

# **An Empirical Estimation and Model Selection of the Short-Term Interest Rates**

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## **ABSTRACT**

*A variety of continuous time series models of the short term riskless rate are estimated using Maximum Likelihood method on discretized models. Then the best model will be found that can fit the data better. A number of well-known models perform poorly in the comparison. Indirect Inference method is used for the best model in order to obtain consistent estimates. At the end, an empirical application of stochastic model for interest rates will be used for pricing the call options of Nokia Company.*

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<sup>1</sup> I am grateful to Professor Isabelle Perrigne, Ms. Ladan Masoudie & Mr. Tomoya Matsumoto for helping me to use the computation facility at the Department of Economics at USC. Also I am grateful to Dr. Levon Goukosian for giving me the data that I used in this study.

## 1-Introduction

The short term interest rate is one of the most fundamental and important prices determined in financial markets. More models have been put forward to explain its behavior than for any other issue in finance. Many of the more popular models currently used by academic researchers and practitioners have been developed in a continuous-time setting, which provides a rich framework for specifying the dynamic behavior of the short term riskless rates. A partial listing of these interest rate models include those by Merton (1973) , Brennan and Schwartz (1977,1979,1980) , Vasicek (1977) , Dothan (1978) , Cox, Ingersoll and Ross (1980 , 1985) , Constant inides and Ingersoll (1984) ,Shaefer and Shwartz (1984) , Sundaresan (1984) , Feldman (1989) , Longstaff (1989a) , Hull and White (1990) , Black and Karasinski (1991) , and Longstaff and Shwartz (1992).

One of the key points in this area is how these models compare in terms of their ability to capture the actual behavior of the short-term riskless rate. The models mentioned above are nested in the general model defined as:

$$d_r = \left( \mathbf{a} + \mathbf{b} r \right) dt + \mathbf{s} r^g dz \quad (1.1)$$

Chan et. al (1992) has compared these different nested models by estimating a discrete-time specification of models. They used the GMM method to find the estimates of discretized model and they implemented the hypothesis testing methods developed by Newey and West (1987) in order to evaluate the restrictions imposed by various models on the unrestricted model defined in (1.1).

This paper uses Maximum Likelihood Estimator for estimation of two Non-Nested models for interest rates:

$$dr = \left( \mathbf{a} + \mathbf{b} r \right) dt + \mathbf{s}_0 \left( r^g + \mathbf{s}_1 \right) dw_t \quad (1.2)$$

$$dr = (\mathbf{a} + \mathbf{b}r)dt + \mathbf{s}_0 \left( e^{\mathbf{g}_1 r + \mathbf{g}_2} + \mathbf{s}_1 \right) dw_t \quad (1.3)$$

The discretized models defined in (1.2) and (1.3) will be used for MLE estimation. Also three other models which are nested in model (1.2) will be estimated in the same way. Vuong (1989) has proposed a method for comparing different models estimated by MLE. By using the time series of monthly Eurodollars rates from 1970-2002, we reject model (1.3) in favor of model (1.2). Then we will estimate some models like CIR, GBM and Vasicek which are nested models in (1.2). CIR and Vasicek Models have been used a lot by researchers in this area to model short term interest rates. Since all of them are nested in model (1.2), we just used the LR test to compare them with the more general model defined in (1.2) in which all of them are rejected.

Estimating discretized models has a bias. Since the discretized model is misspecified, the approximation of ML estimator is generally inconsistent. The indirect inference method will use simulations performed under the initial model, to correct for the asymptotic bias of the estimation.

The remainder of this paper is organized as follows. Section 2 describes the short term interest rate models examined in the paper. Section 3 discusses the econometric approach. Section 4 describes the theory of model selection criterion used in this paper. Section 5 describes the data. Section 6 presents the empirical results. Section 7 shows an example of option pricing for Nokia Company by using the interest rate model. Section 8 summarizes the paper and makes concluding remarks.

## 2- Interest Rate Models

Stochastic differential equations have been used to define the interest rate processes. The general format of a one-factor diffusion model is as below:

$$dr = \boldsymbol{\mu}(r, t)dt + \mathbf{S}(r, t)dw_t$$

Where  $dw_t$  is a Brownian motion process. Implementing different functional forms for  $\mathbf{m}$  and  $\mathbf{s}$ , will deliver various diffusion models. Researchers have tried to propose better models which have the capability to fit the data more efficiently and can be used for pricing the interest rate derivatives. Now some of these models which are used in this study, will be introduced:

*Model 1: Geometric Brownian Motion*

It is one of the interest rate models considered by Marsh and Rosenfeld (1983) and simply defined as:

$$dr = brdt + srdz$$

*Model 2 : Vasicek*

Vasicek model is the Ornstein-Uhlenbeck process used for deriving an equilibrium model of discount bond prices. This Gaussian process has been used extensively by others in valuing bond options, futures and other types of contingent claims. This process is defined as :

$$dr = (\mathbf{a} + \mathbf{br})dt + \mathbf{s} dz$$

*Model 3 : Cox, Ingersoll & Ross Square Root; CIR SR*

This is the square root process which appears in the Cox, Ingersoll and Ross (1985) single factor general equilibrium term structure model. This model has also been used extensively in developing valuation models for interest rate contingent claims. The CIR SR model implies that the conditional volatility of changes is proportional in r. This model is defined as:

$$dr = (\mathbf{a} + \mathbf{br})dt + \mathbf{s} r^{1/2} dz$$

*Model 4 : Generalized 1 Chan model*

This is the generalized format of the model proposed in Chan et al. (1992). The previous 3 models can be nested in this model which is:

$$dr = (\mathbf{a} + \mathbf{b}r)dt + \mathbf{s}_0(r^{\mathbf{g}} + \mathbf{s}_1)dz$$

*Model 5: Generalized 2 Chan model*

This is also another generalized format of Chan et al model in which the diffusion part is different from model 4 and is defined as:

$$dr = (\mathbf{a} + \mathbf{b}r)dt + \mathbf{s}_0(e^{\mathbf{g}_1 r + \mathbf{g}_2} + \mathbf{s}_1)dw_t$$

These different models can be used to define the interest rate patterns. One of the main challenging of the interest rate researches has been comparing different models and selecting the best one. Models 1-3 are classical models which their estimation have been easier than the last two one and the theory of pricing interest rate derivatives using these 3 models, have been developed, which prohibits computational difficulties.

## 4. Econometrics Approach

In this section, we describe the econometric approach used in estimating the parameters of the interest rate models and in examining their explanatory power for the dynamic behavior of short term interest rates.

Let us consider a general form for a continuous time process satisfying stochastic differential equations:

$$dr_t = \mathbf{n}(\mathbf{q}, r_t)dt + \mathbf{s}(\mathbf{q}, r_t)d w_t \quad (4.1)$$

Where W is a standard Brownian motion. If the only available observations correspond to integer dates 1,2,..., T, it is not possible in general to determine the analytical form of the likelihood function of the observations

$$l(r_1, \dots, r_T; \mathbf{q})$$

A common approach consists of replacing the initial continuous time model in (4.1) by its Euler discretization:

$$r_t = r_{t-1} + \mathbf{m}(r_{t-1}; \mathbf{q}) + \mathbf{s}(r_{t-1}; \mathbf{q}) \mathbf{e}_t \quad (4.2)$$

Where  $\mathbf{e}_t$  is Gaussian white noise. Next we estimate  $\mathbf{q}$  by the maximum likelihood method applied to the approximated model in (4.2) and likely conclude that the estimated

parameter  $\hat{\mathbf{q}}$  provides a good approximation of the unknown  $\mathbf{q}$ . The maximum likelihood estimator is defined as:

$$\hat{\mathbf{q}} = \underset{\mathbf{q}}{\text{ArgMin}} \sum_{t=1}^T \left\{ -0.5 \log \mathbf{s}^2(r_{t-1}, \mathbf{q}) - \frac{1}{2} \frac{[r_t - \mathbf{m}(r_{t-1}, \mathbf{q})]^2}{\mathbf{s}^2(r_{t-1}; \mathbf{q})} \right\}$$

However, Since the Euler discretization is an approximation, Model (4.2) is misspecified, causing an asymptotic bias of its estimators, which may be arbitrarily large. The indirect inference uses simulations performed under the initial model to correct for the asymptotic

bias of  $\hat{\mathbf{q}}$  which is Maximum Likelihood estimator of the discretized method (Gourieroux et al (1993)).

The idea is to introduce a second Euler discretization involving a very short time unit (i.e

$d=0.1$ ). More precisely, we define the process  $(y_t^{(d)}, t=kd, k \text{ varying})$  such that

$$r_{(k+1)d}^{(d)} = r_{kd}^{(d)} + d\mathbf{m}(r_{kd}^{(d)}, \mathbf{q}) + d(r_{kd}^{(d)}, \mathbf{q})\sqrt{d} \mathbf{e}_k^{(d)}$$

Where  $\mathbf{e}_k^{(d)}$ ,  $k \text{ varying}$ , is a Gaussian white noise. This finer Euler discretization can be used to simulate the continuous time process. Let us denote by

$$y_{kd}^{(d)s}(\mathbf{q}), \quad k=1, \dots, T/d$$

the simulated path corresponding to the parameter value  $\mathbf{q}$ , and  $r_t^{(d)}(\mathbf{q}) = r_t^{(d)s}(\mathbf{q})$ ,  $t=1, \dots, T$ , the values corresponding to the observation dates. The indirect inference approach includes the three estimation steps outlines below:

1) Estimation of  $\mathbf{q}$  from observations

$$\hat{\mathbf{q}} = \underset{\mathbf{q}}{\text{ArgMin}} \sum_{t=1}^T \left\{ -0.5 \log \mathbf{S}^2(r_{t-1}, \mathbf{q}) - \frac{1}{2} \frac{[r_t - m(r_{t-1}, \mathbf{q})]^2}{\mathbf{S}^2(r_{t-1}; \mathbf{q})} \right\}$$

2) Estimation of  $\mathbf{q}$  from simulations

$$\hat{\mathbf{q}}^S = \underset{\mathbf{q}}{\text{ArgMin}} \sum_{t=1}^T \left\{ -0.5 \log \mathbf{S}^2(r_{t-1}^s(\bar{\mathbf{q}}), \mathbf{q}) - \frac{1}{2} \frac{[r_{t-1}^{(s)}(\bar{\mathbf{q}}) - m(r_{t-1}^{(s)}(\bar{\mathbf{q}}), \mathbf{q})]^2}{\mathbf{S}^2(r_{t-1}^{(s)}(\bar{\mathbf{q}}); \mathbf{q})} \right\}$$

3) Calibration

$$\tilde{\mathbf{q}} = \underset{\bar{\mathbf{q}}}{\text{ArgMin}} \left[ \hat{\mathbf{q}}_T - \hat{\mathbf{q}}_T^S(\bar{\mathbf{q}}) \right] \Omega \left[ \hat{\mathbf{q}}_T - \hat{\mathbf{q}}_T^S(\bar{\mathbf{q}}) \right]$$

In Calibration the minimization is done over  $\bar{\mathbf{q}}$  which is the  $\hat{\mathbf{q}}_T^S$  that is found from step 2. So the optimization has nested another optimization method in its inside, which makes the computational solution difficult and very time consuming.

Gourieroux and Monfort (1996) have shown that for T sufficiently large the choice of  $\Omega$  can be arbitrary. So we can assume  $\Omega$  an Identity matrix.

## 5. Model Selection

Selecting models has been a popular theme in Econometrics. Log Likelihood ratio test (LR) can be used for testing a model which is nested in a more general model. It can be done by testing the hypothesis that the restrictions can be statistically accepted or not.

But in general case, we should compare two models which are defined as:

$$F_q = \{f(y|z; \mathbf{q}; \mathbf{q} \in \Theta)\}$$

$$G_g = \{g(y|z; \mathbf{g}; \mathbf{g} \in \Gamma)\}$$

Based on Kullback\_Leibler information criterion (KLIC) , Voung(1989) has proposed the following statistics under regularity conditions, for strictly non-nested models :

$$\frac{T^{-1/2} LR_T(\hat{\mathbf{q}}_T, \hat{\mathbf{g}}_T)}{\hat{W}_T} \quad (5.1)$$

Where  $\hat{\mathbf{q}}_T, \hat{\mathbf{g}}_T$  are MLE of models  $F_q$  and  $G_g$  and  $\hat{W}_T$  is :

$$\hat{W}_T = \frac{1}{T} \sum_{t=1}^T \left[ \text{Log} \frac{f(y_t|z_t; \hat{\mathbf{q}}_t)}{g(y_t|z_t; \hat{\mathbf{g}}_t)} \right]^2 - \left[ \frac{1}{T} \sum_{t=1}^T \text{Log} \frac{f(y_t|z_t; \hat{\mathbf{q}}_t)}{g(y_t|z_t; \hat{\mathbf{g}}_t)} \right]^2$$

The Vuong statistics is asymptotically distributed as standard normal distribution. So if the statistics is greater than a critical value we can accept the null hypothesis that  $F_q$  is better than  $G_g$ .

In general, this model selection can be used to compare different models estimated by using MLE method.

## 6. Data

The monthly Eurodollar interest rates in the period 1970-2002 are used in this study. Eurodollar deposits are dollar deposits held in banks outside United States and therefore exempt from Federal Reserve regulations. Appendix 1 shows the pattern of these market rates over the 32 years period which is 377 observations. Each observation is the average of the interest rate in each month.

## 7. Empirical Results

In this section, we present our empirical results. The parameters of five models described in section 2 have been estimated by using MLE method on discretized models. Ten Matlab programs for this estimation are written (appendix 2), and the results are as follows:

Model 1- GBM :

$$\beta = -0.0001, s = -0.0796$$

Model 2- Vasicek :

$$a = 0.1623, \beta = -0.0001, s = -0.0796$$

Model 3- CIR SR :

$$a = 0.0525, \beta = -0.0084, s = 0.2350$$

Model 4- Generalized 1 Chan :

$$a = -0.0340, \beta = 0.0070, s_0 = 0.2350, \gamma = 1.7130, s_1 = 8.0720$$

Model 5- Generalized 2 Chan :

$$a = -0.0387, \beta = 0.0077, s_0 = 0.9566, \gamma_1 = 0.0984, s_1 = -0.4510, \gamma_2 = -0.6962$$

Vuong test has been used for comparison of models 4 and 5. The statistics defined in (5.1) is 4.3993 which based on the normality distribution of the statistics, we can claim that model 4 should be preferred to model 5. (A Matlab program named ModelSelect in the appendix 2 calculates the statistics).

Now we can test the three models 1, 2, 3 relative to model 4. Since the 3 models are nested in model 4, we can use likelihood ratio test. This test shows whether the restrictions imposed on model 4 in order to produce the nested models, can be statistically accepted or not. The results are as follows:

$$2LR \text{ for model 1} = 16.2008$$

$$2LR \text{ for model 2} = 283.3434$$

2LR for model 3 = 110.7681

Since the statistics are much greater than significant levels, we can reject the GBM, Vasicek and CIR SR models in favor of Generalized Chan model (Model 4).

Now that we could find the best among the models, we can use the indirect inference method to find asymptotically consistent estimators of model 4. Matlab programs shown in appendix 3 have been used to estimate the consistent estimators. For lowering the variance of the simulated data, the method of Antithetic variable technique<sup>2</sup> has been used (Hull (2003a)). The results are as follows:

$a = -0.0686$  ,  $\beta = 0.0058$  ,  $s_0 = 0.0024$  ,  $\gamma = 1.4934$  ,  $s_1 = 8.1381$

## 8. Pricing the option by using stochastic interest rates

After finding the parameters of a diffusion model for interest rates, we can use it for pricing the contingent claims on interest rates or we can implement it for pricing the derivatives of other securities since interest rate is an important part of all no arbitrage contingent claim pricing.

If we assume the stock price process has a dynamic which can be shown by a Geometric Brownian Motion:

$$ds = \mu s dt + \sigma s dz$$

Then the value of any contingent claim  $f(s)$  at time T can be written as (Black and Sholes (1973)):

$$e^{-rT} \hat{E}(f(s_T)) \quad (8.1)$$

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<sup>2</sup> I am grateful to Dr. Levon Goukasian for mentioning this technique.

Where  $r$  has been assumed as constant interest rate and expectation is the risk neutral measure defined as:

$$ds = r s dt + \mathbf{S} s dz \quad (8.2)$$

Now if we want to be more realistic, we should relax the assumption of constant interest rate and use the stochastic interest rate model. Hull (2003b) has shown that (8.1) pricing still holds but we should just use average interest rate instead of  $r$  in (8.1) and (8.2), which is defined as :

$$\bar{r} = \frac{1}{T} \int_0^T r(s) ds$$

Now we can use Monte Carlo Simulation method for pricing any derivative of a stock. The risk neutral distribution of stock price is

$$S_T = S_0 \exp \left\{ \mathbf{S} W_T + \left( \bar{r} - \frac{1}{2} \mathbf{S}^2 \right) T \right\}$$

Where  $W_T$  is distributed as  $N(0,T)$ . The idea of Monte Carlo Simulation is to generate  $N$  numbers of  $S_T$  and calculating the contingent claim price by

$$e^{-\bar{r}T} \times \frac{1}{N} \sum_{i=1}^N f(S_T^i)$$

In which  $S_T^i$  is generated by

$$S_T^i = S_0 \exp \left\{ \mathbf{S} W_T^i + \left( \bar{r} - \frac{1}{2} \mathbf{S}^2 \right) T \right\}$$

And  $\bar{r}$  is the average of stochastic  $r$  computed by:

$$\bar{r} = \frac{1}{TS} \times \sum_{t=1}^T \sum_{s=1}^S r^s(t)$$

Where  $r^s(t)$  is one realization of interest rate at time  $t$  which, is produced by using the diffusion model found in section 6.

As an example this method has been implemented to price the call options of Nokia Company with expiration date of January 2004 and strike prices 15 USD and 20 USD. The monthly price data of Nokia has been used to estimate the  $\sigma$ , which is the variance of log difference of stock prices (Gurrierex and Jasiak (2001)). Matlab program in appendix 4 is used to price these call options and these prices found:

For strike price 15 USD = 3.7300

For strike price 20 USD = 1.7099

## 9. Conclusion

In this paper, we compared five different models of short-term interest rate dynamics in order to determine which model best fits the short term Eurodollar yield data. Two models were strictly non-nested and the other three, were nested in one of the models.

The results of the tests show that the popular models: GBM, Vasicek and CIR SR perform poorly relative to the Generalized model of Chan. Also, we found consistent estimator of this model by using Indirect Inference method.

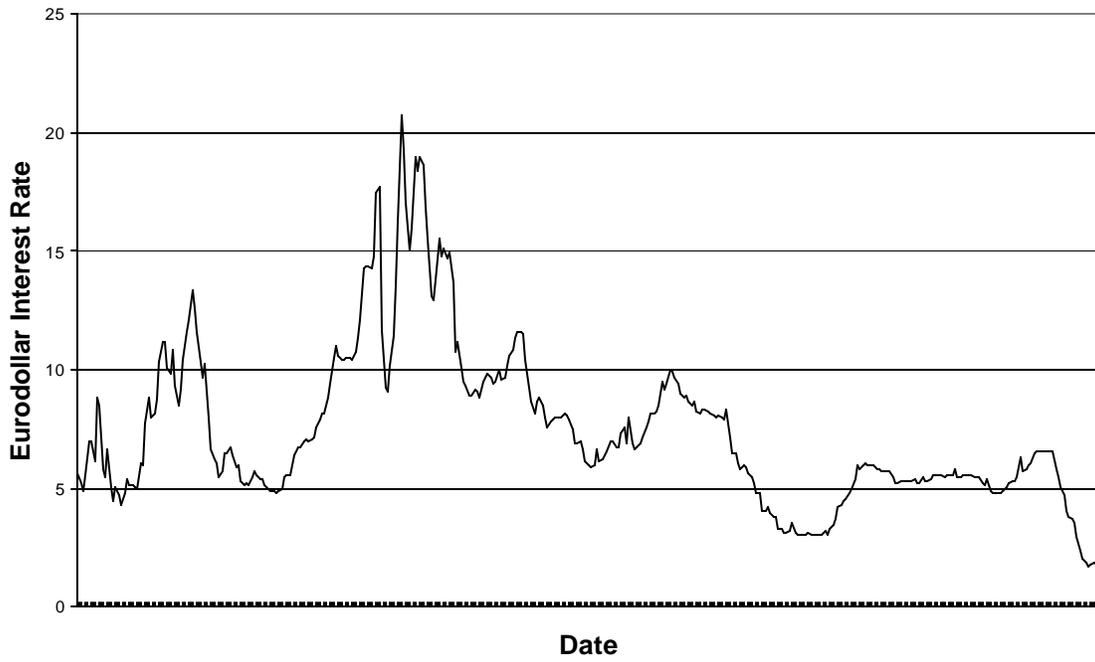
At the end, we used the stochastic model of interest rate to price the call options of Nokia Company. We used the Monte Carlo method for finding the price of call options.

Much works remain to be done. One of the main assumptions of Vuong's model selection criterion is that the data are i.i.d which normally does not hold for time series data. Rivers and Vuong (2002) have extended the model selection criterion to time series data, which should be used in this future studies in this area.

Also with using the data of the interest rate derivatives' prices, we can find the market price of risk for interest rate contingent claims, which can be used for pricing the swaps, futures, options of interest rates.

# Appendix 1

EuroDollar Interest Rate from 1971-2002



## Appendix 2

### Matlab files for Estimation of MLE of Discretized Models

#### *% Program for ML estimation of GBM model*

```
initial=[ -0.0051 -0.7731 ]
options = optimset('maxfunvals',580,'maxiter',7000);
[tetas,fval,exitflag]=fminsearch(@likelim5,initial,options)
```

#### *% Defining the likelihood function for GBM Model*

```
function x=likelim5(param)
```

```
r=wk1read('c:\user\datamonthly');
beta=param(1);
sigma=param(2);
nn=size(r);
n=nn(1,1);
v=zeros(n,1);
for i=2:n,
```

```
    v(i)=.5*log((sigma*r(i-1))^2)+.5*((r(i)-r(i-1)-beta*r(i-1))/(sigma*r(i-1)))^2;
```

```
end
```

```
    wk1write('c:\user\vmod5',v);
end,
x=sum(v)
```

#### *% Program for ML estimation of Vasicek model*

```
initial=[ 0.2623 -0.0232 0.1]
options = optimset('maxfunvals',580,'maxiter',7000);
[tetas,fval,exitflag]=fminsearch(@likelim4,initial,options)
```

#### *% Defining the likelihood function for Vasicek Model*

```
function x=likelim4(param)
```

```
r=wk1read('c:\user\datamonthly');
alfa=param(1);
beta=param(2);
sigma=param(3);
nn=size(r);
n=nn(1,1);
v=zeros(n,1);
```

```
for i=2:n,
```

```
    v(i)=.5*log(sigma^2)+.5*((r(i)-r(i-1)-alfa-beta*r(i-1))/(sigma))^2;
```

```

end
    wk1 write('c:\user\vmod4',v);
end,
x=sum(v)

% Program for ML estimation of CIR SR model

initial=[ 0.0525 -0.0084 0.2351]
options = optimset('maxfunevals',580,'maxiter',7000);
[tetas,fval,exitflag]=fminsearch(@likelim3,initial,options)

% Defining the likelihood function for CIR SR Model
function x=likelim3(param)

r=wk1read('c:\user\datamonthly');
alfa=param(1);
beta=param(2);
sigma=param(3);
nn=size(r);
n=nn(1,1);
v=zeros(n,1);

for i=2:n,
    v(i)=.5*log((sigma*(r(i-1)^0.5))^2)+.5*((r(i)-r(i-1)-alfa-beta*r(i-1))/(sigma*(r(i-1)^0.5)))^2;
end
    wk1 write('c:\user\vmod2',v);
end,
x=sum(v)

%Program for ML estimation of Generalized 2 Model
initial=[-0.0340 0.0070 0.0145 1.7123 8.0456]
options = optimset('maxfunevals',580,'maxiter',5000);
[tetas,fval,exitflag]=fminsearch(@likeli,initial,options)

% Defining the likelihood function for Generalized 1 Model
function x=likeli(param)

r=wk1read('c:\user\datamonthly');
alfa=param(1);
beta=param(2);
sigma0=param(3);
gama=param(4);
sigma1=param(5);
nn=size(r);
n=nn(1,1);
v=zeros(n,1);

for i=2:n,

    v(i)=.5*log((sigma0*(r(i-1)^gama+sigma1))^2)+.5*((r(i)-r(i-1)-alfa-beta*r(i-1))/(sigma0*(r(i-1)^gama+sigma1)))^2;

```

```

end

    wk1write('c:\user\vmod1',v);
end,
x=sum(v);

% Program for ML estimation of Generalized 2 Model

initial=[ -0.0387  0.0077  0.9566  0.0984  -0.4510  -0.6962]
options = optimset('maxfunevals',580,'maxiter',7000);
[tetas,fval,exitflag]=fminsearch(@likelim2,initial,options)

% Defining the likelihood function for Generalized 2 Model
function x=likelim2(param)

r=wk1read('c:\user\datamonthly');
alfa=param(1);
beta=param(2);
sigma0=param(3);
gama1=param(4);
sigma1=param(5);
gama2=param(6);
nn=size(r);
n=nn(1,1);
v=zeros(n,1);

for i=2:n,
    v(i)=.5*log((sigma0*(exp(r(i-1)*gama1+gama2)+sigma1))^2)+.5*((r(i)-r(i-1)-alfa-beta*r(i-1))/(sigma0*(exp(r(i-1)*gama1+gama2)+sigma1)))^2;
end
    wk1write('c:\user\vmod2',v);
end,
x=sum(v);

% Young test for model selection of two different functional forms
% Function name : modelselect

n=377;
vmod1=wk1read('c:\user\vmod1');
vmod2=wk1read('c:\user\vmod2');
lr=vmod1-vmod2;
ws=var(lr)
young=(n^(-.5)*sum(lr))/ws

```

## Appendix 3

### Matlab files for Indirect Inference Estimation of Model 4

*% Function name: endmain*

% This is the main program for finding the parameters  
% from Indirect Inference method  
% This program calls optn function which tries to minimize  
% the objective function defined in calibration step

```
tic;  
initial=[-0.0340 0.0070 0.0145 1.7130 8.0720]  
options = optimset('maxfunvals',500,'maxiter',200)  
[tend,fval,exitflag]=fminsearch(@optn,initial,options)  
toc
```

*% Function name: Simdat*

% This function gets the parameter of the model and generate the simulated data

```
function xx=simdat(param)  
alfa=param(1);  
beta=param(2);  
sigma0=param(3);  
gama=param(4);  
sigma1=param(5);
```

```
d=0.1; % d is the increment of simulation in each step  
m=1/d;  
n=50; % n is the number of simulated paths
```

```
for j=1:n  
    r(1,j)=5.66;  
    for i=2:377  
        t1(1)=r(i-1,j);  
        t2(1)=r(i-1,j);  
        for k=2:m+1  
            eps=randn(1);  
            t1(k)=t1(k-1)+d*(alfa+beta*t1(k-1))+sigma0*(t1(k-1)^gama+sigma1)*d^.5*eps;  
            t2(k)=t2(k-1)+d*(alfa+beta*t2(k-1))-sigma0*(t2(k-1)^gama+sigma1)*d^.5*eps;  
        end  
        r(i,j)=0.5*(t1(k)+t2(k)); % This is the Anthitic variance reduction technique  
        if r(i,j)<0  
            r(i,j)=0;  
        end  
    end  
end  
end  
end  
xx=r;
```

```

% Function name: likelis
% This function gets the parameter of the model and will produce the simulations
% of the model with those parameters (via simdat function) and then produces the
% likelihood function of the simulated data

function x=likelis(param)

alfa=param(1);
beta=param(2);
sigma0=param(3);
gama=param(4);
sigma1=param(5);

f=wk1read('c:\user\sdat');

nn=size(f);
m=nn(1,1);
n=nn(1,2);
v=zeros(m,n);

for i=2:m
    for j=1:n

        v(i,j)=.5*log((sigma0*(f(i-1,j)^gama+sigma1))^2)+.5*((f(i,j)-f(i-1,j)-alfa-beta*f(i-1,j))/(sigma0*(f(i-1,j)^gama+sigma1)))^2;

    end
end

xx=sum(v);
x=sum(xx);

% Function name: optn
% This function calculates the value of the objective function
% in the calibration step

function x=optn(param)

teta=[-0.0340  0.0070  0.0145  1.7130  8.0720];

y=tetas(param);
x=(teta-y)*(teta-y);
wk1write('c:\user\optn',x);
optn=x

% Function name: tetas
% This function finds the MLE estimation of the parameters
% with the data that are simulated by simdat program

function x=tetas(param)

```

```
initial=[-0.0340 0.0070 0.0145 1.7130 8.0720];  
  
options = optimset('maxfunvals',500,'maxiter',60);  
  
gg=simdat(param);  
wk1write('c:\user\sdat',gg);  
  
[t,fval,exitflag]=fminsearch(@likelis,initial,options)  
x=t;
```

## Appendix 4

### Matlab files for Pricing the Nokia Call Options

```
% Function name : Nokiaoption  
% This function calculates the prices of two options of  
% Nokia Company which expires in January 2004 with strike  
% prices 15$ and 20$, on Kune 26th 2003.  
% This method uses stochastic interest rate model and  
% implements the Monte Carlo algorithm
```

```
clc;  
clear;  
% In this part the program reads the time series of prices  
% and finds the standard deviation which will be used for  
% option pricing  
ss1=wk1read('c:\user\nokia');  
ss=flipud(ss1);  
nss=size(ss);  
n=nss(1,1);  
for i=1:n-1  
    u(i)=log(ss(i+1)/ss(i));  
end  
umean=mean(u);  
sigma=sqrt(12)*sqrt((1/(n-1))*sum((u-umean).^2));  
% End of calculation of standard deviation
```

```
% Now the program starts generating n generalization  
% of stochastic interest rates with d increments  
alfa=-0.0686;  
beta=.0058;  
sigma0=.0024;  
gama=1.4934;  
sigma1=8.1381;
```

```
d=0.01;  
m=1/d;  
n=50;  
t=12
```

```
for j=1:n  
    r(1,j)=3.2;  
  
    for i=2:t  
        t1(1)=r(i-1,j);  
        t2(1)=r(i-1,j);  
  
        for k=2:m+1  
            eps=randn(1);
```

```

    t1(k)=t1(k-1)+d*(alfa+beta*t1(k-1))+sigma0*(t1(k-1)^gama+sigma1)*d^.5*eps;
    t2(k)=t2(k-1)+d*(alfa+beta*t2(k-1))-sigma0*(t2(k-1)^gama+sigma1)*d^.5*eps;
end
r(i,j)=0.5*(t1(k)+t2(k));

if r(i,j)<0
    r(i,j)=0;
end
end
end

rm=mean(r');
rmean=mean(rm)*0.01
% End of calculating the mean interest rate until
% options expire

% Start of calculating the Monte Carlo estimation of
% option prices
s0=16.96;
x15=15;
x20=20;
n=5000;
t=7/12;
for i=1:n
    s(i)=s0*exp(rmean*t-0.5*sigma^2*t+sigma*sqrt(t)*randn(1));
    c15(i)=exp(-rmean*t)*(s(i)-x15)*((s(i)-x15)>0);
    c20(i)=exp(-rmean*t)*(s(i)-x20)*((s(i)-x20)>0);
end

callprice15=mean(c15) % Call option price for strike price =15
callprice20=mean(c20) % Call option price for strike price =20

```

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