-sectional no-arbitrage restriction, over an unrestricted VAR for example.

[8] The impact of various assumptions about the dynamics of the convenience yield on real option valuation and investment decisions is also discussed in the last section of Schwartz (1997).

The rest of the article is structured as follows. Section 2 presents the model, Section 3 discusses the specification of risk premia, Section 4 describes the empirical analysis and discusses the results, Section 5 shows the economic implications of the model and Section 6 concludes.

## 2 The 'Maximal' Convenience Yield Model

In this section we develop a general three-factor Gaussian model of (log) futures prices. Following Duffie and Kan (DK 1996), Duffie, Pan and Singleton (DPS 2000) and Dai and Singleton (DS 2000), we first introduce a 'canonical' representation of a three-factor Gaussian state vector driving futures prices.[9] We assume that the spot commodity price $S(t)$ is defined by:

$$X(t) := \log S(t) = \phi_0 + \phi_Y^\top Y(t) \tag{1}$$

$\phi_0$ is a constant, $\phi_Y$ is a $3 \times 1$ vector, and $Y^\top(t) = (Y_1(t), Y_2(t), Y_3(t))$ is a vector of state variables that follows a Gaussian diffusion process under the risk-neutral measure $Q$:[10]

$$dY(t) = -\kappa^Q Y(t)dt + dZ^Q(t) \tag{2}$$

where $\kappa^Q$ is a $3 \times 3$ lower triangular matrix that reflects the degree of mean reversion of the processes, and $dZ^Q$ is a $3 \times 1$ vector of independent Brownian motions. It is well-known (e.g., Duffie (1996)) that the futures price $F^T(t)$ at time $t$ for purchase of one unit of commodity $S(T)$ at time $T$ is simply the expected future spot price under the risk-neutral measure. Using standard results on pricing within the affine framework (e.g., Langetieg (1980), DK (1996), DPS (2000)), we obtain the following expression:

$$F^T(t) = \mathrm{E}_t^Q \left[ e^{X(T)} \right] = e^{A_F(T-t) + B_F(T-t)^\top Y(t)} \tag{3}$$

where $A_F(\tau)$ and $B_F(\tau)$ are the solution to the following system of ODEs:

$$\begin{aligned} \frac{dA_F(\tau)}{d\tau} &= \frac{1}{2} B_F(\tau)^\top B_F(\tau) \\ \frac{dB_F(\tau)}{d\tau} &= -\kappa^{Q\top} B_F(\tau) \end{aligned}$$

[9] The model is in the $A_0(3)$ family using the terminology of DS 2000. They show that N-factor affine model can be classified into N+1 families of models denoted $A_M(N)$ depending on the number, M, of state variables entering the conditional variance covariance structure of the state vector. In Gaussian models the conditional covariance structure is constant, $M = 0$.

[10] We assume the existence of such a risk-neutral measure. See Duffie (1996) for conditions under which the existence of such a measure is equivalent to the absence of arbitrage. If sufficient number of futures contracts are traded, then in general, with such an affine structure markets are complete and this martingale measure is unique. Collin-Dufresne and Goldstein (2000) however build finite dimensional affine models with a continuum of traded derivatives that yield incomplete markets.

with boundary conditions $A_F(0) = \phi_0$ and $B_F(0) = \phi_Y$ which can be solved in closed form (see appendix A).

Such a model is maximal in the sense that, conditional on observing only futures prices (and not the state variables $Y_1, Y_2, Y_3$ themselves), it has the maximum number of identifiable parameters. This result follows directly from the analysis in DS 2000. However, unlike in DS 2000 where bonds are derivatives of the non-traded short rate, in our framework, the underlying process $S(t)$ is a traded commodity. We emphasize that the assumption that we observe all futures prices implies that the spot price, which is but one particular futures price, is 'observable.'[11] Absence of arbitrage therefore implies:

$$E_t^Q[dS(t)] = (r(t) - \delta(t))S(t)dt \tag{4}$$

where $r(t)$ is the instantaneous risk-free rate and $\delta(t)$ is the instantaneous convenience yield. The latter has the standard interpretation of a dividend flow, net of storage costs, which accrues to the holder of the commodity in return for immediate ownership (e.g., Hull (1997)). As discussed in the introduction, convenience yields also arise endogenously in models based on the "theory of storage" (e.g., RSS 2000) as a result of the interaction between supply, demand and storage decisions. Augmenting the data set with bond prices and making an identifying assumption about the short-rate model driving the term structure of interest rates, we can recover the process for the convenience yield from equation (4), effectively viewing the latter as **defining** the convenience yield.[12] Following previous empirical papers on commodity futures, we assume the risk-free rate follows a one-factor Gaussian process:[13]

$$r(t) = \psi_0 + \psi_1 Y_1(t) \tag{5}$$

Zero-coupon bond prices may be computed explicitly by solving for $P^T(t) = \mathrm{E}_t^Q[e^{-\int_t^T r(s)ds}]$ as in Vasicek (1977) (see appendix B).

Using the definitions for $X(t)$ and $r(t)$ given in equations (1) and (5), the arbitrage restriction (4), and applying Itô's lemma, we obtain the following expression for the maximal convenience yield model implied by our model:

$$\delta(t) = r(t) - \frac{\mathrm{E}_t^Q[dX(t)] + \frac{1}{2}V_t^Q[dX(t)]}{dt} = \psi_0 - \frac{1}{2}\phi_Y^\top \phi_Y + \psi_1 Y_1(t) + \phi_Y^\top \kappa^Q Y(t) \tag{6}$$

[11] All that is really needed is that it can be 'inverted' from the cross-section of futures prices, which implies that at least three futures prices be observed in our model. This is similar to the special role played by the short rate for identification of parameters in affine term structure models (e.g., Collin-Dufresne, Goldstein and Jones (2002)).

[12] If the spot price is actually not a traded asset (as would be the case for electricity futures for example), then the process $\delta$ defined by equation 4 is still of interest, as it reflects, per definition, how much the spot price dynamics differ from that of a traded asset.

[13] This model is maximal in the $A_0(1)$ family, i.e., conditional on observing only bond prices, it has the maximum number of parameters identifiable for a one-factor Gaussian model.

Noting that equations for $X, r, \delta$ given in (1), (5) and (6) above specify a unique transformation from the latent variables $\{Y_1, Y_2, Y_3\}$ to $\{r, \delta, X\}$ we may derive the dynamics of the convenience yield implied by the model. We summarize the results in the following proposition:

**Proposition 1** *Assume the risk-free interest rate follows an autonomous one-factor Ornstein-Uhlenbeck process as in equation (5), then the 'maximal' model of futures prices and convenience yields defined in equations (1-6) can equivalently be represented by:*

$$
\begin{aligned}
dr(t) &= \kappa_r^Q \left(\theta_r^Q - r(t)\right) dt + \sigma_r dZ_r^Q(t) & (7)\\
d\delta(t) &= \left(\kappa_{\delta 0}^Q + \boxed{\kappa_{\delta r}^Q} r(t) + \kappa_\delta^Q \delta(t) + \boxed{\kappa_{\delta X}^Q} X(t)\right) dt + \sigma_\delta dZ_\delta^Q(t) & (8)\\
dX(t) &= \left(r(t) - \delta(t) - \frac{1}{2}\sigma_X^2\right) dt + \sigma_X dZ_X^Q(t) & (9)
\end{aligned}
$$

*where $Z_X^Q, Z_\delta^Q, Z_r^Q$ are standard correlated Brownian motions.*

**Proof:**

The proof follows immediately from applying Itô's Lemma to $X, r, \delta$ defined in equations equations (1), (5) and (6) above and noting that these equations specify a unique transformation from the latent variables $\{Y_1, Y_2, Y_3\}$ to $\{r, \delta, X\}$. In the appendix we provide the relation between the parameters of the latent model and the parameters of the $\{r, \delta, X\}$ representation. For future reference we define the correlation coefficients:

$$
dZ_X^Q(t) dZ_\delta^Q(t) = \rho_{X\delta} dt \qquad dZ_X^Q(t) dZ_r^Q(t) = \rho_{Xr} dt \qquad dZ_\delta^Q(t) dZ_r^Q(t) = \rho_{\delta r} dt \qquad (10)
$$

□

In the class of three-factor Gaussian models of futures (and spot) commodity prices, where the short rate is driven by one factor, this is the most general specification of the convenience yield that is also identifiable. By analogy to the terminology of DS (2000), we call it the 'maximal' convenience yield model.[14]

The proposition shows that the drift of the convenience yield process in general may depend on both the interest rate and the spot rate. This contrasts with the specifications analyzed in the existing literature which, in general, assume that the convenience yield follows an autonomous process, i.e., that the highlighted coefficients in equation (8) are zero. The following proposition

[14]Note that unlike in DS (2000), we have a three state variable model of two types of securities, bond and futures prices. Even though the two models are separately maximal, one may wonder if together they form a maximal model, as the joint observation of the two securities may allow the empiricist to recover more information about the state vector than observing the two separately. It turns out that in the Gaussian case the joint observation of two types of securities does not help identifying more parameters. The model above is thus maximal, conditional on observing bond and futures prices, and restricting the term structure to be driven by only one-factor.

provides a better understanding for the significance of imposing restrictions on the parameters $\kappa^Q_{\delta r}, \kappa^Q_{\delta X}$.

**Proposition 2** *The maximal convenience yield of proposition 1 can be decomposed as*

$$\delta(t) = \widehat{\delta}(t) + \alpha_r\, r(t) + \alpha_X\, X(t) \tag{11}$$

*where $\widehat{\delta}$ follows an autonomous Ornstein-Uhlenbeck process:*

$$d\widehat{\delta}(t) = \kappa^Q_{\widehat{\delta}} \left(\theta^Q_{\widehat{\delta}} - \widehat{\delta}(t)\right) dt + \sigma_{\widehat{\delta}} dZ^Q_{\widehat{\delta}}(t) \tag{12}$$

*There is a unique such decomposition such that* $\begin{cases} \alpha_r = 0 & \iff & \kappa^Q_{\delta r} = 0 \\ \alpha_X = 0 & \iff & \kappa^Q_{\delta X} = 0 \end{cases}$

*Using that decomposition the dynamics of the spot price process become:*

$$dX(t) = \left(\alpha_X(\theta^Q_X - X(t)) + (\alpha_r - 1)(\theta^Q_r - r(t)) + \theta^Q_{\widehat{\delta}} - \widehat{\delta}(t)\right) dt + \sigma_X dZ^Q_X(t) \tag{13}$$

*where the long-term mean of the log spot price is given by* $\theta^Q_X = \frac{1}{\alpha_X}\left((1-\alpha_r)\theta^Q_r - \theta^Q_{\widehat{\delta}} - \frac{1}{2}\sigma^2_X\right)$.

**Proof:** Applying Itô's lemma to the right hand side of equation (11) and equating drift and diffusion of the resulting process with those of equation (8) shows that there exist two possible proposed decomposition given by:

$$\begin{aligned}
\alpha^{\pm}_X &= \frac{1}{2}\left(-\kappa^Q_\delta \pm \sqrt{(\kappa^Q_\delta)^2 - 4\kappa^Q_{\delta X}}\right) && (14)\\
\alpha^{\pm}_r &= \frac{\alpha^{\pm}_X - \kappa^Q_{\delta r}}{\alpha^{\pm}_X + \kappa^Q_r + \kappa^Q_\delta} && (15)\\
\kappa^{Q\pm}_{\widehat{\delta}} &= -\kappa^Q_\delta - \alpha^{\pm}_X && (16)\\
\kappa^{Q\pm}_{\widehat{\delta}}\theta^{Q\pm}_{\widehat{\delta}} &= \kappa^Q_{\delta 0} + \alpha^{\pm}_X\sigma^2_X/2 - \alpha^{\pm}_r\kappa^Q_r\theta^Q_r && (17)\\
\sigma^{\pm}_{\widehat{\delta}} dZ^{Q\pm}_{\widehat{\delta}} &= \sigma_\delta dZ^Q_\delta - \alpha^{\pm}_X\sigma_X dZ^Q_X - \alpha^{\pm}_r\sigma_r dZ^Q_r && (18)\\
(\sigma^{\pm}_{\widehat{\delta}})^2 &= \sigma^2_\delta + (\alpha^{\pm}_X)^2\sigma^2_X + (\alpha^{\pm}_r)^2\sigma^2_r && (19)\\
&\quad -2\rho_{\delta X}\sigma_\delta\sigma_X\alpha^{\pm}_X + 2\rho_{rX}\alpha^{\pm}_r\alpha^{\pm}_X\sigma_r\sigma_X - 2\rho_{r\delta}\sigma_\delta\alpha^{\pm}_r\sigma_r && (20)
\end{aligned}$$

Defining $\varsigma = \text{sign}(\kappa^Q_\delta)$ we see that only the solution $\alpha^\varsigma_X, \alpha^\varsigma_r, \kappa^{Q\varsigma}_{\widehat{\delta}}, \theta^{Q\varsigma}_{\widehat{\delta}}$ satisfies the condition $\alpha_X = 0 \iff \kappa^Q_{\delta X} = 0$ and $\alpha_r = 0 \iff \kappa^Q_{\delta r} = 0$. Further, we note that $\alpha_r, \alpha_X$ are real if and only if ${\kappa^Q_\delta}^2 - 4\kappa^Q_{\delta X} \geq 0$ which corresponds to the condition that eigenvalues of the mean-reversion matrix be real.

Finally, we note that $Z^Q_{\widehat{\delta}}$ defined by equations (18) and (20) is a standard Brownian motion which is correlated with $Z^Q_X, Z^Q_r$. For future reference we define the correlation coefficients as:

$$dZ^Q_X(t)dZ^Q_{\widehat{\delta}}(t) = \rho_{\widehat{\delta}X}dt \qquad dZ^Q_{\widehat{\delta}}(t)dZ^Q_r(t) = \rho_{r\widehat{\delta}}dt \tag{21}$$

$\square$

The two propositions above show that once we assume the short rate follows an autonomous one-factor process, then the arbitrage restriction (4) delivers a convenience yield process which has its own specific stochastic component $\widehat{\delta}$ but is also linearly affected by the short rate and the log spot price.[15] Proposition 2 also makes apparent that the maximal model nests the three models analyzed in Schwartz (1997), as well as the models of Ross (1997), Brennan and Schwartz (1985), Gibson and Schwartz (1990) and Schwartz and Smith (2000).[16] For example, Schwartz's model 1 corresponds to a one factor ($X$) model with $\alpha_r = 0$. Schwartz's model 2 corresponds to a two-factor model $(X, \widehat{\delta})$ with $\alpha_r = \alpha_X = 0$. Schwartz's model 3 corresponds to a three factor model with $\alpha_r = \alpha_X = 0$.

One simple insight of the maximal convenience yield model is that the one-factor models of Ross (1997) and Schwartz (1997), which allow for mean-reversion under the risk-neutral measure of spot prices, can simply be interpreted as arbitrage-free models of commodity spot prices, where the convenience yield is a function of the log-spot price. A positive relation between the convenience yield and the (log) spot price, i.e. a positive $\alpha_X$, leads to a mean-reverting spot price under the risk-neutral measure. The latter feature seems to be empirically desirable to fit the cross-section of futures prices. A positive relation between convenience yield and spot price also seems consistent with the predictions of theoretical models. Several papers (Working (1949), Brennan (1958), Deaton and Laroque (1992), Routledge, Seppi and Spatt (2000)) have shown that convenience yields arise endogenously as a result of the interaction between supply, demand and storage decisions. In particular, Routledge, Seppi and Spatt (RSS 2000) show that, in a competitive rational expectations model of storage, when storage in the economy is driven to its lower bound, e.g. in periods of relative scarcity of the commodity available for trading, convenience yields should be high. This provides some economic rational for allowing the convenience yield to depend on spot prices as in our maximal model. In fact, assuming that periods of low inventory and relative scarcity of the commodity coincide with high spot prices, the theory of storage predicts a positive relation between the convenience yield and spot prices.

Further, RSS 2000 note that the correlation structure between spot prices and convenience yields should be time-varying, in contrast to the prediction of standard commodity derivatives pricing models such as Brennan (1991), Gibson and Schwartz (1997), Amin, Ng and Pirrong

[15]Note that it seems economically sensible to assume that a market wide variable such as the short rate follows an autonomous process, i.e. is not driven by the convenience yield or the spot price of a specific commodity. The model could easily be extended to allow for multi-factor term structure models. However, to maintain the assumption that interest rate risk is 'autonomous' and at the same time have convenience yield and spot price be specific sources of risk would require a four-factor model.

[16]Some of these are actually nested in the models analyzed by Schwartz (1997).

(1995) and Schwartz (1997). Since it is a Gaussian model, the maximal convenience yield has a constant instantaneous correlation structure. However, since all state variables enter the drift of convenience yield and spot price, it allows for a richer unconditional correlation structure than previous specifications.[17]

Finally, note that previous models restrict $\alpha_r$ to be zero, i.e. convenience yields to be independent of the level of interest rates. While most theoretical models assume zero interest rates (e.g., RSS 2000) and thus do not deliver empirical predictions about that coefficient, relaxing this assumption seems desirable. If we expect interest rates and inventory to be correlated, then, following the "theory of storage" argument, we may expect a significant non-zero coefficient. In fact, to the extent that holding inventory becomes more costly in periods of high interest rates, we may expect a negative correlation between interest rates and inventory and thus a positive $\alpha_r$.

Of course, our model is a reduced-form model which makes no predictions about these relations. However, it is the natural framework to investigate empirically these questions. In the next sections we discuss the specification of risk-premia and empirical implementation.

## 3 Specification of Risk Premia

Our discussion above is entirely cast in terms of the risk-neutral dynamics of state variables. These are useful to price the cross-section of futures prices. To explain the historical time-series dynamics of prices and subject our model to empirical scrutiny we need a specification of risk-premia. In contrast to previous empirical research (e.g., Schwartz (1997)) which assumes constant risk-premia, we allow risk-premia to be a linear function of the state variables following Duffee (2002) and Dai and Singleton (2002).

In terms of the canonical representation, this amounts to defining the relation between the historical and risk-neutral measure using the following specification of the Girsanov factor:

$$dZ^Q(t) = dZ(t) + (\beta_{0Y} + \beta_{1Y} Y(t))\, dt \tag{22}$$

Here $dZ$ is a $3 \times 1$ vector of independent Brownian motions under the physical martingale measure, $\beta_{0Y}$ is a $3 \times 1$ vector of constants and $\beta_{1Y}$ is a $3 \times 3$ matrix of constants. With this specification the dynamics of the state variables is Gaussian under both the historical and risk-neutral measure. The process under the physical measure is given by:

$$dY(t) = (\beta_{0Y} - (\kappa^Q - \beta_{1Y})Y(t))dt + dZ(t) \tag{23}$$

[17]In RSS 2000, the correlation is derived endogenously and is a function of the level of inventory. To the extent that spot prices proxy for inventories, the maximal convenience yield model may be able to capture that feature. It is a reduced form model, however, and inventory is not an explicit state variable of the model.

Note that both the mean reversion coefficient and the long-run mean differ under both measures. Under the physical measure the mean reversion matrix is $(\kappa^Q - \beta_{1Y})$ and the long-run mean vector is $(\kappa^Q - \beta_{1Y})^{-1}\beta_{0Y}$. Traditional models assume that $\beta_{1Y} = 0$.[18]

For ease of economic interpretation we prefer to study the $\{r, \widehat{\delta}, X\}$ representation obtained in proposition 2.[19] The risk premium specification of equation (22) can equivalently be rewritten in terms of the rotated Brownian motion basis (see appendix D) as:

$$d\begin{pmatrix} Z_r^Q \\ Z_{\widehat{\delta}}^Q \\ Z_X^Q \end{pmatrix} = d\begin{pmatrix} Z_r \\ Z_{\widehat{\delta}} \\ Z_X \end{pmatrix} + \Sigma^{-1}\left(\begin{pmatrix} \beta_{0r} \\ \beta_{0\widehat{\delta}} \\ \beta_{0X} \end{pmatrix} + \begin{pmatrix} \beta_{rr} & \beta_{r\widehat{\delta}} & \beta_{rX} \\ \beta_{\widehat{\delta}r} & \beta_{\widehat{\delta}\widehat{\delta}} & \beta_{\widehat{\delta}X} \\ \beta_{Xr} & \beta_{X\widehat{\delta}} & \beta_{XX} \end{pmatrix}\begin{pmatrix} r(t) \\ \widehat{\delta}(t) \\ X(t) \end{pmatrix}\right) dt \quad (24)$$

where

$$\Sigma = \begin{pmatrix} \sigma_r & 0 & 0 \\ 0 & \sigma_{\widehat{\delta}} & 0 \\ 0 & 0 & \sigma_X \end{pmatrix} \quad (25)$$

Given the representation adopted it seems natural to impose the following restrictions on the risk premia:

$$\begin{cases} \beta_{r\widehat{\delta}} = \beta_{rX} = 0 \\ \beta_{\widehat{\delta}r} = \beta_{\widehat{\delta}X} = 0 \end{cases} \quad (26)$$

The first set of restrictions basically guarantees that the risk-free interest rates term premia do not depend on the level of convenience yield or commodity spot price. This insures that the short rate follows an autonomous process under both measures. It is also consistent with our applying this model to different commodities which all share the same interest rate model. The second set of restrictions simply guarantees that the component of the convenience yield ($\widehat{\delta}$) which is linearly independent of interest rate and spot price level under the risk-neutral measure, remains so under the historical measure. With these restrictions, the dynamics of the state variables $\{r, \widehat{\delta}, X\}$ under the historical measure have the same form as under the risk-neutral measure, but with different risk-adjusted drift coefficients.

**Proposition 3** *If risk-premia are given by equations (24)-(26) then the state variables* $\{r, \widehat{\delta}, X\}$

[18]Duffee (2002) shows that the more general, "essentially affine" specification, improves the ability of term structure models at capturing the predictability of bond price returns under the historical measure, while retaining their ability at pricing the cross-section of bonds (i.e. fitting the shape of the term structure).

[19]As is apparent from the proof of proposition 2, studying this particular decomposition of the 'maximal' convenience yield model of proposition 1 effectively restricts the model to the parameter set for which the eigenvalues of the mean-reversion matrix are real. We checked empirically (by directly estimating the model of proposition 1) that this restriction was never binding for our data.

*introduced in proposition 2 have the following dynamics under the physical measure:*

$$dr(t) = \kappa_r^P \left(\theta_r^P - r(t)\right) dt + \sigma_r dZ_r(t) \tag{27}$$

$$d\widehat{\delta}(t) = \kappa_{\widehat{\delta}}^P \left(\theta_{\widehat{\delta}}^P - \widehat{\delta}(t)\right) dt + \sigma_{\widehat{\delta}} dZ_{\widehat{\delta}}(t) \tag{28}$$

$$dX(t) = \left(\mu(t) - \delta(t) - \frac{1}{2}\sigma_X^2\right) dt + \sigma_X dZ_X(t) \tag{29}$$

$$:= \left(\kappa_{Xr}^P(\theta_r^P - r(t)) + \kappa_{X\widehat{\delta}}^P(\theta_{\widehat{\delta}}^P - \widehat{\delta}(t)) + \kappa_X^P(\theta_X^P - X(t))\right) dt + \sigma_X dZ_X(t) \tag{30}$$

*where $\delta$ is as defined in equation (11) and $Z_X, Z_{\widehat{\delta}}, Z_r$ are standard Brownian Motions defined in equation (24).*

*The relation between the $P$ and $Q$ parameters expressed in terms of the risk-premia is:*

$$\kappa_r^P = \kappa_r^Q - \beta_{rr} \quad \theta_r^P = \frac{\kappa_r^Q\theta_r^Q + \beta_{0r}}{\kappa_r^Q - \beta_{rr}} \tag{31}$$

$$\kappa_{\widehat{\delta}}^P = \kappa_{\widehat{\delta}}^Q - \beta_{\widehat{\delta}\widehat{\delta}} \qquad \theta_{\widehat{\delta}}^P = \frac{\kappa_{\widehat{\delta}}^Q\theta_{\widehat{\delta}}^Q + \beta_{0\widehat{\delta}}}{\kappa_{\widehat{\delta}}^Q - \beta_{\widehat{\delta}\widehat{\delta}}} \tag{32}$$

$$\mu(t) = r(t) + \left(\beta_{0X} + \beta_{Xr} r(t) + \beta_{X\widehat{\delta}}\widehat{\delta}(t) + \beta_{XX} X(t)\right) \tag{33}$$

$$\kappa_{Xr}^P = \kappa_{Xr}^Q - \beta_{Xr} \qquad \kappa_{X\widehat{\delta}}^P = 1 - \beta_{X\widehat{\delta}} \qquad \kappa_X^P = \kappa_X^Q - \beta_{XX} \tag{34}$$

$$\theta_X^P = \frac{\theta_X^Q\kappa_X^Q + \beta_{0X}}{\kappa_X^Q - \beta_{XX}} + \frac{\kappa_{Xr}^Q\theta_r^Q - \kappa_{Xr}^P\theta_r^P + \theta_{\widehat{\delta}}^Q - \kappa_{X\widehat{\delta}}^P\theta_{\widehat{\delta}}^P}{\kappa_X^Q - \beta_{XX}} \tag{35}$$

Proposition 3 above shows that allowing for essentially affine risk-premia allows to disentangle the level of mean-reversion in spot commodity prices under the risk-neutral measure from the level of mean-reversion under the historical measure. The former is essential to capture the term structure of futures prices (i.e. the cross section), whereas the latter captures the time-series properties of spot commodity prices. Fama and French (1987, 1988) argue that negative correlation between risk-premia and spot prices can generate mean-reversion in spot prices. Our model captures this feature as is apparent from equations (29) and (33). A negative $\beta_{XX}$ implies negative correlation between risk-premia and spot prices and generates mean-reversion in spot prices. Thus, our model has the ability to distinguish two sources of mean-reversion. First, mean-reversion in (log) spot prices can be due to level dependence in convenience yield (a positive $\alpha_X$) which is consistent with the theory of storage. Second, mean-reversion can appear as a result of negative correlation between risk-premia and spot prices (a negative $\beta_{XX}$). Only the convenience yield component affects the cross section of futures prices, i.e. enters the risk-neutral measure dynamics. Both drive the time-series of commodity prices, i.e., enter the historical measure price dynamics. In addition, the instantaneous correlation of the spot price with interest rate and $\widehat{\delta}$ combined with the signs of respectively $\kappa_{Xr}$ and $\kappa_{X\widehat{\delta}}$ may contribute to ‘mean-reversion like’ behavior in commodity prices. Distinguishing between the various sources (if any) of mean-reversion may have important consequences for valuation and investment decision, as well as risk-management, as we document below. We first turn to the empirical estimation of the model.

# 4 Empirical implementation

We estimate our model for four types of commodity futures using maximum likelihood. We first describe the data, then the empirical methodology and discuss the results.

## 4.1 Description of the Data

Our data set consists of futures contracts on crude oil, copper, gold and silver and zero-coupon bond prices.[20] For all commodities we use weekly data from 8/1/1995 to 3/25/2002. Table 1 contains the summary statistics of the four commodities. The maturities of the contracts studied differ across commodity. We use short-term contracts with maturities 1, 3, 6, 9, 12, 15 and 18 months (labeled from F1 to F18), and depending on availability we also include longer maturity contracts. For crude oil, gold and silver we use long-term contracts with maturities up to 36, 42 and 48 months, respectively. If a specific contract is not available we select the one with the nearest maturity. In general, the long-term data is fully available for the whole period studied (348 weeks) except for silver, because contracts with maturities above 24 months started trading on April of 1996. A special characteristic of futures contracts is that the last trading day is a specific day of each month, implying that the maturity of the contracts varies over time.[21] For interest rates we use constant maturity Treasury yields to build zero-coupon bonds with maturities of 0.5, 1, 2, 3, 5, 7 and 10 years.

Figure 1 shows the price of the F1 and F18 contracts for crude oil, copper, gold and silver. We can see a decreasing tendency on copper and gold prices during the period analyzed. Also, copper, oil and gold reached their lowest price in the period during the first half of 1999. Finally, if we "casually" compare the F1 and F18 contracts, there appears to be mean-reversion (under the risk-neutral measure) in copper and crude oil prices. Indeed the difference between the F18 and F1 futures prices alternates signs.[22] Because of convergence, the F1 futures price should be close

[20]The data for the commodities is from the New York Mercantile Exchange. The crude oil data is from the NYMEX Division, while copper, gold and silver data is from the COMEX Division. The interest rate data is from The Federal Reserve Board.

[21]The last trading day is different across commodities. For copper, gold and silver the last trading day is the close of the third last business day of the maturing delivery month, while for crude oil it is the close of the third business day prior to the 25th calendar day of the month preceding the delivery month.

[22]Suppose that $d\ln S_t = (r - \delta - \kappa \ln S_t - \frac{1}{2}\sigma^2)dt + \sigma dZ_t^Q$ where all coefficients are constant. Then simple calculations show that $F^T(t) = \mathrm{E}_t^Q[S_T] = \exp\left(\theta + (\ln S_t - \theta)e^{-\kappa(T-t)} + \frac{\sigma^2}{2}B_{2\kappa}(T-t)\right)$ where $\kappa\theta = r - \delta -$

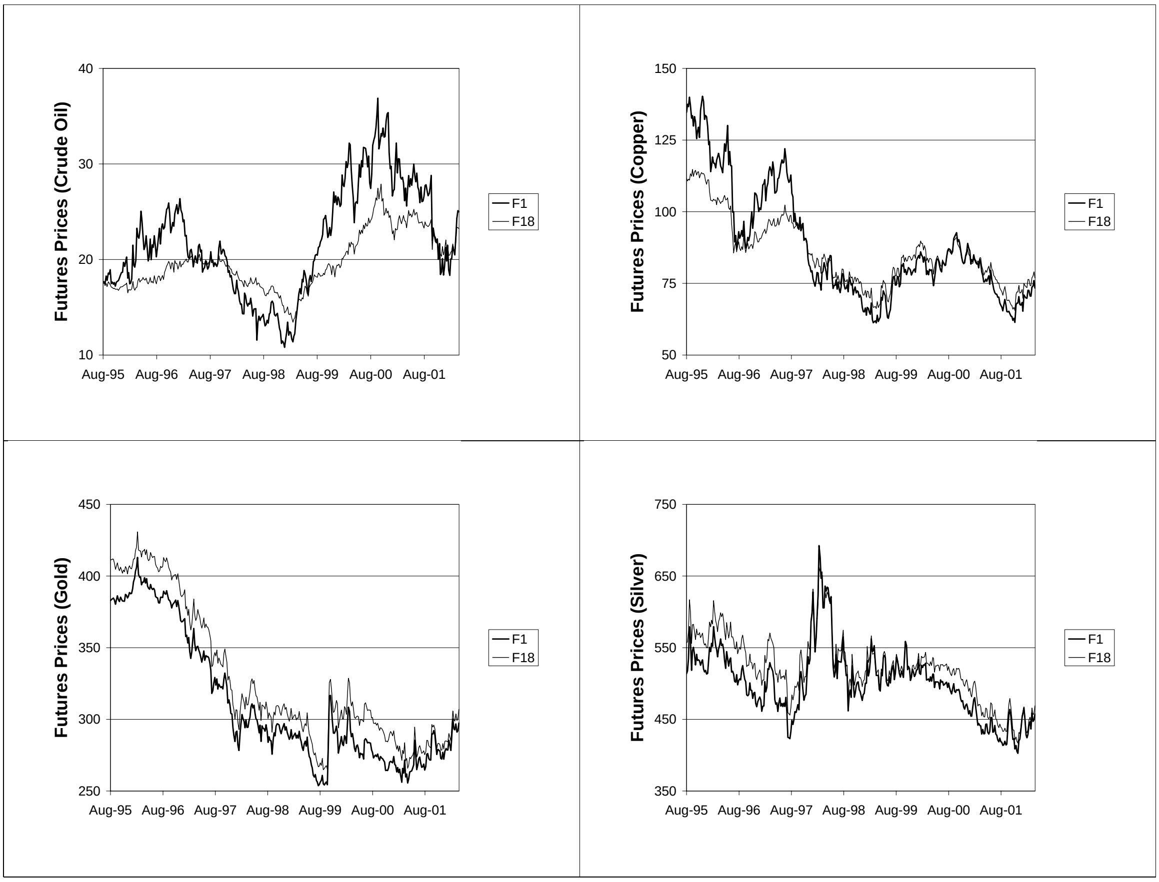


Figure 1: F1 and F18 futures contracts on crude oil, copper, gold and silver between 8/1/1995 to 3/25/02. Oil prices are in dollars per barrel, copper prices are in cents per pound, gold prices are in dollars per troy ounce and silver prices are in cents per troy ounce.

to the spot price. Thus alternating signs in $F18 - F1$ suggests periods of strong backwardation in oil and copper markets as documented in Litzenberger and Rabinowitz (1995). Gold and silver exhibit fewer episodes of strong backwardation. The 'basis' estimated by $F18 - F1$ appears to be more stable and mostly positive. Figure 2 plots the term structures for each commodity and confirms these findings. It seems that oil and copper has higher degrees of mean-reversion (under the risk-neutral measure) than gold and silver. Figure 3 presents the historical evolution of the 6-month and 60-month interest rates used for the estimation.

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$\frac{1}{2}\sigma^2$ and $B_\kappa(\tau) = (1 - e^{-\kappa\tau})/\kappa$. It is thus clear that $F18 > F1 \;\Leftrightarrow\; \ln S_t < \theta + \frac{\sigma^2(e^{-2\kappa T_1} - e^{-2\kappa T_{18}})}{4\kappa(e^{-\kappa T_1} - e^{-\kappa T_{18}})}$. Further in the absence of mean-reversion under the risk-neutral measure ($\kappa = 0$) we observe that $F18 - F1$ has the same (constant) sign as $r - \delta$.

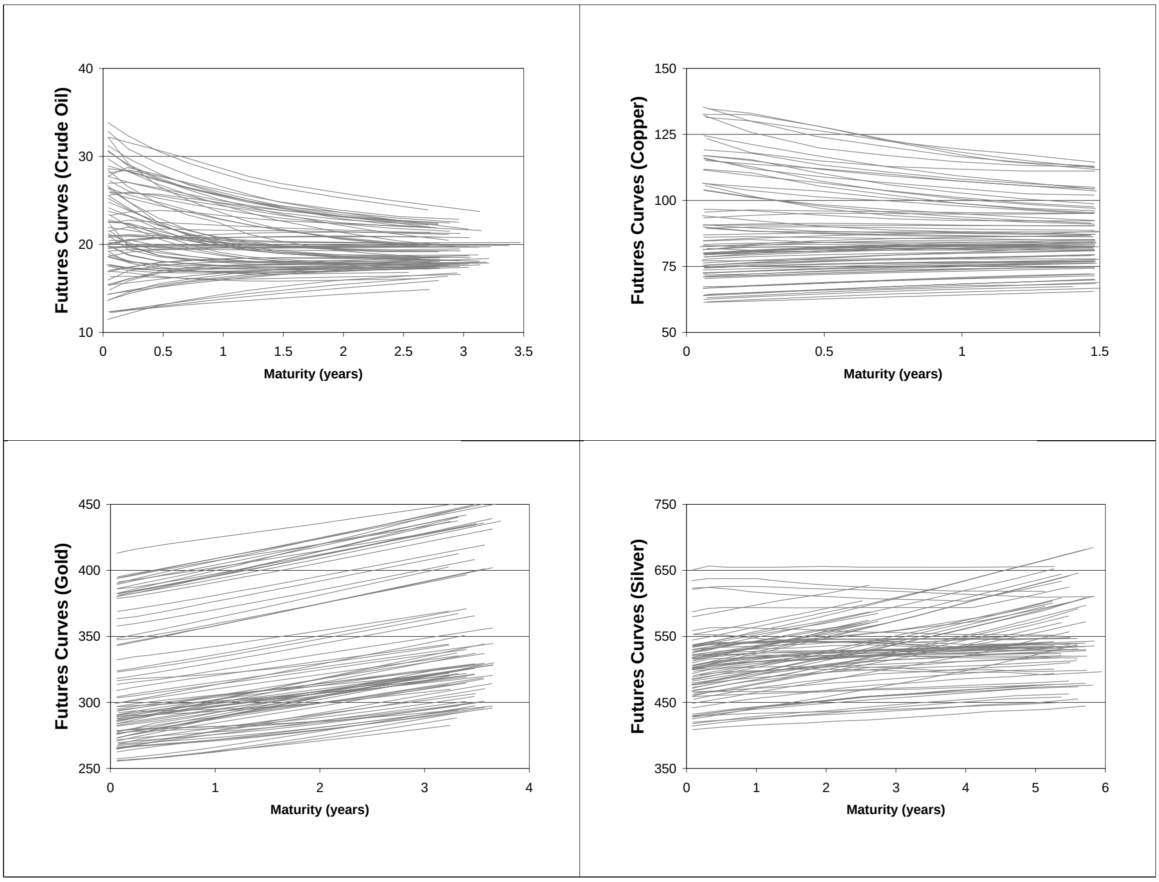


Figure 2: Monthly term structures of futures prices on crude oil, copper, gold and silver between 8/1/1995 to 3/25/02. Oil prices are in dollars per barrel, copper prices are in cents per pound, gold prices are in dollars per troy ounce and silver prices are in cents per troy ounce.

### 4.2 Empirical Methodology

We use maximum-likelihood estimation using both time-series and cross-sectional data in the spirit of Chen and Scott (1993) and Pearson and Sun (1994).[23] Since the three state variables $\{r, \widehat{\delta}, X\}$ are not directly observed in our data set, their approach consists in arbitrarily choosing three securities to pin down the state variables. Instead, we follow Collin-Dufresne, Goldstein and Jones (2002) and choose to fit the first principal component of the term structure of interest rates and the first two principal components of the futures curve. Since the principal components remain affine in the state variables, they can easily be inverted for the state variables using the closed form formulas given in appendices A and B which depend on the risk-neutral parameters. The remaining principal components of the term structure and of futures prices, which, at any point

[23]Duffie and Singleton (1997), Collin-Dufresne and Solnik (2001), Duffee (2002) use a similar method. Schwartz (1997) uses a Kalman-filter approach.


Figure 3: 6-month and 60-month interest rates from constant maturity treasury bills between 8/1/1995 to 3/25/02.

in time, are also deterministic functions of the state variables are then over identified.[24] Following Chen and Scott (1993), we assume they are priced or measured with 'measurement errors,' which we assume follow an AR(1) process. For simplicity, we assume that measurement errors in the futures prices principal components have the same auto-correlation coefficient. Similarly, we estimate only one auto-correlation coefficient for risk-free term structure errors. Given the known Gaussian transition density for the state variables and the distribution for the error terms, the likelihood can be derived.[25] We note that the transition density depends on the historical measure parameters. Apart from the likelihood value itself, the resulting properties of the "measurement errors" provide direct (mis-)specification tests for the model. Since silver long-term contracts were not traded before April 1996, we decided to back out the factors from the principal components of the data up to 24 months of maturity. When long maturity contracts are available we assume that they are observed with measurement errors.

[24]The principal components can be thought of as portfolios of contracts with different maturities. The first principal component is in general an equally weighted portfolio of contracts, while the second principal component is a portfolio with weights that are linearly decreasing with maturity. See Collin-Dufresne, Goldstein and Jones (2002) for further details on the procedure.

[25]Further details are provided in appendix E.

### 4.3 Empirical Results

Table 2 presents the maximum-likelihood estimates of the ‘maximal’ convenience model presented in propositions 2 and 3. For each commodity we present the risk-neutral parameters which affect the drift of spot price, convenience yield and interest rate processes under the risk-neutral measure, the risk-premia parameters, the volatility and correlation parameters, and the autocorrelation coefficients of the measurement errors of futures ($\rho_F$) and Treasury rates ($\rho_P$). Table 3 presents the likelihood-ratio test results for three different sets of restrictions compared to the maximal model. In table 4 we also report the point estimates of the drift parameters of the various processes under the historical measure. As shown in proposition 2, these are simple transformations of the risk-neutral and risk-premia parameters given in table 2 (for example, $\kappa_X^P = \alpha_X - \beta_{XX}$). Finally, table 5 reports point estimates for the unconditional first and second moments (long-term mean and covariances) of convenience yield and log spot prices. In the same table, we also present the long-term spot prices.[26]

Table 2 shows that all risk-neutral parameters are significant except for some of the correlation coefficients $\rho_{r\widehat{\delta}}, \rho_{rX}$. This suggests that three factors are indeed necessary to explain the dynamics of each of the four commodities[27] and, further, that innovations in the risk-free interest rates are uncorrelated with innovations in commodity spot prices and convenience yields (i.e, the assumption that the risk-free rate is an autonomous process seems appropriate).

The coefficient $\alpha_X$ is significant across all commodities. It is positive for oil and copper which is consistent with the theory of storage and indicates mean-reversion in spot prices under the risk-neutral measure. The estimated $\alpha_X$ is negative for gold and silver, which is evidence against this type of mean-reversion in these commodities. The sensitivity of convenience yields to interest rates $\alpha_r$ is significant and positive across commodities, which is consistent with the theory of storage.[28] Interestingly it is higher for gold and silver than for crude oil and copper. Performing a likelihood ratio test to jointly test for the significance of $\alpha_r$ and $\alpha_X$, we find that they are highly significant (see table 3).

The significance of the risk-premia parameters varies across commodities, but there are some consistent patterns.[29] For all commodities, the risk-premia coefficients related to the spot price

[26]Given the Gaussian nature of our model it is straightforward to calculate the exact moments for the state variables $\{r, \widehat{\delta}, X\}$. For the long-term spot price we use $E\left[\exp(X)\right] = \exp\left(E\left[X\right] + \frac{1}{2}\mathrm{VAR}[X]\right)$.

[27]Although for gold and silver the third factor appears to be very highly correlated with respectively interest rates and spot prices.

[28]As described in the introduction, most theoretical models do not allow for stochastic interest rates. However, assuming that costs of holding inventory increase with interest rates suggests a negative correlation between inventory and interest rates. We thus expect a positive relation between interest rates and convenience yield.

[29]For simplicity, in the estimation results presented we dropped the time-varying risk-premia parameters that had a t-ratio less than 1.0, which corresponds to a level of significance of 31.7%. This was the case for $\beta_{\widehat{\delta}\widehat{\delta}}$ and $\beta_{Xr}$ for crude oil and $\beta_{X\widehat{\delta}}$ for silver.

(i.e., $\beta_{0X}, \beta_{XX}$) are significant. Furthermore, $\beta_{XX}$ is always negative implying that the premium is time-varying and, in fact, negatively correlated with the spot price. All spot commodity prices exhibit mean reversion under the physical measure as evidenced by the positive coefficient of mean-reversion $\alpha_X - \beta_{XX}$. When performing a likelihood ratio test for the significance of time-variation in risk-premia (i.e., a joint test that all coefficients in the $\beta_{1Y}$ matrix are zero) we find that they are jointly highly significant (see table 3).

Overall the results show that the maximal model, which allows convenience yields to be a function of the interest rate and spot price, associated with the more flexible time varying risk-premia specification is a significant improvement over nested models proposed in the literature. The joint likelihood ratio tests of table 3 suggest that allowing more general dynamics of the convenience yield is the more important feature. This may be due to the fact that spot price dependence in convenience yield results in mean-reversion under both the risk-neutral and historical measures, whereas the time-variation in risk-premia only affects the strength of mean-reversion under the physical measure.

We first provide more detailed discussions of the individual commodities, then summarize the implications for the dynamics of convenience yields and the sources of mean-reversion in commodity spot prices as well as the evidence on model (mis-)specification.

#### 4.3.1 Crude Oil

We find that the oil price has a significant positive effect on the convenience yield ($\alpha_X = 0.162$). This implies strong mean reversion of log spot prices under the risk-neutral measure. Also, there is evidence of negative correlation between risk-premia and spot prices. The parameter $\beta_{XX}$ is $-1.107$ and significant, implying that the mean reversion under the physical measure is significantly higher from the mean reversion under the risk-neutral measure ($\kappa_X^P > \kappa_X^Q$).[30] The (historical) mean reversion in oil prices is due to both, the convenience yield and the time-variation in risk-premia. The relation between the convenience yield and interest rates is significant and positive ($\alpha_r = 0.977$) which is consistent with the 'prediction' of theory of storage. All risk-neutral coefficients are significant for oil, except for some correlations, indicating that three factors are necessary to capture the dynamics of oil futures prices. The 'idiosyncratic' component of the convenience yield $\widehat{\delta}$ has high volatility $\sigma_{\widehat{\delta}} = 0.302$, low persistence $\kappa_{\widehat{\delta}} = 0.992$ and is positively correlated with the spot price $\rho_{\widehat{\delta}X} = 0.762$. While this third factor is clearly a significant component of the convenience yield, it seems to be driven by innovations that are correlated with the spot market and are short lived. The long-term maximal convenience yield is 0.151 which is the highest among the commodities studied (see table 5). Also from this table the estimate for the

[30]Recall from proposition 2 that $\kappa_X^Q = \alpha_X$.

long-term spot price is 24.21 dollars per barrel.

### 4.3.2 Copper

Copper has a similar behaviour than crude oil. This is not surprising since both commodities share the characteristic of being input for productive processes. We find a statistically significant positive relation between the spot price of copper and its convenience yield ($\alpha_X = 0.122$). This implies mean reversion in spot prices under the risk-neutral measure, which is consistent with the theory of storage. We find a significant negative correlation between risk-premia and spot prices ($\beta_{XX} = -2.972$). This implies that the mean-reversion is stronger under the historical measure than under the risk-neutral measure. Table 4 give the point estimates of $\kappa_X^P = 3.094$ vs. $\kappa_X^Q = 0.122$ (in Table 2). The relation between convenience yields and interest rates is positive and statistically significant as before. The idiosyncratic component of the convenience yield$\widehat{\delta}$ is quite volatile $\sigma_{\widehat{\delta}} = 0.184$, not persistent $\kappa_{\widehat{\delta}} = 1.165$ and positively correlated with the spot price $\rho_{\widehat{\delta}X} = 0.582$. As for oil, convenience yield in the copper market is primarily driven by the spot price itself and economic factors that are correlated with spot price innovation and are short lived.[31] Finally, table 5 gives a long-term mean for the convenience yield of 0.035 and long-term average copper price of 79.97 cents per pound.

### 4.3.3 Gold

We find that there is a small negative relation between the convenience yield and gold spot prices ($\alpha_X = -0.022$). This suggests that gold tends to be mean-averting under the risk-neutral measure. However, the price of gold exhibits mean-reversion under the historical measure ($\kappa_X^P = 1.953$), because of the negative correlation between the time-varying risk-premia and the spot price ($\beta_{XX} = -1.976$). Interest rates seem to be more important in driving the convenience yield of gold than for oil and copper ($\alpha_r = 2.213$). The idiosyncratic factor seems to have a small effect on the convenience yield. Its long-term mean $\theta_{\widehat{\delta}}^Q$ is not significant, its volatility is very low $\sigma_{\widehat{\delta}} = 0.026$ and it is highly persistent $\kappa_{\widehat{\delta}}^Q = 0.157$. Further, the correlation between this factor and the interest rates is very high in absolute terms $\rho_{r\widehat{\delta}} = -0.858$, suggesting that the convenience yield of gold is mainly driven by the interest rate. Overall the convenience yield of gold is quite small and not very variable. Table 5 shows that the convenience yield has a long-term mean of 0.013 and an unconditional standard deviation of only 0.008. The long-term price of gold is 302.73 dollars per troy ounce.

[31] Of course the convenience yield is also affected by the interest rate through the parameter $\alpha_r$ but to a lesser extent. Even though, $\alpha_r$ is greater than $\alpha_X$, recall from equation (11) that the effect in the convenience yields are through the magnitude of $\alpha_r r(t)$ and $\alpha_X X(t)$.

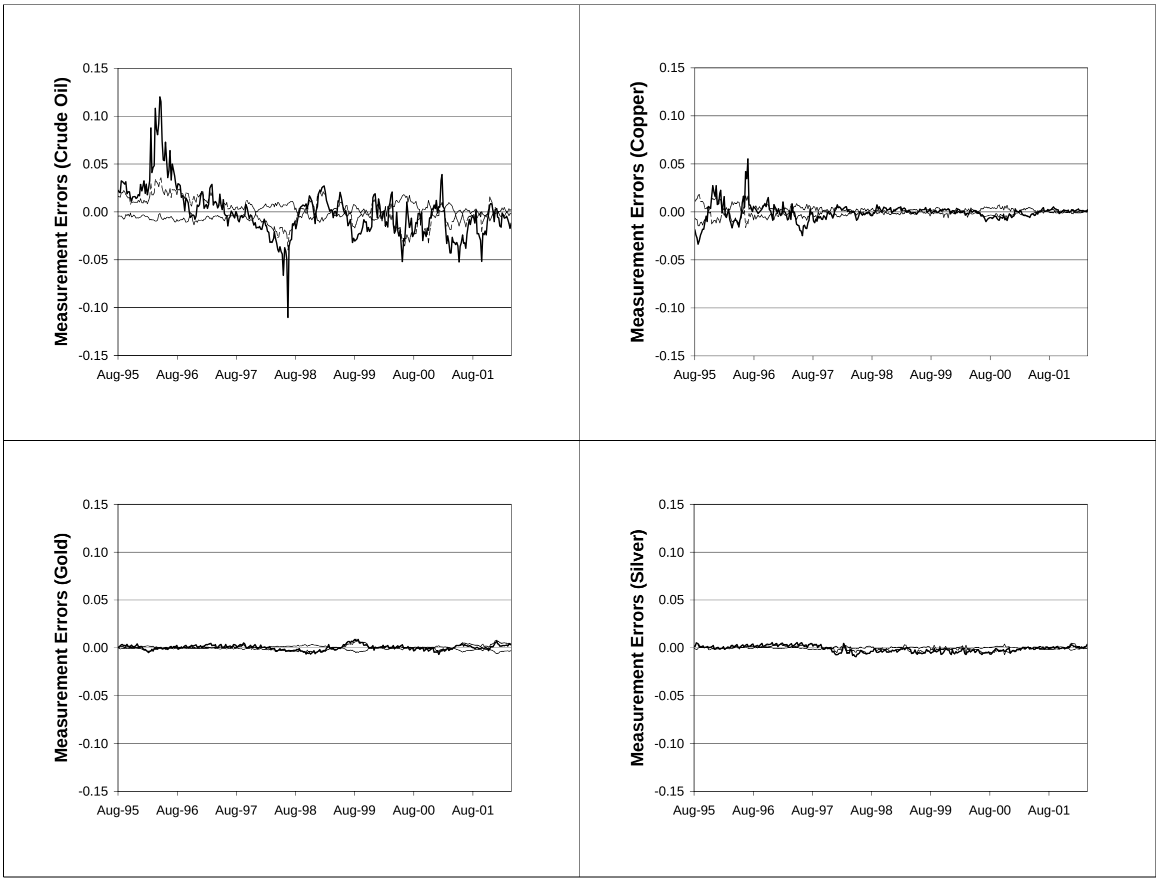


Figure 4: Difference between true and estimated (log) futures prices using the maximal model for the period between 8/1/1995 to 3/25/02. The thick line and the continuous thin line correspond to the F1 and F18 contracts, respectively. The discontinuous line corresponds to the F36, F12, F42 and F24 for crude oil, copper, gold and silver, respectively.

#### 4.3.4 Silver

The dynamics of silver share some characteristics with the behaviour of gold. We find that silver is mean-averting under the risk-neutral measure ($\alpha_X = -0.187$). The negative $\alpha_X$ reflects that in general there is no backwardation in silver futures curves. Silver prices exhibit mean-reversion under the historical measure due to the negative correlation between spot prices and risk-premia ($\beta_{XX} = -2.128$). Interest rates have an important effect in convenience yields ($\alpha_r = 2.244$). The high correlation between the idiosyncratic factor $\widehat{\delta}$ and spot prices ($\rho_{\widehat{\delta}X} = 0.920$) and its low volatility ($\sigma_{\widehat{\delta}} = 0.067$) suggest that the convenience yield of silver is mainly driven by spot prices and interest rates. Table 5 shows that the long-term convenience yield is 0.018 and the unconditional standard deviation is 0.019, which are of the same order of magnitude as for gold. Finally, table 5 gives a long-term average silver price of 477.99 cents per troy ounce.

### 4.3.5 Mis-specification

Since we estimate the parameters for each commodity separately, we obtain four different estimates for interest rate parameters. In general, the estimates seem reasonable (e.g., in line with estimation of single factor models found in the literature) $\kappa_r^Q = 0.03$, $\theta_r^Q = 0.099$ and $\sigma_r = 0.009$, and do not vary significantly across estimation, which is weak evidence that the model correctly captures the relation between interest rates and convenience yield and commodity process. Not surprisingly, the auto-correlation coefficient for the term structure 'measurement errors' is quite high $\approx 0.98$ indicating that at least a second factor is needed to capture the dynamics of the term structure. This is well-known (Litterman and Scheinkman (1991)), but our primary focus is to analyze the term structure of commodity futures, and we expect an additional term structure factor to have only limited explanatory power for commodity prices. More important for our study are the 'measurement errors' for the commodity futures. The auto-correlation coefficients are lower than for interest rates (around 0.7), but very significant. Figure 4 graphs time series of the pricing errors of some futures contract for the four commodities. In table 6 we present some summary statistics about these pricing errors for the maximal model. There does not seem to be a systematic bias in the fit of the model. Not surprisingly, the analysis of the unconditional pricing errors ($u_t$) show that the model performs (in terms of MSE) slightly less well with the two commodities that exhibit higher volatility (i.e., oil and copper).

Inspection of the time series of future prices indicates that perhaps some of these errors are attributable to the inability of the pure diffusion model to accommodate jumps. For example, over the sample period considered gold prices have, on average, consistently declined, but experienced a +25% jump in prices during September-October 1999. This jump followed an announcement made by the European central banks, in response to increased pressures of gold producers, to cut sales of gold reserves. Further, demand for gold at that time may have been fueled by the Y2K uncertainty. Since our Gaussian model does not allow for jumps, some of the empirical findings may be due to the model trying to accommodate for the presence of jumps in spot prices.

To make sure the presence of jumps in the spot time series does not affect our conclusions, we re-estimate the model by allowing for jumps in the underlying spot price dynamics.

## 4.4 Estimation of the Jump component in commodity spot prices

We allow for jumps in commodity prices by considering the model introduced in proposition 2 where the dynamics of $X(t)$ are modified as follows:

$$dX(t) = \left( r(t) - \delta(t) - \frac{1}{2}\sigma_X^2 - \sum_{i=1}^{3} (\varphi_i - 1)\lambda_i \right) dt + \sigma_X dZ_X^Q(t) + \sum_{i=1}^{3} \nu_i(t) dN_i(t) \quad (36)$$

where $N_i(t) = \sum_j \mathbf{1}_{\{\tau_i^j \leq t\}}$ is the counting process associated with a sequence of stopping times $\tau_i^1, \tau_i^2, \ldots$ generated by a standard Poisson process with Q-measure intensity $\lambda_i^Q$ (see Bremaud (1981) for a rigorous exposition of point processes). The $\nu_i(\tau_i^j)$ $\forall j = 1, 2\ldots$ are i.i.d. random variables that are independent of the Poisson process and the Brownian motions. Further we assume $\nu_1$ is Gaussian with mean jump size $m_1$ and standard deviation $v_1$, while $\nu_2$ and $\nu_3$ have constant jump sizes $m_2$ and $m_3$, respectively (i.e., $v_2 = v_3 = 0$).[32] We denote the Laplace transform of the random variable $\nu_i$ by $\varphi_i = e^{m_i + \frac{v_i^2}{2}}$ $i = 1, .., 3$.

Applying Itô's lemma to the spot price defined as before by $S(t) = e^{X(t)}$ we obtain:

$$\frac{dS(t)}{S(t^-)} = (r(t) - \delta(t))dt + \sigma_X dZ_X^Q(t) + \sum_i dM_i^Q(t) \tag{37}$$

where $M_i^Q(t) := \int_0^t (e^{\nu_i(s)} - 1)dN_i(s) - (\varphi_i - 1)\lambda_i^Q t$ is a $Q$-Martingale. Thus

$$\mathrm{E}^Q\left[\frac{dS(t)}{S(t^-)}\right] = (r(t) - \delta(t))dt$$

and as before $\delta$ retains the interpretation of a 'convenience' yield that accrues to the holder of the commodity similarly to a dividend yield.

To empirically implement the model we need a specification of risk-premia for both, Brownian motion and Jump risk. We use the same 'essentially affine' risk-premium structure for Brownian motions as in equation (24). We also studied the risk-premia for the jump intensities. If jump risk is systematic (e.g., if there is a common jump in the pricing kernel) then intensities need to be risk-adjusted. If jump risk is non-systematic (for example because it is conditionally diversifiable as in Jarrow, Lando and Yu (2000) or 'extraneous' as in Collin-Dufresne and Hugonnier (1999)) then intensities are not risk-adjusted and remain the same under both measures. We empirically found that allowing intensities to change did not improve the fit of the model significantly.[33] We report the case where the jump intensities are not risk-adjusted, i.e. $N_i$ are Poisson processes with the same intensities under both measures. Further, we assume jump size risk is not priced, i.e. that the jump distribution is the same under both measures.[34]

Following Duffie and Kan (1996), futures prices may be computed in closed form for this Gaussian jump-diffusion model. We report the closed form formulas in appendix F. We estimate the model using maximum likelihood as exposed in section 4.2. The only change is that the transition density of the log-spot price is no longer Gaussian. Following Ball and Torous (1983),

[32] We found that allowing for more than one jump to have a stochastic jump size did not improve the likelihood and thus choose to report only the constant jump size case.

[33] In fact, futures prices are very insensitive to the presence of these jumps (because they have almost zero mean and futures prices are Q-expectations of the future spot price). As a result, most improvement in the likelihood is due to the improvements in the P-measure distribution for this jump component. See the discussion below and appendix F.

[34] Obviously, for the constant jump size cases this is a requirement since both measures must be equivalent.

Jorion (1988), and Das (2002) we approximate the transition density by a mixture of Gaussian (the approximation would be exact if the time interval was infinitesimal). Several problems arise when implementing this approach. Mainly, the likelihood function is unbounded if the model is estimated without any restrictions. We use Honore's (1998) approach to obtain consistent estimates of the parameters. More details about the estimation procedure and approximation to the likelihood function are presented in appendix E.2:.

Results for the parameter estimates are reported in table 7.[35] For all commodities except for copper, we find significant evidence for the presence of a frequent stochastic jump component with mean $m_1$ close to zero. This reflects small variable frequent jumps in the spot price that are unaccounted for by the pure diffusion model.[36] For copper this type of jump is not significant and thus not reported. We also find evidence for the presence of a positive and a negative less frequent jumps. For copper and silver the positive jump has a mean around 6% and occurs on average twice and once a year, respectively. For gold this jump is much more frequent and its size is $m_2 = 0.016$. For crude oil this jump is not significant. The negative jumps have different means and intensities across commodities. For crude oil the jump size is $m_3 = -0.171$ and it is very infrequent (once every six years on average). For copper and gold these jumps have means on -9.6% and -6.3%, and they occur, on average, every two and six years, respectively. For silver the jump size is small ($m_3 = -0.034$) but it is more frequent (once a year on average). In table 8 we carry out a likelihood ratio test for the hypothesis of having no jumps which is clearly rejected. The inclusion of jumps appears especially significant for copper and silver.

Overall however, our previous results seem mostly robust to the inclusion of jumps. When we compare the parameter estimates to those obtained without jumps in table 2 we see that the estimates of risk-neutral drift parameters are almost unchanged. Including jumps mainly affects the estimates of the volatility coefficients and the risk-premia parameters. These results can be explained by the fact that jumps in the spot price have little impact on the predicted cross-section of futures prices. Indeed, we show in appendix F that futures prices are unchanged if $\alpha_X = 0$. The intuition is that futures prices are martingales under the risk-neutral measure. The no-arbitrage restriction on the risk-neutral drift of the spot price (i.e., equation 4) implies that jumps in the spot price can only 'matter' if there is a common jump in the spot rate and/or the convenience yield.[37] Further, we also show in the appendix that for the estimated jump intensity and jump distribution, the impact of jumps on futures prices is negligible. However, accounting for jumps helps better capture the historical measure dynamics of futures prices.

[35] As before, we have considered jumps whose parameter estimates have a t-ratio greater than 1.

[36] Small and frequent jumps may be suggestive of Levy processes, see e.g. Bakshi and Madan (2000).

[37] Hilliard and Reis (1998) for example, find that, in their model, jumps have no impact on futures prices. Their convenience yield model is not maximal however.

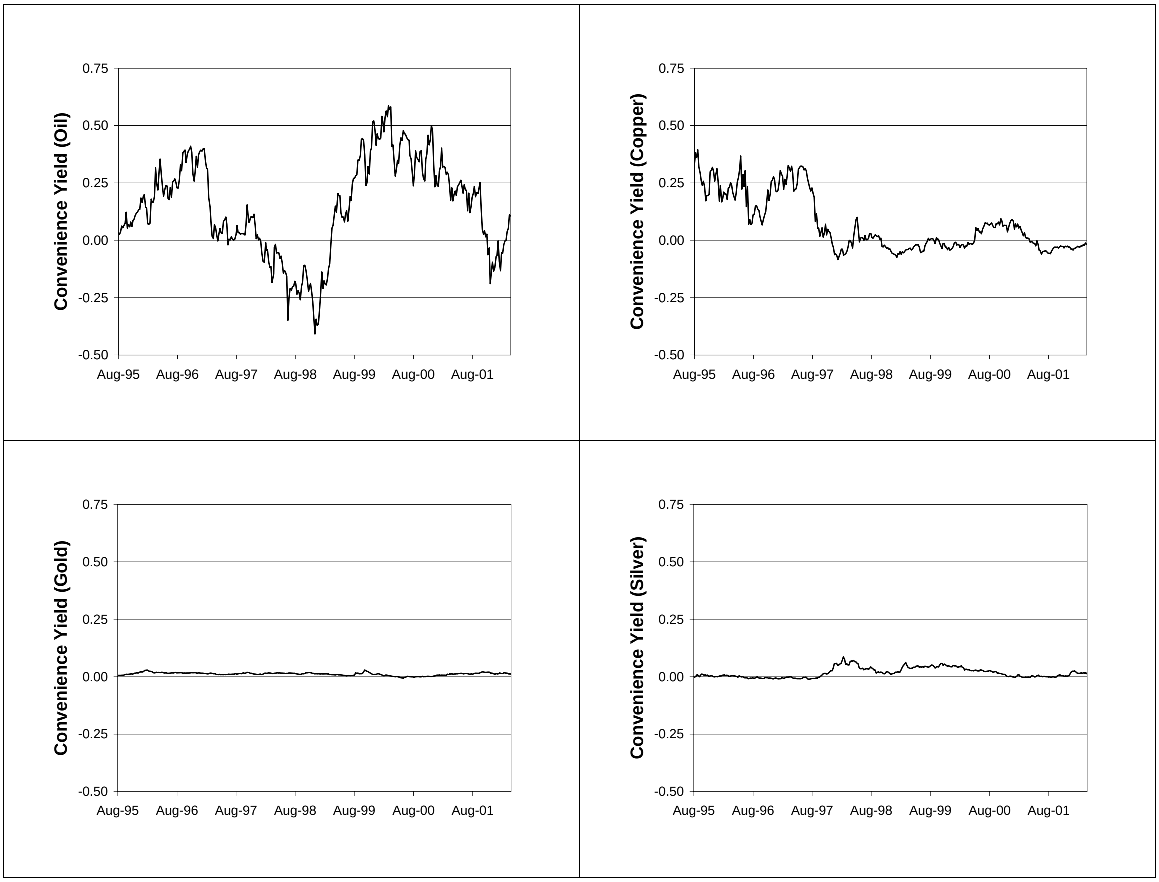


Figure 5: Implied maximal convenience yield from the maximal model for crude oil, copper, gold and silver between 8/1/1995 to 3/25/02.

#### 4.4.1 Summary of the results

**Implied convenience yields**

In figure 5 we present the implied convenience yields for the four commodities. These graphs were obtained using the estimated $\{r, \widehat{\delta}, X\}$ state variables and then calculating the implied convenience yield for each time-series observation.[38] The figure clearly distinguishes oil and copper which have highly volatile implied convenience yields from silver and gold whose convenience yields are close to zero and exhibit little variability. This is in part attributable to a higher standard deviation of the spot commodity prices for oil and copper, as well as a higher volatility of the residual third factor, $\sigma_{\widehat{\delta}}$ (see table 2). Table 5 confirms these results. Gold and silver have implied convenience yield of about 1.3% and 1.8%, whereas copper and oil have convenience yield of

[38] We have presented the four implied convenience yields with the same scale for comparison purposes.

respectively 3.5% and 15.1%.

**Sources of Mean-reversion: convenience yield and Time-varying risk-premia**

Overall our results suggest that the maximal convenience yield model improves upon all nested specifications tested in the literature (such as the models studied by Schwartz (1997)). We find that for all commodities level-dependence in convenience yield is significant, but it is higher for assets that tend to be used as inputs to production, such as oil and copper. Time variation in risk-premia, on the other hand, seems to be highest for assets which serve as a store of value and resemble more financial assets, such as gold and silver. Our results are consistent both with the option theoretic models of convenience yields (Litzenberger and Rabinowitz (1995), Deaton and Laroque (1988), RSS 2000), and with the time-varying risk-premium models proposed in Fama and French (1987, 1988). Both explanations contribute to explaining mean-reversion in commodity prices with more or less impact depending on the nature of the commodity.

Aside from their econometric interest, these results have also economic implications. In the following section we offer two simple applications that demonstrate the impact on valuation and risk-management of ignoring the various sources of mean-reversion in commodity prices.

# 5 Implication of mean-reversion for option pricing and value at risk.

Schwartz (1997) states that the stochastic behavior of commodity prices may have important implications for valuation of commodity related securities. We have documented that allowing convenience yields to be a function of spot prices and interest rates, and allowing risk-premia to be time-varying better captures dynamics of commodity futures prices. Both features have largely been ignored by previous pricing models. We focus on two simple examples to document how significant the implications are for economic applications, namely (i) valuation of options, and (ii) computation of VAR.

## 5.1 Option Pricing

As discussed previously, allowing convenience yields to be a function of the spot price, effectively induces mean-reversion under the risk-neutral measure. Since the latter 'matters' for valuation, we expect this to affect the cross section of option prices. Using the Fourier inversion approach introduced by Heston (1993) we can compute in closed-form (up to a Fourier transform inversion) European option values within our three-factor affine framework.[39] We compute the option value

[39]See Duffie, Pan and Singleton (2000) for a thorough exposition of this option valuation approach.

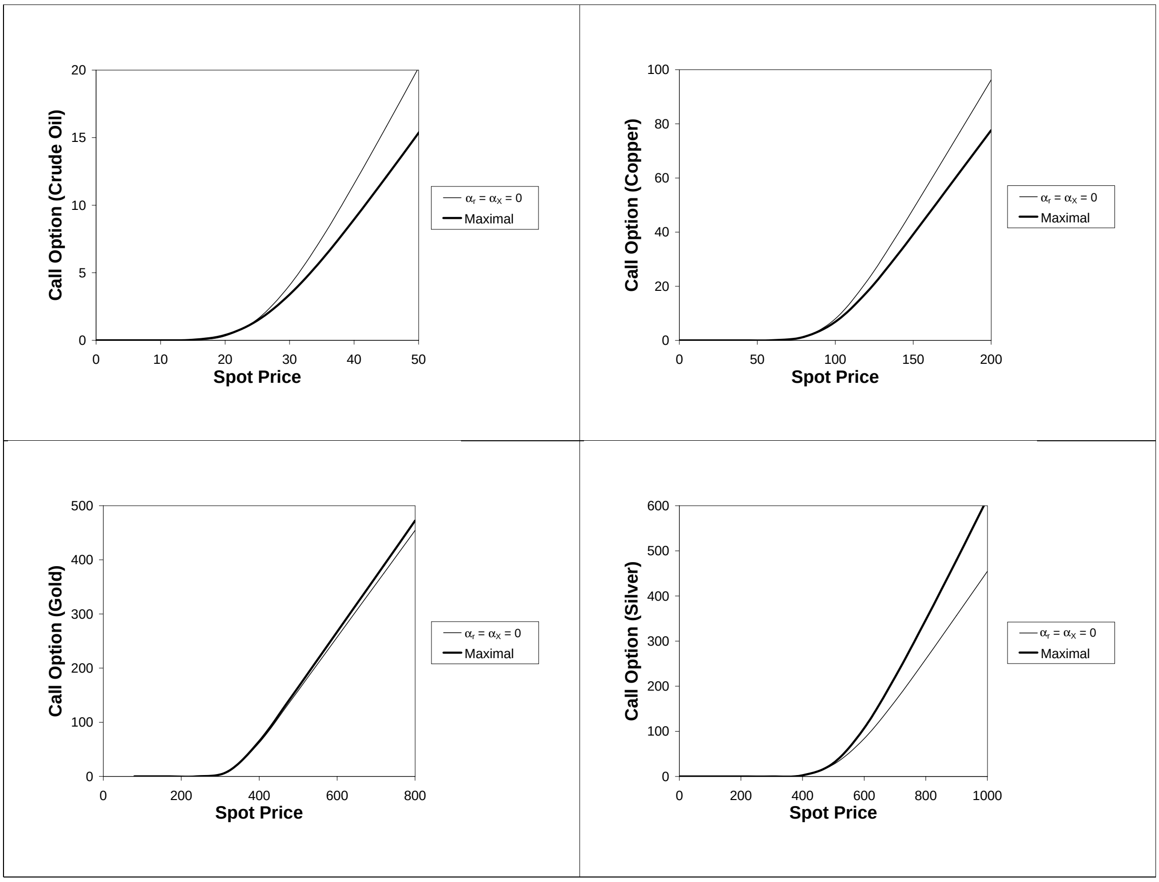


Figure 6: Two-year maturity European Call Option prices using the maximal model. The strike price for oil is \$25 per barrel, for copper is 100 cents per pound, for gold is \$350 per troy ounce and for silver is 550 cents per troy ounce. Spot and options prices are in the same unit as strike prices. Each line corresponds to a different set of parameters. The bold line is corresponds to the maximal model, while the thin line is assuming $\alpha_r = \alpha_X = 0$ (and re-estimation of parameters).

using two sets of parameters. First, we use the parameters corresponding to our ‘maximal’ convenience yield model as given in table 2. Second, we re-estimate the parameters assuming one were to ignore the level dependence in convenience yields (i.e., setting $\alpha_X = \alpha_r = 0$ in our model). This corresponds basically to estimating model 3 of Schwartz (1997), but with a more flexible specification of risk-premia and an AR(1) representation of the measurement errors. The option prices obtained for each commodity with the two sets of parameters are shown in figure 6. For each commodity we value European call options written on a unit of the asset with a maturity of two years, and strike prices of \$25 per barrel for oil, 100 cents per pound for copper, \$350 per troy ounce for gold and 550 cents per troy ounce for silver. The figure shows that the difference can become quite important for oil, copper and silver, especially for options that are at and in the money. For gold the difference in option values is small indicating that the coefficient $\alpha_X$ while statistically significant has a small economic impact.

For commodities with a positive relation between convenience yields and spot prices (i.e. crude oil and copper) ignoring the level dependence in convenience yields, leads to overestimation of call option values. The direction of the bias for a positive $\alpha_X$ is expected for two reasons. First, the 'maximal' convenience yield model effectively introduces mean-reversion under the risk-neutral measure and thus leads to reduced term volatility which reduces option prices. Second, a positive $\alpha_X$ implies a convenience yield which is stochastic and increasing in the spot price. This contributes to decreasing call option prices for in the money options.[40] For commodities with a negative relation between convenience yields and spot prices (gold and silver), we find the opposite bias, i.e., in the money call options are underestimated with existing models.

The size of the error for the estimated parameters is quite dramatic. For example, the error is close to 20% for in the money options written on crude oil. This suggests that appropriately modeling the dynamics of convenience yields may have important consequences on investment decisions within real-option models. For natural resource investments related to commodities like crude oil and copper, our results suggest that in a typical 'waiting to invest' (Majd and Pindyck (1987)) framework that ignores the spot price dependence in convenience yields, the optimal investment rule would have a tendency to postpone investment sub-optimally.

## 5.2 Value at Risk

Our results indicate that for metals such as gold, silver and copper a substantial part of the mean-reversion in spot prices is due to negative covariation between spot prices and risk-premia. This implies that ignoring the time-variation of risk-premia during estimation will lead to mis-estimation of the holding period return distribution of commodities. To illustrate the latter for each commodity we compute the Value at Risk (VAR) of the five year return on a portfolio invested in one unit of the commodity. By definition, the VAR is computed from the total (i.e., 'cum dividend') return under the historical measure. It thus allows us to focus on the effect of time-varying risk-premia. As before we compute the VAR for two different sets of parameters. One corresponds to our maximal model, i.e. table 2. For the other we re-estimate the three factor model, but constraining risk-premia to be constant (i.e., setting $\beta_{XX} = \beta_{rr} = \beta_{\hat{\delta}\hat{\delta}} = \beta_{Xr} = \beta_{X\hat{\delta}} = 0$). We note that in both cases the return on the commodity in our maximal model of proposition 1 and 2 is Gaussian, which is consistent with the usual assumptions of the VAR framework. Figure 7 shows the five year holding period return distributions corresponding to the two sets of parameters, and

[40]This intuition is analogue to a call option on a dividend-paying stock in the Black-Scholes world. The higher the dividend rate, the lower the price of the option.

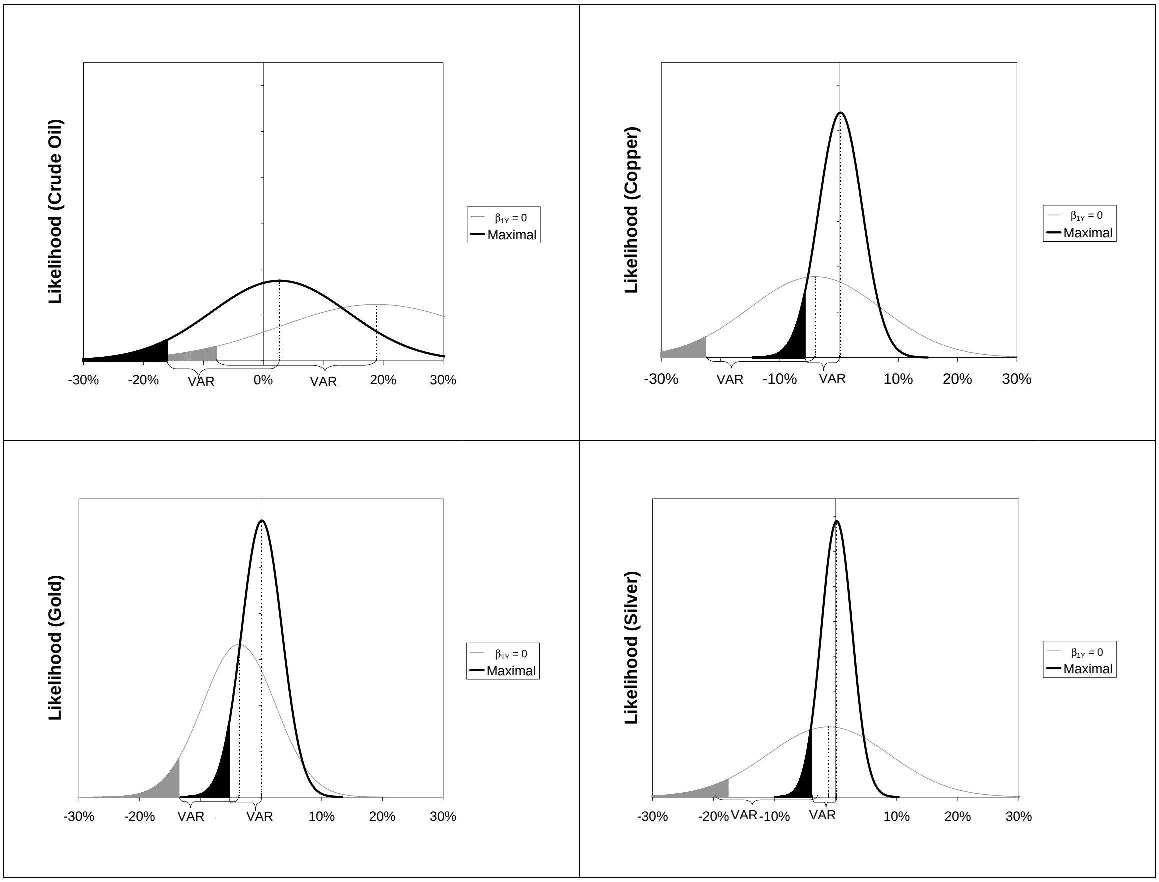


Figure 7: Distribution of returns and Value at Risk for holding one unit of commodity for 5 years using the maximal model. Value at Risk is calculated at a 5% significance level. The two distributions correspond to different set of parameters. The bold line is using copper estimates from the maximal model, while the thin line is assuming constant risk premia (and re-estimation of parameters).

graphs the corresponding 5% VAR.[41] The figure clearly shows that accounting for time-variation in risk-premia has a substantial impact on the dispersion of the holding period return, especially for copper, gold and silver. Not only is the expected return lower for the model with constant risk-premia, but, also the return distribution is more spread out. Consequently, the VAR (the potential loss corresponding to a 5% tail event) for the three metals more than doubles when the distribution is estimated without accounting for time variation in risk-premia. This suggests that economic capital required to cover holdings in precious metals are significantly reduced when appropriately taking into account the dynamics of risk-premia. The same figure shows that the VAR for oil is much less significantly affected.

[41] As before, to facilitate the comparison across commodities, we keep the same scale for all distributions.

## 6 Conclusions

We develop a three-factor model of commodity spot prices, convenience yields and interest rates, which extends previous research in two ways. First, the model nests several (e.g., Brennan (1991), Gibson and Schwartz (1990), Schwartz (1997), Ross (1997), Schwartz and Smith (2000)) proposed specifications. Second, it allows for time-varying risk-premia. We show that previous models have implicitly imposed unnecessary restrictions on the unconditional correlation structure of commodity prices, convenience yields and interest rates. In particular, the present model allows for convenience yields to be a function of spot commodity prices, which leads to mean-reversion in spot prices. Mean-reversion in spot prices can also be generated by negative correlation between risk-premia and spot prices. The former affects the risk-neutral dynamics of commodity prices, i.e. the cross-section of futures prices. The latter affects only the historical measure dynamics of prices, i.e. the time series of futures prices. Both components can thus be identified with panel data on futures prices. Using data on crude oil, copper, gold and silver commodity futures, we empirically estimate the model using maximum likelihood. We find both features of the model to be economically and empirically significant. In particular, we find strong evidence for spot-price level dependence in convenience yields of crude oil and copper, which implies mean-reversion in spot prices under the risk-neutral measure, and is consistent with the "theory of storage." We find evidence for time-varying risk-premia, which implies mean-reversion of commodity prices under the physical measure albeit with different strength and long-term mean. We also document the presence of a jump component in commodity prices.

The results suggest that the relative contribution of both effects to mean reversion (level dependent convenience yield vs. time-varying risk-premia) depends on the nature of the commodity. In particular, it depends on extent to which the commodity serves as an input to production (e.g., a consumption good) versus as a store of value (e.g., a financial asset). We find that for metals like gold and silver, negative correlation between risk-premia and spot prices explains most of the mean reversion, whereas for oil and copper some of the mean-reversion in spot prices is attributable also to convenience yields. The analysis of various examples suggests that disentangling the sources of mean-reversion and careful modeling of the dynamics of the convenience yield can have a substantial impact on (real) option valuation, investment decisions and risk-management.

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# A Closed-form solution for futures prices

The value of a futures contract with maturity $\tau$ is given by:

$$F(Y,\tau) = \exp\left[A_F(\tau) + B_F(\tau)^T Y\right] \tag{A.1}$$

where the closed-form solution for $A_F(\tau)$ and $B_F(\tau)$ are:

$$\begin{aligned} A_F(\tau) \;=\;& \phi_0 + \frac{1}{2}M_{11}^2 \frac{1-e^{-2\tau\kappa_{11}^Q}}{2\kappa_{11}^Q} + \frac{1}{2}\left(M_{12}^2 + M_{22}^2\right)\frac{1-e^{-2\tau\kappa_{22}^Q}}{2\kappa_{22}^Q} \\ &+\frac{1}{2}\left(M_{13}^2 + M_{23}^2 + M_{33}^2\right)\frac{1-e^{-2\tau\kappa_{33}^Q}}{2\kappa_{33}^Q} \\ &+M_{11}M_{12}\frac{1-e^{-\tau\left(\kappa_{11}^Q+\kappa_{22}^Q\right)}}{\kappa_{11}^Q+\kappa_{22}^Q} + M_{11}M_{13}\frac{1-e^{-\tau\left(\kappa_{11}^Q+\kappa_{33}^Q\right)}}{\kappa_{11}^Q+\kappa_{33}^Q} \\ &+\left(M_{12}M_{13} + M_{22}M_{23}\right)\frac{1-e^{-\tau\left(\kappa_{22}^Q+\kappa_{33}^Q\right)}}{\kappa_{22}^Q+\kappa_{33}^Q} \end{aligned} \tag{A.2}$$

$$B_F(\tau) = \begin{pmatrix} M_{11} & M_{12} & M_{13} \\ 0 & M_{22} & M_{23} \\ 0 & 0 & M_{33} \end{pmatrix}\begin{pmatrix} e^{-\tau\kappa_{11}^Q} \\ e^{-\tau\kappa_{22}^Q} \\ e^{-\tau\kappa_{33}^Q} \end{pmatrix} \tag{A.3}$$

with

$$M_{11} = \phi_1 + \alpha_1\left(\phi_2 + \alpha_2\phi_3\right) + \alpha_3\phi_3 \tag{A.4}$$

$$M_{12} = -\alpha_1\left(\phi_2 + \alpha_2\phi_3\right) \tag{A.5}$$

$$M_{13} = -\alpha_3\phi_3 \tag{A.6}$$

$$M_{22} = \phi_2 + \alpha_2\phi_3 \tag{A.7}$$

$$M_{23} = -\alpha_2\phi_3 \tag{A.8}$$

$$M_{33} = \phi_3 \tag{A.9}$$

and

$$\alpha_1 = \frac{\kappa_{21}^Q}{\kappa_{11}^Q - \kappa_{22}^Q} \tag{A.10}$$

$$\alpha_2 = \frac{\kappa_{32}^Q}{\kappa_{22}^Q - \kappa_{33}^Q} \tag{A.11}$$

$$\alpha_3 = \frac{\kappa_{31}^Q}{\kappa_{11}^Q - \kappa_{33}^Q} - \frac{\kappa_{21}^Q}{\kappa_{11}^Q - \kappa_{33}^Q}\frac{\kappa_{32}^Q}{\kappa_{22}^Q - \kappa_{33}^Q} \tag{A.12}$$

## B Closed-form solution for zero-coupon bonds

The value of a zero-coupon bond with maturity $\tau$ is given by:

$$P(Y_1, \tau) = \exp\left[A_P(\tau) + B_P(\tau)Y_1\right] \tag{B.1}$$

where the closed-form solution for $A_P(\tau)$ and $B_P(\tau)$ are:

$$\begin{aligned} A_P(\tau) &= -\left(\psi_0 - \frac{1}{2}\left(\frac{\psi_1}{\kappa_{11}^Q}\right)^2\right)\tau - \left(\frac{\psi_1}{\kappa_{11}^Q}\right)^2 \frac{1-e^{-\tau\kappa_{11}^Q}}{\kappa_{11}^Q} \\ &\quad + \frac{1}{2}\left(\frac{\psi_1}{\kappa_{11}^Q}\right)^2 \frac{1-e^{-2\tau\kappa_{11}^Q}}{2\kappa_{11}^Q} \qquad \text{(B.2)} \\ B_P(\tau) &= -\psi_1 \frac{1-e^{-\tau\kappa_{11}^Q}}{\kappa_{11}^Q} \qquad \text{(B.3)} \end{aligned}$$

## C The $\{r, \delta, X\}$ representation

We apply an invariant transformation to the canonical base to get the economic representation $\{r, \delta, X\}$ (see Dai and Singleton (2000)). This transformation rotates the state variables, but all the initial properties of the model all maintained, i.e. the resulting model is a three-factor Gaussian model that is maximal. We have the transformations for $X(t)$, $r(t)$ and $\delta(t)$ from equations (1), (5) and (6), respectively.

$$\delta(t) = \eta_0 + \eta_Y^\top Y(t) \tag{C.1}$$

where

$$\begin{aligned} \eta_0 &= \psi_0 - \frac{1}{2}\phi_Y^\top \phi_Y \qquad \text{(C.2)} \\ \eta_Y &= \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} \psi_1 + \kappa_{11}^Q \phi_1 + \kappa_{21}^Q \phi_2 + \kappa_{31}^Q \phi_3 \\ \kappa_{22}^Q \phi_2 + \kappa_{32}^Q \phi_3 \\ \kappa_{33}^Q \phi_3 \end{pmatrix} \qquad \text{(C.3)} \end{aligned}$$

We define the transformed state vector $W^\top(t) = (r(t), \delta(t), X(t))$. The linear transformation in matrix form is

$$W(t) = \vartheta + LY(t) \tag{C.4}$$

where $Y(t)$ follows the process in (2). The matrices for the linear transformations are:

$$\vartheta = \begin{pmatrix} \psi_0 \\ \eta_0 \\ \phi_0 \end{pmatrix} \quad \text{and} \quad L = \begin{pmatrix} \psi_1 & 0 & 0 \\ \eta_1 & \eta_2 & \eta_3 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \tag{C.5}$$

From equation (C.4) and Ito's lemma we have

$$dW(t) = L\kappa^Q L^{-1}(\vartheta - W(t))dt + LdZ^Q(t) \tag{C.6}$$

The mean-reversion and long-run parameters under the equivalent martingale measure are given by $\kappa_r^Q = \kappa_{11}^Q$, $\kappa_{\delta r}^Q = -\left[L\kappa^Q L^{-1}\right]_{21}$, $\kappa_\delta^Q = -\kappa_{22}^Q - \kappa_{33}^Q$, $\kappa_{\delta X}^Q = \kappa_{22}^Q\kappa_{33}^Q$, $\theta_r^Q = \psi_0$ and $\kappa_{\delta 0}^Q = \eta_Y^\top \kappa^Q L^{-1}\vartheta$. This transformation was done under the equivalent martingale measure. Using the specification of the risk premia in equation (22) and equation (C.4) we get the rotation under the physical measure

$$\begin{aligned} dW(t) &= L\kappa^Q L^{-1}(\vartheta - W(t))dt + LdZ(t) + L(\beta_{0Y} + \beta_{1Y}Y(t))dt \\ &= L\kappa^Q L^{-1}(\vartheta - W(t))dt + LdZ(t) + (L\beta_{0Y} - L\beta_{1Y}L^{-1}\vartheta + L\beta_{1Y}L^{-1}W(t))dt \end{aligned}$$

The risk-premia parameters for the $\{r, \delta, X\}$ representation are

$$\begin{pmatrix} \beta_{0r} \\ \beta_{0\delta} \\ \beta_{0X} \end{pmatrix} = L\beta_{0Y} - L\beta_{1Y}L^{-1}\vartheta \qquad \text{and} \qquad \begin{pmatrix} \beta_{rr} & \beta_{r\delta} & \beta_{rX} \\ \beta_{\delta r} & \beta_{\delta\delta} & \beta_{\delta X} \\ \beta_{Xr} & \beta_{X\delta} & \beta_{XX} \end{pmatrix} = L\beta_{1Y}L^{-1} \tag{C.7}$$

To get the covariance matrix we match the instantaneous covariance matrices of the state variables from the model equation (C.6) and the model in Proposition 1

$$LL^T = \begin{pmatrix} \sigma_r^2 & \rho_{r\delta}\sigma_r\sigma_\delta & \rho_{rX}\sigma_r\sigma_X \\ \rho_{r\delta}\sigma_r\sigma_\delta & \sigma_\delta^2 & \rho_{\delta X}\sigma_\delta\sigma_X \\ \rho_{rX}\sigma_r\sigma_X & \rho_{\delta X}\sigma_\delta\sigma_X & \sigma_X^2 \end{pmatrix} \tag{C.8}$$

From equation (C.8) we get that $\sigma_r^2 = \psi_1^2$, $\sigma_\delta^2 = \eta_Y^\top\eta_Y$, $\sigma_X^2 = \phi_Y^\top\phi_Y$, $\rho_{r\delta} = \frac{\eta_1}{\sigma_\delta}$, $\rho_{rX} = \frac{\phi_1}{\sigma_X}$ and $\rho_{\delta X} = \frac{\eta_Y^\top\phi_Y}{\sigma_\delta\sigma_X}$.

# D The $\{r, \widehat{\delta}, X\}$ representation

We follow the same approach as in C. We have the transformations for $X(t)$ and $r(t)$ from equations (1) and (5), respectively. From equation (11) in Proposition 2 and equation (6) we can back out the idiosyncratic component of the convenience yield,$\widehat{\delta}(t)$, as a function of the canonical state variables[42]

$$\widehat{\delta}(t) = \widehat{\eta}_0 + \widehat{\eta}_Y^\top Y(t) \tag{D.1}$$

where

$$\widehat{\eta}_0 = -\frac{1}{2}\phi_Y^\top\phi_Y + (1-\alpha_r)\psi_0 - \alpha_X\phi_0 \tag{D.2}$$

$$\widehat{\eta}_Y = \begin{pmatrix} \widehat{\eta}_1 \\ \widehat{\eta}_2 \\ \widehat{\eta}_3 \end{pmatrix} = \begin{pmatrix} -\phi_2\kappa_{21}^Q\frac{\kappa_{22}^Q - \kappa_{33}^Q}{\kappa_{11}^Q - \kappa_{22}^Q} - \phi_3\frac{\kappa_{21}^Q\kappa_{32}^Q}{\kappa_{11}^Q - \kappa_{22}^Q} \\ \phi_2(\kappa_{22}^Q - \kappa_{33}^Q) + \phi_3\kappa_{32}^Q \\ 0 \end{pmatrix} \tag{D.3}$$

[42] There are two possible decompositions for this representation equivalent to the ones in the proof of Proposition 2. We present the unique decomposition that satisfies the conditions of that proposition.

and

$$\alpha_r = 1 + \frac{\phi_1(\kappa^Q_{11} - \kappa^Q_{33}) + \phi_2\kappa^Q_{21} + \phi_3\kappa^Q_{31} - \widehat{\eta}_1}{\psi_1} \tag{D.4}$$

$$\alpha_X = \kappa^Q_{33} \tag{D.5}$$

We define the transformed state vector $\widehat{W}^\top(t) = \Big(r(t), \widehat{\delta}(t), X(t)\Big)$. The linear transformation in matrix form is

$$\widehat{W}(t) = \widehat{\vartheta} + \widehat{L}Y(t) \tag{D.6}$$

where $Y(t)$ follows the process in (2). The matrices for the linear transformations are:

$$\widehat{\vartheta} = \begin{pmatrix} \psi_0 \\ \widehat{\eta}_0 \\ \phi_0 \end{pmatrix} \qquad \text{and} \qquad \widehat{L} = \begin{pmatrix} \psi_1 & 0 & 0 \\ \widehat{\eta}_1 & \widehat{\eta}_2 & \widehat{\eta}_3 \\ \phi_1 & \phi_2 & \phi_3 \end{pmatrix} \tag{D.7}$$

From equation (D.6) and Ito's lemma we have

$$d\widehat{W}(t) = \widehat{L}\kappa^Q\widehat{L}^{-1}(\widehat{\vartheta} - \widehat{W}(t))dt + \widehat{L}dZ^Q(t) \tag{D.8}$$

The remaining mean-reversion and long-run parameters under the equivalent martingale measure are given by $\kappa^Q_r = \kappa^Q_{11}$, $\kappa^Q_{\widehat{\delta}} = \kappa^Q_{22}$, $\theta^Q_r = \psi_0$ and $\theta^Q_{\widehat{\delta}} = \widehat{\eta}_0$. This transformation was done under the equivalent martingale measure. Using the specification of the risk premia in equation (22) and equation (D.6) we get the rotation under the physical measure

$$\begin{aligned} d\widehat{W}(t) &= \widehat{L}\kappa^Q\widehat{L}^{-1}(\widehat{\vartheta} - \widehat{W}(t))dt + \widehat{L}dZ(t) + \widehat{L}(\beta_{0Y} + \beta_{1Y}Y(t))dt \\ &= \widehat{L}\kappa^Q\widehat{L}^{-1}(\widehat{\vartheta} - \widehat{W}(t))dt + \widehat{L}dZ(t) + (\widehat{L}\beta_{0Y} - \widehat{L}\beta_{1Y}\widehat{L}^{-1}\widehat{\vartheta} + \widehat{L}\beta_{1Y}\widehat{L}^{-1}\widehat{W}(t))dt \end{aligned}$$

The risk-premia parameters for the $\{r,\widehat{\delta}, X\}$ representation are

$$\begin{pmatrix} \beta_{0r} \\ \beta_{0\widehat{\delta}} \\ \beta_{0X} \end{pmatrix} = \widehat{L}\beta_{0Y} - \widehat{L}\beta_{1Y}\widehat{L}^{-1}\widehat{\vartheta} \qquad \text{and} \qquad \begin{pmatrix} \beta_{rr} & \beta_{r\widehat{\delta}} & \beta_{rX} \\ \beta_{\widehat{\delta}r} & \beta_{\widehat{\delta}\widehat{\delta}} & \beta_{\widehat{\delta}X} \\ \beta_{Xr} & \beta_{X\widehat{\delta}} & \beta_{XX} \end{pmatrix} = \widehat{L}\beta_{1Y}\widehat{L}^{-1} \tag{D.9}$$

To get the covariance matrix we match the instantaneous covariance matrices of the state variables from the model equation (D.8) and the model in Proposition 2

$$\widehat{L}\widehat{L}^T = \begin{pmatrix} \sigma_r^2 & \rho_{r\widehat{\delta}}\sigma_r\sigma_{\widehat{\delta}} & \rho_{rX}\sigma_r\sigma_X \\ \rho_{r\widehat{\delta}}\sigma_r\sigma_{\widehat{\delta}} & \sigma^2_{\widehat{\delta}} & \rho_{\widehat{\delta}X}\sigma_{\widehat{\delta}}\sigma_X \\ \rho_{rX}\sigma_r\sigma_X & \rho_{\widehat{\delta}X}\sigma_{\widehat{\delta}}\sigma_X & \sigma_X^2 \end{pmatrix} \tag{D.10}$$

From equation (D.10) we get that $\sigma_r^2 = \psi_1^2$, $\sigma^2_{\widehat{\delta}} = \widehat{\eta}_Y^\top\widehat{\eta}_Y$, $\sigma_X^2 = \phi_Y^\top\phi_Y$, $\rho_{r\widehat{\delta}} = \frac{\widehat{\eta}_1}{\sigma_{\widehat{\delta}}}$, $\rho_{rX} = \frac{\phi_1}{\sigma_X}$ and $\rho_{\widehat{\delta}X} = \frac{\widehat{\eta}_Y^\top\phi_Y}{\sigma_{\widehat{\delta}}\sigma_X}$.

# E Maximum Likelihood

## E.1: Estimation of the Gaussian Model

We follow the maximum-likelihood approach of Collin-Dufresne, Goldstein and Jones (2002) which extends Chen and Scott (1993) and Pearson and Sun (1994) in the following way: instead of assuming that we observe without error some futures contracts and bond prices, we choose to fit the principal components (PCs) of futures and bonds.[43] From the perfectly observed data and using the closed-form solutions for futures and bonds in Appendices A and B we can invert for the state variables $Y(t)$. We assume that the first two PCs of the futures curve and the first PC of the yield curve are observed without error. The rest of the PCs are assumed to be observed with measurement errors that are jointly normally distributed and follow an AR(1) process. Using the principal component approach instead of single contracts has some advantages. First, it guarantees (by construction) that we fit perfectly the first two PCs of futures and the first PC of the yield curve. Second, it orthogonalizes the matrix of measurement errors. Finally, dispenses the arbitrariness of what contracts are perfectly observed.

Our cross-sectional data set is composed by $m$ futures contracts and $n$ zero-coupon bonds. The $i^{th}$ principal components of the futures curve is

$$PC_F^i(\widehat{Y};\Theta) = {\omega_F^i}^\top \left(\overline{A_F}(\Theta) + \overline{B_F}(\Theta)^\top \widehat{Y}\right) \tag{E.1}$$

where $\omega_F^i$ is an $m \times 1$ eigenvector corresponding to the $i^{th}$ principal component, $\overline{A_F}(\Theta)$ is an $m \times 1$ vector and $\overline{B_F}(\Theta)$ is an $3 \times m$ matrix that determine the theoretical value of the log futures prices for different maturity contracts, i.e. $\overline{A_F}(\Theta)^\top = \left(A_F(\tau_F^1;\Theta),\ldots,A_F(\tau_F^m;\Theta)\right)$ and $\overline{B_F}(\Theta) = \left(B_F(\tau_F^1;\Theta),\ldots,B_F(\tau_F^m;\Theta)\right)$ (see Appendix A). $\Theta$ is the parameter space of the model.

In the same way we obtain the $i^{th}$ principal component of the yield curve

$$PC_P^i(\widehat{Y}_1;\Theta) = {\omega_P^i}^\top \left(\overline{A_P}(\Theta) + \overline{B_P}(\Theta)\widehat{Y}_1\right) \tag{E.2}$$

To invert the state variables $\widehat{Y}$ we perfectly observe the first two principal components of the futures curve and the first principal component of the yield curve. The relation between the data and the state variables is

$$G(t) = A(\Theta) + B(\Theta)\,\widehat{Y}(t) \tag{E.3}$$

where $G(t)^\top = \left({\omega_P^1}^\top \overline{LnP}(t), {\omega_F^1}^\top \overline{LnF}(t), {\omega_F^2}^\top \overline{LnF}(t)\right)$ is the $1 \times 3$ vector of perfectly

[43] These principal components are linear in the state variables and can be thought of being portfolios of single contracts.

observed PCs at time t,[44] $A(\Theta)^\top = \left(\omega_P^{1\top}\overline{A_P}(\Theta), \omega_F^{1\top}\overline{A_F}(\Theta), \omega_F^{2\top}\overline{A_F}(\Theta)\right)$ is a $1 \times 3$ vector and $B$ is a $3 \times 3$ matrix given by

$$B(\Theta) = \left( \begin{array}{ccc} \omega_P^{1\top}\overline{B_P}(\Theta) & 0 & 0 \\ & \omega_F^{1\top}\overline{B_F}(\Theta)^\top & \\ & \omega_F^{2\top}\overline{B_F}(\Theta)^\top & \end{array} \right) \tag{E.4}$$

At any given point in time $t$, we can invert equation (E.3) to back out the state variables $\widehat{Y}(t)^\top = (\widehat{Y}_1(t), \widehat{Y}_2(t), \widehat{Y}_3(t))$:

$$\widehat{Y}(t) = B(\Theta)^{-1}(G(t) - A(\Theta)) \tag{E.5}$$

The other bonds and futures principal components are priced with error

$$\begin{aligned} \omega_P^{i\top}\overline{LnP}(t) &= \omega_P^{i\top}\left(\overline{A_P}(\Theta) + \overline{B_P}(\Theta)\widehat{Y}_1(t)\right) + u_P^i(t) \quad \text{for} \quad i = 2, .., n \\ \omega_F^{i\top}\overline{LnF}(t) &= \omega_F^{i\top}\left(\overline{A_F}(\Theta) + \overline{B_F}(\Theta)^\top\widehat{Y}(t)\right) + u_F^i(t) \quad \text{for} \quad i = 3, .., m \end{aligned}$$

We assume that the measurement errors $u_j^i(t)$ follow and AR(1) process, i.e. $u_j^i(t) = \rho_j u_j^i(t-1) + e_j^i(t)$ for $j \in \{P, F\}$, and the errors $e_j^i(t)$ are jointly normally distributed with zero mean and covariance matrix $E\left[e_j^i e_j^{i\top}\right]$.

The conditional likelihood function for every time $t$ will be given by the likelihood function of $G(t)$ times the likelihood function of the measurement errors, $f_u(u(t) \mid u(t-1)) = f_e(e(t))$. We don't know the conditional density function of $G(t)$, but since it is an affine function of the state vector $\widehat{Y}(t)$ and we know the conditional distribution of $\widehat{Y}(t)$[45], we can get it using the relation in equation (E.5):

$$f_G(G(t) \mid G(t-1)) = abs(J_{\widehat{Y}}) f_{\widehat{Y}}\left(\widehat{Y}(t) \mid \widehat{Y}(t-1)\right) \tag{E.6}$$

where $J_{\widehat{Y}}$ is the Jacobian of the transformation from $G(t)$ to $\widehat{Y}(t)$, i.e. $J_{\widehat{Y}} = \det\left(B^{-1}\right)$.

The estimated parameters will be the ones that maximize the log-likelihood function:

$$\max_{\Theta} L(\Theta) = \sum_{t=1}^{T} f_G(G(t) \mid G(t-1)) + f_e(e(t)) \tag{E.7}$$

where $f_G(G(1) \mid G(0))$ is the unconditional density function.

[44] $\overline{LnP}(t)$ and $\overline{LnF}(t)$ are the vectors of the logarithm of the observed bonds and futures contracts at time t, respectively.

[45] In our Gaussian model we can calculate the exact moments for the distribution of $\widehat{Y}(t)$.

### E.2: Estimation of the Model with Jumps in Spot Prices

For the case with a triple-jump component in spot prices we use a similar approach than the one in Section E.1:. Since the jumps are in the spot price it is easier to work with the economic representation $\{r, \widehat{\delta}, X\}$, than with the canonical form $\{Y_1, Y_2, Y_3\}$. Using equation (C.4) from Appendix D and equation (E.5), the conditional likelihood of the perfectly observed principal components is

$$f_G\left(G(t) \mid G(t-1)\right) = abs(J_{\widehat{W}}) f_{\widehat{W}}\left(\widehat{W}(t) \mid \widehat{W}(t-1)\right) \tag{E.8}$$

where $\widehat{W}$ is the vector of the economic state variables $\{r, \widehat{\delta}, X\}$ implied from the perfectly observed data and $J_{\widehat{W}}$ is the Jacobian of the transformation from $G(t)$ to $\widehat{W}(t)$. Equation (E.8) is similar to equation (E.6), but depends on the economic state variables $\widehat{W}$, instead of the canonical vector $\widehat{Y}$. We also use the close-form solution for futures prices with jumps from Appendix F instead of the one in Appendix A.

Since the transition density with jumps is no longer Gaussian, we follow Ball and Torous (1983), Jorion (1988), and Das (2002) and approximate it by a mixture of Gaussian. This approximation is as follows

$$\begin{aligned} f_{\widehat{W}}\left(\widehat{W}(t) \mid \widehat{W}(t-1)\right) \quad = \quad & \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{k_3=0}^{\infty} p_1(N_1(\Delta t) = k_1) p_2(N_2(\Delta t) = k_2) p_3(N_3(\Delta t) = k_3) \times \\ & f_{\widehat{W}}\left(\widehat{W}(t) \mid \widehat{W}(t-1), N_1(\Delta t) = k_1, N_2(\Delta t) = k_2, N_3(\Delta t) = k_3\right) \end{aligned}$$

where the last term is the likelihood function conditional on a fixed number of jumps $k_1$ and $k_2$ which is Gaussian, and the Poisson probabilities are

$$p_i(N_i(\Delta t) = k_i) = e^{-\lambda_i \Delta t} \frac{(\lambda_i \Delta t)^{k_i}}{k_i} \tag{E.9}$$

This approximation would be exact if the time interval was infinitesimal.

## F Closed-form solution for futures prices with jumps

The model is given by

$$\delta(t) = \alpha_r\, r(t) + \widehat{\delta}(t) + \alpha_X\, X(t) \tag{F.1}$$

where the state variables $\{r, \widehat{\delta}, X\}$ have the following risk-neutral dynamics:

$$
\begin{aligned}
dr(t) &= \kappa_r^Q \left(\theta_r^Q - r(t)\right) dt + \sigma_r dZ_r^Q(t) && \text{(F.2)}\\
d\widehat{\delta}(t) &= \kappa_{\widehat{\delta}}^Q \left(\theta_{\widehat{\delta}}^Q - \widehat{\delta}(t)\right) dt + \sigma_{\widehat{\delta}} dZ_{\widehat{\delta}}^Q(t) && \text{(F.3)}\\
dX(t) &= \left(r(t) - \delta(t) - \frac{1}{2}\sigma_X^2 - \sum_{i=1}^{3}(\varphi_i - 1)\lambda_i^Q\right) dt \\
&\qquad + \sigma_X dZ_X^Q(t) + \sum_{i=1}^{3} \nu_i(t) dN_i(t) && \text{(F.4)}
\end{aligned}
$$

The futures price $F^T(t) = \mathrm{E}_t^Q[S(T)] = \mathrm{E}_t^Q[e^{X(T)}]$. We show that the expectation has the following exponential affine form:

$$
F^T(t) = \exp\left(A_0(T-t) + B_X(T-t)X(t) + B_r(T-t)r(t) + B_{\widehat{\delta}}(T-t)\widehat{\delta}(t)\right) \tag{F.5}
$$

where the functions $A_0, B_X, B_r, B_{\widehat{\delta}}$ are given by:

$$
\begin{aligned}
B_X(\tau) &= e^{-\alpha_X \tau} && \text{(F.6)}\\
B_{\widehat{\delta}}(\tau) &= \frac{1}{\alpha_X - \kappa_{\widehat{\delta}}}\left(e^{-\alpha_X \tau} - e^{-\kappa_{\widehat{\delta}} \tau}\right) && \text{(F.7)}\\
B_r(\tau) &= \frac{\alpha_r - 1}{\alpha_X - \kappa_r}\left(e^{-\alpha_X \tau} - e^{-\kappa_r \tau}\right) && \text{(F.8)}\\
A_0(\tau) &= \int_0^\tau \Big\{\frac{1}{2}(B_X^2(s)\sigma_X^2 + B_{\widehat{\delta}}^2(s)\sigma_{\widehat{\delta}}^2 + B_r^2(s)\sigma_r^2) && \text{(F.9)}\\
&\quad - \sum_i \lambda_i \left(B_X(s)\left(\varphi_i - 1\right) - \left(\varphi_i(B_X(s)) - 1\right)\right) + B_{\widehat{\delta}}(s)\kappa_{\widehat{\delta}}\theta_{\widehat{\delta}} + B_r(s)\kappa_r\theta_r \\
&\quad + \rho_{r\widehat{\delta}}\sigma_r\sigma_{\widehat{\delta}}B_r(s)B_{\widehat{\delta}}(s) + \rho_{rX}\sigma_r\sigma_X B_r(s)B_X(s) + \rho_{\widehat{\delta}X}\sigma_X\sigma_{\widehat{\delta}}B_X(s)B_{\widehat{\delta}}(s)\Big\} ds
\end{aligned}
$$

where we define $\varphi_i(\alpha) = \exp\left(\alpha m_i + \frac{\alpha^2\sigma_i^2}{2}\right)$. The proof consists in verifying that the candidate solution given in equation (F.5) is a Q-martingale. Indeed applying itô's lemma to $F$ defined in (F.5) and using equations (F.6) to (F.9) we see that

$$
\begin{aligned}
F^T(t) &= F^T(T) - \int_t^T \Big\{B_X(T-s)\sigma_X dZ_X(s) + B_{\widehat{\delta}}(T-s)\sigma_{\widehat{\delta}} dZ_{\widehat{\delta}}(s) + B_r(T-s)\sigma_r dZ_r(s)\Big\} \\
&\quad + \sum_i \int_t^T \Big\{(e^{B_X(s)\nu_i(s)} - 1)dN_i(s) - \left(\varphi_i(B_X(s)) - 1\right)\lambda_i^Q ds\Big\} && \text{(F.10)}
\end{aligned}
$$

Thus

$$
F^T(t) = \mathrm{E}_t^Q[F^T(T)] = \mathrm{E}_t^Q[e^{X(T)}]
$$

where for the second equality we have used the fact that $B_X(0) = 1$ and $A_0(0) = B_{\widehat{\delta}}(0) = B_r(0) = 0$.

Inspecting the solution we see that the only impact of jumps on the prices of futures is through the term

$$J_i := \lambda_i^Q \left(B_X(s)(\varphi_i - 1) - \varphi_i(B_X(s)) - 1\right)$$

in the expression for $A_0$.

Note that if $\alpha_X = 0$ then $B_X(s) = 1$ and thus we have $J_i = 0$. We conclude that *If* $\alpha_X = 0$ *then futures prices are not affected by the presence of jumps*.

Next we show that if $\alpha_X \neq 0$ then the impact of jumps is likely to be small if the jump intensity, and jump size volatility are 'small.' Indeed if a Taylor series approximation is appropriate we have $\varphi_i \approx 1 + (m_i + \frac{\sigma_i^2}{2})$ and $\varphi_i(B_X) \approx 1 + (m_i B_X + \frac{v_i^2 B_X^2}{2})$. Substituting in the expression for $J_i$ we obtain:

$$J_i \approx \lambda_i^Q \frac{v_i^2}{2} B_X(1 - B_X)$$

We thus see that the impact of jumps for the cross section of futures prices is minimal if the jump intensity and jump size volatility are small. Thus, jumps are mainly helpful in capturing time series properties of futures prices. Of course, jumps would have a significant impact for the cross-section of option prices.

**Crude Oil Data**

| Contract | Obs. | Mean Price (Std. Error) | Maturity (Std. Error) |
|---|---|---|---|
| F1 | 348 | 21.85 (5.65) | 0.038 (0.024) |
| F3 | 348 | 21.43 (5.10) | 0.204 (0.026) |
| F6 | 348 | 20.90 (4.47) | 0.454 (0.025) |
| F9 | 348 | 20.48 (3.98) | 0.705 (0.025) |
| F12 | 348 | 20.15 (3.58) | 0.954 (0.029) |
| F15 | 348 | 19.88 (3.23) | 1.205 (0.028) |
| F18 | 348 | 19.67 (2.94) | 1.455 (0.027) |
| F24 | 348 | 19.43 (2.52) | 1.953 (0.035) |
| F30 | 348 | 19.30 (2.22) | 2.418 (0.111) |
| F36 | 348 | 19.24 (2.00) | 2.864 (0.207) |

**Copper Data**

| Contract | Obs. | Mean Price (Std. Error) | Maturity (Std. Error) |
|---|---|---|---|
| F1 | 348 | 88.06 (19.93) | 0.037 (0.024) |
| F3 | 348 | 87.62 (18.49) | 0.203 (0.025) |
| F6 | 348 | 87.07 (16.55) | 0.454 (0.025) |
| F9 | 348 | 86.58 (14.91) | 0.703 (0.025) |
| F12 | 348 | 86.19 (13.57) | 0.954 (0.027) |
| F15 | 348 | 85.92 (12.58) | 1.203 (0.027) |
| F18 | 348 | 85.76 (11.76) | 1.453 (0.028) |

**Gold Data**

| Contract | Obs. | Mean Price (Std. Error) | Maturity (Std. Error) |
|---|---|---|---|
| F1 | 348 | 311.35 (45.43) | 0.038 (0.026) |
| F3 | 348 | 313.04 (45.61) | 0.205 (0.034) |
| F6 | 348 | 316.00 (46.06) | 0.458 (0.063) |
| F9 | 348 | 318.91 (46.59) | 0.706 (0.062) |
| F12 | 348 | 321.80 (47.15) | 0.956 (0.065) |
| F15 | 348 | 324.79 (47.69) | 1.208 (0.067) |
| F18 | 348 | 327.82 (48.28) | 1.458 (0.071) |
| F24 | 348 | 333.35 (49.23) | 1.905 (0.115) |
| F30 | 348 | 340.18 (50.62) | 2.434 (0.149) |
| F36 | 348 | 346.69 (51.85) | 2.931 (0.150) |
| F42 | 348 | 353.16 (52.74) | 3.425 (0.148) |

**Silver Data**

| Contract | Obs. | Mean Price (Std. Error) | Maturity (Std. Error) |
|---|---|---|---|
| F1 | 348 | 500.94 (48.57) | 0.039 (0.027) |
| F3 | 348 | 503.95 (48.65) | 0.203 (0.031) |
| F6 | 348 | 508.55 (47.92) | 0.443 (0.063) |
| F9 | 348 | 512.53 (47.35) | 0.698 (0.066) |
| F12 | 348 | 516.11 (47.08) | 0.946 (0.063) |
| F15 | 348 | 519.60 (46.90) | 1.199 (0.067) |
| F18 | 348 | 523.12 (47.00) | 1.444 (0.073) |
| F24 | 348 | 529.34 (47.61) | 1.902 (0.107) |
| F30 | 312 | 528.24 (46.44) | 2.454 (0.167) |
| F36 | 312 | 534.48 (48.69) | 2.966 (0.173) |
| F42 | 312 | 541.15 (52.71) | 3.464 (0.159) |
| F48 | 312 | 548.33 (57.81) | 3.963 (0.155) |

Table 1: Statistics for weekly observations of crude oil, copper, gold and silver futures contracts between 8/1/1995 and 3/25/02. Oil prices are in dollars per barrel, copper prices are in cents per pound, gold prices are in dollars per troy ounce, silver prices are in cents per troy ounce and maturities are in years.

| Parameter | Crude Oil Estimate (Std. Error) | Copper Estimate (Std. Error) | Gold Estimate (Std. Error) | Silver Estimate (Std. Error) |
|---|---|---|---|---|
| $\kappa_r^Q$ | 0.031 (0.010) | 0.031 (0.010) | 0.027 (0.010) | 0.031 (0.010) |
| $\kappa_{\hat{\delta}}^Q$ | 0.992 (0.032) | 1.165 (0.066) | 0.157 (0.013) | 0.208 (0.011) |
| $\alpha_r$ | 0.977 (0.161) | 1.147 (0.254) | 2.213 (0.241) | 2.244 (0.227) |
| $\alpha_X$ | 0.162 (0.011) | 0.122 (0.004) | -0.022 (0.002) | -0.187 (0.009) |
| $\theta_r^Q$ | 0.099 (0.020) | 0.099 (0.020) | 0.148 (0.035) | 0.080 (0.014) |
| $\theta_{\hat{\delta}}^Q$ | -0.520 (0.033) | -0.567 (0.027) | 0.005 (0.018) | 1.077 (0.049) |
| $\beta_{0r}$ | 0.019 (0.020) | 0.021 (0.017) | 0.031 (0.017) | 0.024 (0.025) |
| $\beta_{0\hat{\delta}}$ | 0.123 (0.118) | -0.595 (0.411) | 0.030 (0.017) | 1.678 (0.464) |
| $\beta_{0X}$ | 4.758 (1.014) | 14.678 (2.383) | 7.041 (2.134) | 12.362 (3.262) |
| $\beta_{rr}$ | -0.456 (0.397) | -0.505 (0.323) | -0.763 (0.345) | -0.538 (0.464) |
| $\beta_{\hat{\delta}\hat{\delta}}$ | | -1.122 (0.758) | -0.585 (0.395) | -1.575 (0.420) |
| $\beta_{Xr}$ | | 20.250 (9.204) | 69.012 (26.141) | 15.986 (5.751) |
| $\beta_{X\hat{\delta}}$ | 2.902 (0.624) | 4.707 (1.471) | 27.099 (9.442) | |
| $\beta_{XX}$ | -1.107 (0.274) | -2.972 (0.333) | -1.976 (0.569) | -2.128 (0.557) |
| $\sigma_r$ | 0.009 (0.000) | 0.009 (0.000) | 0.009 (0.000) | 0.009 (0.000) |
| $\sigma_{\hat{\delta}}$ | 0.302 (0.015) | 0.184 (0.010) | 0.026 (0.002) | 0.067 (0.004) |
| $\sigma_X$ | 0.355 (0.013) | 0.246 (0.009) | 0.132 (0.005) | 0.222 (0.009) |
| $\rho_{\hat{\delta}X}$ | 0.762 (0.025) | 0.582 (0.036) | 0.386 (0.051) | 0.920 (0.010) |
| $\rho_{r\hat{\delta}}$ | 0.050 (0.052) | 0.101 (0.062) | -0.858 (0.031) | -0.135 (0.064) |
| $\rho_{rX}$ | 0.065 (0.050) | 0.196 (0.052) | -0.046 (0.047) | 0.115 (0.051) |
| $\rho_F$ | 0.820 (0.013) | 0.670 (0.020) | 0.845 (0.012) | 0.788 (0.014) |
| $\rho_P$ | 0.981 (0.005) | 0.981 (0.005) | 0.988 (0.005) | 0.981 (0.005) |
| Log-likelihood | 28954.2 | 24900.4 | 37064.5 | 34585.1 |

Table 2: Maximum-likelihood parameter estimates for the maximal model for crude oil, copper, gold and silver weekly prices and interest rate data between 8/1/1995 to 3/25/02.

| Restriction | Crude Oil | Copper | Gold | Silver |
|---|---|---|---|---|
| $\alpha_r = \alpha_X = 0$ [a] | 223.08 | 44.55 | 125.02 | 105.93 |
| $\beta_{1Y} = 0$ [b] | 13.10 | 15.07 | 17.26 | 15.75 |
| $\alpha_r = \alpha_X = 0$ and $\beta_{1Y} = 0$ [c] | 235.39 | 62.92 | 143.53 | 121.64 |

[a] $Prob\{\chi_2^2 \geq 5.99\} = 0.05$
[b] $Prob\{\chi_5^2 \geq 11.07\} = 0.05$
[c] $Prob\{\chi_7^2 \geq 14.07\} = 0.05$

Table 3: Likelihood ratios for the maximal model and three sets of restrictions.

| Parameter | Crude Oil Estimate | Copper Estimate | Gold Estimate | Silver Estimate |
|---|---|---|---|---|
| $\kappa_r^P$ | 0.487 | 0.535 | 0.791 | 0.569 |
| $\kappa_{\hat{\delta}}^P$ | 0.992 | 2.287 | 0.742 | 1.783 |
| $\kappa_{Xr}^P$ | -0.023 | -20.102 | -67.799 | -14.743 |
| $\kappa_{X\hat{\delta}}^P$ | -1.902 | -3.707 | -26.099 | 1.000 |
| $\kappa_X^P$ | 1.269 | 3.094 | 1.953 | 1.940 |
| $\theta_r^P$ | 0.045 | 0.045 | 0.045 | 0.047 |
| $\theta_{\hat{\delta}}^P$ | -0.396 | -0.549 | 0.042 | 1.067 |
| $\theta_X^P$ | 3.106 | 4.369 | 5.700 | 6.162 |

Table 4: Maximal model estimates of historical parameters for crude oil, copper, gold and silver using weekly prices and interest rate data between 8/1/1995 to 3/25/02.

| Unconditional Moments | Crude Oil | Copper | Gold | Silver |
|---|---|---|---|---|
| $E[\delta]$ | 0.151 | 0.035 | 0.013 | 0.018 |
| Stdev $(\delta)$ | 0.274 | 0.104 | 0.008 | 0.019 |
| $E[X]$ | 3.106 | 4.369 | 5.700 | 6.162 |
| Stdev $(X)$ | 0.402 | 0.162 | 0.163 | 0.124 |
| Corr $(\delta, X)$ | 0.920 | 0.845 | 0.775 | 0.682 |
| $E[e^X]$ | 24.21 | 79.97 | 302.73 | 477.99 |

Table 5: Unconditional first and second moment estimates from the maximal model of $\delta(t)$ and $X(t)$ for crude oil, copper, gold and silver using weekly prices and interest rate data between 8/1/1995 to 3/25/02.

| Error | Crude Oil | Copper | Gold | Silver |
|---|---|---|---|---|
| Mean$u_t$ | 0.000 | -0.001 | 0.000 | -0.001 |
| Stdev$u_t$ | 0.027 | 0.008 | 0.003 | 0.003 |
| Max$u_t$ | 0.120 | 0.055 | 0.009 | 0.009 |

Table 6: Mis-specification statistics for the error terms of the F1 futures contract using the maximal model. The error term $u_t$ is $\log(F1) - \log(\widetilde{F1})$, where $\widetilde{F1}$ is the estimated F1 futures contract. The statistics are for the four commodities using data between 8/1/1995 and 3/25/02.

| Parameter | Crude Oil Estimate (Std. Error) | Copper Estimate (Std. Error) | Gold Estimate (Std. Error) | Silver Estimate (Std. Error) |
|---|---|---|---|---|
| $\kappa_r^Q$ | 0.031 (0.010) | 0.031 (0.010) | 0.027 (0.007) | 0.031 (0.010) |
| $\kappa_{\hat{\delta}}^Q$ | 0.992 (0.032) | 1.103 (0.056) | 0.150 (0.012) | 0.210 (0.011) |
| $\alpha_r$ | 0.987 (0.148) | 1.396 (0.287) | 2.379 (0.233) | 2.371 (0.194) |
| $\alpha_X$ | 0.163 (0.005) | 0.078 (0.004) | -0.026 (0.002) | -0.198 (0.006) |
| $\theta_r^Q$ | 0.099 (0.020) | 0.099 (0.020) | 0.150 (0.028) | 0.082 (0.014) |
| $\theta_{\hat{\delta}}^Q$ | -0.522 (0.010) | -0.392 (0.019) | 0.014 (0.036) | 1.142 (0.031) |
| $\beta_{0r}$ | 0.019 (0.022) | 0.021 (0.019) | 0.030 (0.018) | 0.024 (0.021) |
| $\beta_{0\hat{\delta}}$ | 0.122 (0.119) | -0.313 (0.252) | 0.038 (0.023) | 1.726 (0.581) |
| $\beta_{0X}$ | 3.291 (1.828) | 10.957 (3.765) | 5.625 (2.250) | 10.861 (3.321) |
| $\beta_{rr}$ | -0.456 (0.434) | -0.499 (0.364) | -0.746 (0.364) | -0.544 (0.392) |
| $\beta_{\hat{\delta}\hat{\delta}}$ | | -0.900 (0.687) | -0.565 (0.395) | -1.530 (0.484) |
| $\beta_{Xr}$ | | 22.519 (10.143) | 48.110 (29.157) | 15.096 (5.278) |
| $\beta_{X\hat{\delta}}$ | 2.497 (0.824) | 3.894 (1.657) | 17.045 (10.331) | |
| $\beta_{XX}$ | -0.678 (0.514) | -2.419 (0.798) | -1.535 (0.665) | -1.874 (0.573) |
| $\sigma_r$ | 0.009 (0.000) | 0.009 (0.000) | 0.009 (0.000) | 0.009 (0.000) |
| $\sigma_{\hat{\delta}}$ | 0.302 (0.015) | 0.183 (0.009) | 0.028 (0.002) | 0.070 (0.003) |
| $\sigma_X$ | 0.284 (0.022) | 0.230 (0.010) | 0.109 (0.006) | 0.228 (0.010) |
| $\rho_{\hat{\delta}X}$ | 0.943 (0.047) | 0.765 (0.025) | 0.476 (0.056) | 0.966 (0.007) |
| $\rho_{r\hat{\delta}}$ | 0.049 (0.050) | 0.090 (0.061) | -0.870 (0.028) | -0.146 (0.054) |
| $\rho_{rX}$ | 0.074 (0.060) | 0.229 (0.053) | -0.050 (0.050) | 0.098 (0.052) |
| $m_1$ | 0.000 (0.000) | | -0.010 (0.008) | -0.001 (0.001) |
| $v_1$ | 0.019 (0.010) | | 0.013 (0.011) | 0.008 (0.002) |
| $\lambda_1$ | 103.710 (64.019) | | 7.313 (5.338) | 46.630 (11.129) |
| $m_2$ | | 0.068 (0.008) | 0.016 (0.003) | 0.054 (0.004) |
| $\lambda_2$ | | 1.961 (0.721) | 13.926 (5.548) | 0.738 (0.523) |
| $m_3$ | -0.171 (0.022) | -0.096 (0.013) | -0.063 (0.015) | -0.034 (0.002) |
| $\lambda_3$ | 0.155 (0.153) | 0.476 (0.277) | 0.174 (0.181) | 1.014 (0.563) |
| $\rho_F$ | 0.820 (0.012) | 0.669 (0.020) | 0.844 (0.012) | 0.787 (0.014) |
| $\rho_P$ | 0.981 (0.005) | 0.981 (0.005) | 0.988 (0.005) | 0.981 (0.005) |
| Log-likelihood | 28966.5 | 24923.1 | 37084.8 | 34631.3 |

Table 7: Maximum-likelihood parameter estimates for the triple-jump model for crude oil, copper, gold and silver weekly prices and interest rate data between 8/1/1995 to 3/25/02.

| Restriction | Crude Oil | Copper | Gold | Silver |
|---|---|---|---|---|
| $m_i = v_i = \lambda_i = 0$ [a] | 24.67 | 45.270 | 40.538 | 92.420 |

[a] $Prob\{\chi^2_7 \geq 14.07\} = 0.05$

Table 8: Likelihood ratio for the jump parameters.