OPTION PRICING BASED ON THE GENERALIZED LAMBDA DISTRIBUTION

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ABSTRACT

This paper proposes the generalized lambda distribution as a tool for modeling non-lognormal security price distributions. Known best as a facile model for generating random variables with a broad range of skewness and kurtosis values, potential financial applications of the generalized lambda distribution include Monte Carlo simulations, estimation of option-implied state price densities, and almost any situation requiring a flexible density shape. A multivariate version of the generalized lambda distribution is developed to facilitate stochastic modeling of portfolios of correlated primary and derivative securities.

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INTRODUCTION

Option pricing theory has advanced along many fronts since the pioneering breakthroughs by Black and Scholes (1973) and Merton (1973). At the forefront of these advances has been the development of option pricing solutions within the original framework of Black-Scholes-Merton, in which security return volatility is a deterministic function of the underlying security price and time, that is, $\sigma = \sigma(S,t)$. Within this classic framework, options can be used to form a dynamic, riskless hedge with the underlying security. An absence of arbitrage opportunities in financial markets implies that the resulting option prices are independent of investor risk preferences (Cox and Ross (1976), Harrison and Kreps (1979)).

The deterministic volatility approach to option valuation has been productive. The famous Black-Scholes (1973) formula represents the special case of a constant volatility diffusion process. Merton (1976) shows that substituting average volatility into the Black-Scholes formula yields the solution where volatility is solely a function of time. Cox and Ross (1976) derive the option pricing formula for the case of a constant elasticity relationship between volatility and the underlying security price. These are well-known solutions. The actual number of known solutions is too large for review in even comprehensive texts. Though large, this number is a fraction of the plausible forms for the deterministic volatility function $\sigma = \sigma(S,t)$, for which no general solution exists.

Flanking the traditional approach of developing specific solutions from exact parametric assumptions, several researchers have recognized the need for general solutions based on limited assumptions. These solutions are intended to apply across varied stochastic process specifications within the Black-Scholes-Merton framework, yet still provide sufficient accuracy to be useful to market participants. In an early effort, Jarrow and Rudd (1982) develop an option pricing formula that supplements the Black-Scholes formula with Edgeworth expansion terms related to the skewness and kurtosis of the underlying security price distribution. More recently, Rubinstein (1994) and Jackwerth and Rubinstein (1996) implement a nonparametric method to fit a security price distribution to observed options price data. In concert, Buchen and Kelley (1996) and Stutzer (1996) suggest methods of fitting a maximum entropy distribution to historical security returns data and currently observed option prices. Still another approach, well exemplified by Ait-Sahalia and Lo (1998), makes use of growing computing power to extract a price structure consistent with market data using kernel regression techniques.

Broadly described, a parametric option pricing model specifies in advance a stochastic process for a security price from which a future security price distribution can be derived. Evaluating this distribution on one side of a strike price yields the expected payoff to a European option. This approach dates at least to Bachelier (1900), and has advanced to provide a rich inventory of models with varied stochastic process assumptions. Most recent contributions focus on stochastic volatility and interest rate assumptions; a partial list includes Amin and Jarrow (1992), Amin and Ng (1993), Bailey and Stulz (1989), Bakshi and Chen (1997a,b), Bates (1996), Heston (1993), Hull and White (1987), Scott (1997) and Stein and Stein (1991). Bakshi, Cao and Chen (1997) provide extensive empirical tests based on an extended formula that nests varied models as special cases. They conclude that each nested model is to some degree misspecified by the criterion of internal parameter consistency, but each does in fact yield improved performance relative to a constant interest rate and volatility specification.

By contrast, nonparametric models typically begin with a future security price distribution, often estimated from observed market data. It is well known that a theoretically justified estimate of a future security price distribution, or at least its risk-neutral cousin, can be obtained from European options on the underlying security (Ross (1976), Breeden and Litzenberger (1978)). Derman and Kani (1994), Dupire (1994) and Rubinstein (1994) take this approach a step further to infer a stochastic process for a security price from its estimated price distribution. In particular, Rubinstein (1994) develops an implied binomial tree algorithm that yields a deterministic-volatility stochastic process from almost any plausible security price distribution.

The nonparametric methods cited immediately above share the practical advantage of being data-driven, with a minimal reliance on specific parametric assumptions. Notwithstanding, each approach has specific advantages and trade-offs in actual applications. The Jarrow-Rudd method is one of the simplest to implement, since it adds only two parameters to the Black-Scholes formula. Moreover, these two parameters are straightforward measures describing the skewness and kurtosis of the underlying security price distribution. Similar simplicity is achieved by the entrenched ‘street’ method of fitting a volatility smile to options price data. Dumas, Fleming and Whaley (1998) conclude that fitting a quadratic function to a volatility smile works as well in practical terms as more complex specifications. A quadratic implied volatility function also adds only two parameters to the Black-Scholes formula, in this case describing the shape of the volatility smile. By contrast, other methods cited earlier are parameter intensive, typically requiring a lengthy vector of parameters that individually provide an incremental description of the underlying price distribution. Nevertheless, parameter intensity is hardly a problem in actual implementations, since modern computing power is fully adequate to the task. Rather, the issue of parameter intensity pivots on the practical trade-off between underfitting a model to data with too few parameters and overfitting with too many.

Table I reproduces a data set originally presented in Rubinstein (1994) comprising 16 S&P 500 index call option prices and the underlying dividend-adjusted index level. This is an interesting data set because it reveals well some nettlesome characteristics of observed options price data. In Table I, the first column lists strike prices, while the second and third columns list the corresponding bid and ask call option prices for each strike. The next column reports bid-ask price averages. The last column presents prices of butterfly positions centered on three adjacent strike prices, reported only in instances where adjacent strikes are $5 apart. These butterfly prices are calculated using bid-ask average prices. Each butterfly price represents a long butterfly position created by buying a call for each of the two outer strikes and writing two calls for the inner strike.

As shown in Table I, butterfly positions centered on the 345 and 380 strikes have negative prices. These indicate points of non-convexity in the observed relationship between option prices and strike prices. Such non-convexities are common in archival options price data but do not necessarily represent arbitrage violations, since they may not reflect the full cost of trading at bid and ask prices.[1] Nonetheless, bid-ask average prices are standard inputs for estimating an implied distribution of future security prices from options price data (Jackwerth and Rubinstein (1996), Dumas, Fleming and Whaley (1998)). Consequently, an estimation method that allows overfitting an implied distribution function to options price data might yield unduly small or possibly even negative density values over strikes with negative observed butterfly prices. Indeed, Jackwerth and Rubinstein (1996) report that much of the difficulty encountered in estimating an implied price distribution from options price data stems from the trade-off between overfitting, which tends to produce jagged density estimates, and smoothing procedures designed to avoid overfitting. The jaggedness of the butterfly prices reported in Table I, typical in observed options price data, suggests that this not a trivial task.

This paper proposes the generalized lambda distribution as a flexible tool for modeling non-lognormal security price distributions. Originally developed as a facile method for generating random variables in Monte Carlo simulations, the generalized lambda distribution was later suggested by Ramberg et al. (1979) as a supple model for fitting a distribution to data since it permits a broad range of skewness and kurtosis values. Applied to the problem of fitting an implied security price distribution to options price data, it combines elements of the simplicity of Jarrow-Rudd and the versatility of Buchen-Kelley-Stutzer and Jackwerth-Rubinstein. Well known among statisticians, the generalized lambda distribution has not yet been advanced as an empirical tool in the finance literature.

The remainder of the paper is organized into six sections. The section immediately below describes the generalized lambda distribution and highlights its more salient details. The next section derives option pricing formulas based on the generalized lambda distribution. These formulas are applied in the following section to the problem of recovering a probability distribution implied by option prices. A procedure for exactly calibrating an implied binomial tree to the generalized lambda distribution comes next. Then the following section develops a multivariate version of the generalized lambda distribution. Finally, a summary and conclusion reviews the major advantages of the generalized lambda distribution in financial applications.

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[1] Other empirical anomalies in option prices have been documented. For example, Bakshi, Cao and Chen (2000) show that option prices do not always move with underlying stock prices in the theoretically predicted direction. Bergman, Grundy and Wiener (1996) provide a general discussion of rational option pricing.

THE GENERALIZED LAMBDA DISTRIBUTION

The generalized lambda distribution is primarily defined by its percentile function. For this reason, it is frequently applied to algorithms generating random variables in Monte Carlo simulations. The percentile function $s(p)$ immediately below defines a generalized lambda distribution with a zero mean and unit variance. The parameters λ_3 and λ_4 determine the shape of the generalized lambda density, and p represents a probability measure.

$$s(p) \quad = \quad \frac{1}{\lambda_2(\lambda_3, \lambda_4)} \left[p^{\lambda_3} - (1-p)^{\lambda_4} + \frac{1}{\lambda_4 + 1} - \frac{1}{\lambda_3 + 1} \right] \tag{1}$$

The scaling function $\lambda_2 = \lambda_2(\lambda_3,\lambda_4)$ specified in equation (2), in which $\beta(x,y)$ denotes the complete beta function, assumes the conditions $\lambda_3 \times \lambda_4 > 0$ and $min(\lambda_3,\lambda_4) > -1/2$ are satisfied.

$$\begin{aligned} \lambda_2(\lambda_3, \lambda_4) &= sign(\lambda_3) \times \sqrt{B - A^2} \\ A &= \frac{1}{\lambda_3 + 1} - \frac{1}{\lambda_4 + 1} \\ B &= \frac{1}{2\lambda_3 + 1} + \frac{1}{2\lambda_4 + 1} - 2\beta(\lambda_3 + 1, \lambda_4 + 1) \end{aligned} \tag{2}$$

Ramberg et al. (1979) point out that the conditions $\lambda_3 \times \lambda_4 > 0$ and $min(\lambda_3,\lambda_4) > -1/2$ suffice to ensure that $s(p)$ is a continuous, monotonically increasing function of the probability measure p. In Monte Carlo simulations, p-values drawn from a uniform random number generator enter the generalized lambda percentile function $s(p)$ to become random variables with population skewness and kurtosis determined by the shape parameters λ_3 and λ_4.

Ramberg et al. (1979) show that existence of the n^{th} moment of $s(p)$ requires that $min(\lambda_3,\lambda_4) > -1/n$. Thus, the condition $min(\lambda_3,\lambda_4) > -1/4$ ensures that fourth and lower moments exist. In this case, coefficients of skewness and kurtosis are well-defined functions of the shape parameters λ_3 and λ_4. Ramberg et al. (1979) provide these expressions for coefficients of skewness and kurtosis, in which A and B are defined in equation (2) above:

$$
\begin{aligned}
Skewness \;&=\; \frac{C-3AB+2A^3}{\left(B-A^2\right)^{3/2}} \qquad Kurtosis \;=\; \frac{D-4AC+6A^2B-3A^4}{\left(B-A^2\right)^{2}} \\
C \;&=\; \frac{1}{3\lambda_3+1}-\frac{1}{3\lambda_4+1}+3\beta\left(2\lambda_4+1,\,\lambda_3+1\right)-3\beta\left(2\lambda_3+1,\,\lambda_4+1\right) \\
D \;&=\; \frac{1}{4\lambda_3+1}+\frac{1}{4\lambda_4+1}-4\beta\left(3\lambda_4+1,\,\lambda_3+1\right)-4\beta\left(3\lambda_3+1,\,\lambda_4+1\right) \\
&\qquad +6\beta\left(2\lambda_3+1,2\lambda_4+1\right)
\end{aligned} \tag{3}
$$

Inverting the first derivative of the percentile function $s(p)$ yields the generalized lambda density function dp/ds:

$$
dp\,/\,ds \;=\; \lambda_2\left(\lambda_3,\,\lambda_4\right) / \left(\lambda_3 p^{\lambda_3-1} + \lambda_4(1-p)^{\lambda_4-1}\right)
$$

The condition $\lambda_3 \times \lambda_4 > 0$ along with the appropriate scaling function ensures that this density function is non-negative everywhere on the probability measure *p*.

Figure 1 graphs four generalized lambda densities with zero means, unit variances and increasing coefficients of kurtosis and negative skewness. The panel immediately below states the skewness and kurtosis values for these densities, along with corresponding values for their shape parameters λ_3 and λ_4. For any generalized lambda density, interchanging values for the parameters λ_3 and λ_4 reverses the direction of skewness. The equality $\lambda_3 = \lambda_4$ yields a symmetric density function with zero skewness.

Skewness = 0	Skewness = -1	Skewness = -3	Skewness = -6
Kurtosis = 3	Kurtosis = 6	Kurtosis = 30	Kurtosis = 2000
λ_3 = .134912	λ_3 = .038800	λ_3 = -.174915	λ_3 = -.248020
λ_4 = .134912	λ_4 = .018738	λ_4 = -.049470	λ_4 = -.047853

An analytic expression for the cumulative distribution function $p(s)$ of the generalized lambda distribution does not exist, except as the inverse of its percentile function $s(p)$. This is not a practical concern, since the same is true of the normal distribution and many popular distributions for which numerical methods enable function evaluation. An iterative bisection algorithm is perhaps the simplest method to evaluate the generalized lambda distribution function $p(s)$. That is, to obtain $p(c)$ first compare the value of *c* with $s(.5)$ and next with $s(.75)$ or $s(.25)$ as $c > s(.5)$ or $c < s(.5)$, respectively. Iterate this bisection procedure until $|s(p) - c| < \varepsilon$, with ε determining the desired level of precision. Fortran programs for all numerical procedures implemented in this paper are available from the author upon request.

The generalized lambda percentile function in equation (1) may be located and scaled to any desired mean and variance without affecting the density shape. In a security market context, setting the mean to $s_0 e^{rt}$ and the variance to $\left(s_0 e^{rt}\right)^2\left(e^{\sigma^2 t}-1\right)$ yields the percentile function shown in equation (4) below, in which s_0 is a current security price, r an appropriate interest rate, t the time to a future price realization and σ^2 a variance parameter.

$$s(p) \quad = \quad s_0 e^{rt}\left[1+\frac{\sqrt{e^{\sigma^2 t}-1}}{\lambda_2 \sigma(\lambda_3, \lambda_4)}\left(p^{\lambda_3}-(1-p)^{\lambda_4}+\frac{1}{\lambda_4+1}-\frac{1}{\lambda_3+1}\right)\right] \qquad (4)$$

The mean and variance of $s(p)$ in equation (4) have a form identical to that of the lognormal distribution adapted to a security market context (Jarrow and Rudd (1982)).

The percentile function in equation (4) generates random security prices from distributions with arbitrary means, variances and coefficients of skewness and kurtosis. Moreover, skewness and kurtosis values can be specified from a wide range of possible values.

There are several potentially interesting alternatives to the generalized lambda distribution that share some of its flexible shape characteristics. Classical examples are the Pearson family and the Johnson and Burr systems of distributions (Stuart and Ord (1987)). A well-known example from the finance literature is the generalized beta distribution of the second kind suggested by Bookstaber and McDonald (1987). Among these alternatives, only the Burr system possesses a percentile function in closed-form similar to equation (1) and is therefore well-suited to Monte Carlo simulation procedures. However, the Burr system has a limited range of negative skewness values (Rodriguez (1977), Tadikamalla (1980)), which may be an impediment in some applications.

Option pricing formulas consistent with the percentile function $s(p)$ in equation (4) might usefully supplement financial Monte Carlo simulations and other procedures involving options by allowing arbitrary assumptions regarding the skewness and kurtosis of the underlying security price distribution. Option pricing formulas for the generalized lambda distribution are developed in the next section.

OPTION PRICING FORMULAS BASED ON THE GENERALIZED LAMBDA DISTRIBUTION

The basic formulas

Pricing formulas for European call and put options based on the generalized lambda distribution (GLD) are readily derived from the security price percentile function in equation (4). With $p(s)$ representing a security price distribution on an option expiration date, equation (5) below states the expected expiration-date payoff to a European call option with strike price k, wherein the change of variable $s(p) = s$ is applied in moving from left- to right-hand sides of the equation.

$$\int_{k}^{\infty}(s-k)dp(s) \quad = \quad \int_{p(k)}^{1}(s(p)-k)dp \tag{5}$$

In equation (5), $p(k)$ denotes the probability that a call option expires out of the money. Substituting equation (4) into the right-hand side of equation (5), completing the integration and discounting the expected payoff yields this call price formula:

$$Call \quad = \quad s_0\, G_1 \quad - \quad e^{-rt} k\, G_2$$

$$G_1 \quad = \quad 1 - p(k) \quad + \quad \frac{\sqrt{e^{s^2 t} - 1}}{l_2\,(l_3, l_4)} \left(\frac{p(k) - p(k)^{l_3+1}}{l_3+1} + \frac{1 - p(k) - (1-p(k))^{l_4+1}}{l_4+1} \right) \tag{6.a}$$

$$G_2 \quad = \quad 1 - p(k)$$

Applying put-call parity results in this put price formula:

$$Put \quad = \quad e^{-rt} k [1 - G_2] \quad - \quad s_0 [1 - G_1] \tag{6.b}$$

In actual applications, the first four moments of a target security price distribution are matched to those of the generalized lambda percentile function in equation (4). Option prices are then calculated using the corresponding formulas in equation (6.a) or (6.b). Intuition suggests that the first four moments of a security price distribution will account for most of the stochastic influences affecting the underlying security price.

Comparisons with Black-Scholes formula values

A pertinent concern in many applications is how well the GLD option pricing formulas derived above might perform in applications where the Black-Scholes (1973) formula yields exact results. In the Black-Scholes model, future security prices are lognormally distributed with mean $s_0 e^{rt}$ and variance $\left(s_0 e^{rt}\right)^2 \left(e^{\sigma^2 t} - 1\right)$. These match the mean and variance of the generalized lambda percentile function in equation (4). To adapt the generalized lambda distribution to approximate the Black-Scholes model, skewness and kurtosis are matched to those of the target lognormal distribution.

Expressions for lognormal skewness and kurtosis are these, in which $q^2 = \left(e^{\sigma^2 t} - 1\right)$ (Jarrow and Rudd (1982)):

$$Skewness \;=\; 3q + q^3 \qquad Kurtosis \;=\; 3 + 16q^2 + 15q^4 + 6q^6 + q^8$$

For example, the volatility and time parameters $\sigma = .4$ and $t = .167$ imply lognormal skewness and kurtosis values of .4975 and 3.4433, respectively. Identical skewness and kurtosis values for the generalized lambda distribution are obtained with the parameter values $\lambda_3 = .054989$ and $\lambda_4 = .012446$ in equation (4). The IMSL Fortran routine BCPOL efficiently computes values for the parameters λ_3 and λ_4 from given skewness and kurtosis values, and is similar to the algorithm used by Ramberg et al. (1979) to produce their reference tables.

Table II reports option price and price derivative values based on the Black-Scholes formula and the GLD call price formulas in equation (6.a), with appropriately matched mean, variance, skewness and kurtosis values. All formulas share common parameter values for the security price $s_0 = 50$, volatility $\sigma = .4$, time $t = .167$ and interest rate $r = .05$.

Table II reveals that call price formulas derived from the generalized lambda distribution yield option price and price derivative values that differ from corresponding Black-Scholes formula values by at most a few cents. These deviations are similar in magnitude to what is obtainable using a binomial tree approximation of the Black-Scholes formula (Cox and Rubinstein (1985), Breen (1991)) and small relative to the price ticks of exchange traded options.

RECOVERING A PROBABILITY DISTRIBUTION IMPLIED BY OPTION PRICES

Option pricing formulas based on the generalized lambda distribution (GLD) adapt well to the problem of recovering a probability distribution from observed option prices. The effectiveness of such a recovery procedure is demonstrated here and compared with several alternative procedures using the options price data listed in Table I. For these price data, option maturity is 164 days, the interest rate is 9 percent and the dividend-adjusted index level is 349.21.

Fitting a generalized lambda distribution to the call option price data in Table I involves estimating the three parameters σ, λ_3, λ_4 by minimizing the sum of squares stated in equation (7) immediately below. In equation (7), $C(k_j,\sigma,\lambda_3,\lambda_4)$ represents the GLD call price formula from equation (6.a) with the arguments s_0, t and r suppressed. $C(k_j)$ denotes an observed call price for the strike price k_j.

$$\min_{\sigma,\lambda_3,\lambda_4} \sum_j \left(C(k_j,\sigma,\lambda_3,\lambda_4) - C(k_j)\right)^2 \qquad (7)$$

Minimizing this sum of squares using the GLD call price formula from equation (6.a) yields the volatility and shape parameters $\sigma = .168$, $\lambda_3 = -.065042$ and $\lambda_4 = -.020756$. These estimates for the parameters λ_3 and λ_4 imply skewness and kurtosis values of -1.58 and 8.55, respectively.

Table III compares fitted option prices obtained via this least squares procedure with fitted values obtained by Rubinstein (1994). Rubinstein's algorithm produces an implied security price distribution satisfying the restriction that all fitted option prices lie within observed bid-ask spreads. This algorithm was later abandoned in Jackwerth and Rubinstein (1996), as it did not yield solutions with some data sets. However, the fitted prices in the third column of Table III represent a solution for this data set. For the reader's convenience, strike prices and bid-ask average prices from Table I are listed again in the first and second columns, respectively, of Table III.

Table III also presents fitted option prices obtained using a volatility smile technique and two density expansion models. The volatility smile technique is based on the quadratic regression equation stated in equation (8) below, in which $IV(k)$ denotes a fitted implied volatility for an option with strike price k. Using this function adds just two parameters to the

Black-Scholes model. Dumas, Fleming and Whaley (1998) demonstrate the effectiveness of this parsimonious specification in modeling an implied volatility function.

$$IV(k) \quad = \quad .482 \quad - \quad .212\, k / s_0 \quad - \quad .096 \left(k / s_0\right)^2 \qquad R^2 = .995 \tag{8}$$

Summing coefficients in this regression yields an at-the-money ($k = s_0$) volatility of $\sigma = .173$. The high regression R^2 value indicates that the volatility smile fits a parabolic shape quite well. The fourth column of Table III reports fitted option prices for each strike computed by substituting fitted volatility values $IV(k)$ from equation (8) into the Black-Scholes formula.

The two density expansion methods are based on equation (9) below, in which C_X represents an expanded formula composed by a Black-Scholes formula price, denoted by C_{BS}, and skewness and kurtosis adjustment terms derived from the density function $a(S_t)$ for the expiration date security price. Coefficients c_3 and c_4 are measures related to skewness and kurtosis deviations, respectively, from a normal or lognormal distribution depending on the expansion employed.

$$C_X \quad = \quad C_{BS} + \frac{c_3}{3!}\frac{da(K)}{dS_t} + \frac{c_4}{4!}\frac{d^2a(K)}{dS_t^2} \tag{9}$$

In the Jarrow and Rudd (1982) model, equation (9) represents an Edgeworth expansion of the lognormal density. Alternatively, equation (9) can represent an expansion of the normal density (Rubinstein (1998)). Corrado and Su (1996, 1997a,b) show that both expansions are effective in flattening empirical volatility smiles implied by S&P 500 index option prices.

Following least squares procedures to simultaneously estimate implied volatilities and coefficients of skewness and kurtosis, the lognormal expansion model produces an implied volatility of $\sigma = .161$ and skewness and kurtosis values of -1.253 and 3.188, respectively. The normal expansion model produces an implied volatility of $\sigma = .178$ and skewness and kurtosis values of -1.648 and 4.934, respectively. The fifth and sixth columns of Table III report fitted call option prices for each strike using the lognormal and normal density expansion models, respectively.

Beneath each column of fitted option prices in the third through seventh columns of Table III are reported root mean squared deviations (RMSD) between fitted prices listed in that column and corresponding bid-ask average prices listed in the second column. These root mean squared deviations of fitted prices from bid-ask average prices represent a goodness of fit measure. The RMSD obtained from Rubinstein's fitted prices and the fitted volatility smile are $.27 and $.25, respectively. RMSDs based on the lognormal and normal density expansion models are $.19 and $.21, respectively. In comparison, the GLD call price formula in equation (6.a) produces call prices resulting in a RMSD of $.23. Differences in magnitude among these RMSDs appear insignificant relative to the average bid-ask spread of $.78 for these data.

Figure 2 presents probability densities inferred from the Rubinstein data using the three recovery methods described earlier. In Figure 2, the smooth, but distinguishably bimodal density is based on an expansion of the lognormal density. The bimodal density shape reported in Figure 4 of Rubinstein (1994) is similar, though less pronouncedly bimodal. Not shown here, an expansion of the normal density yields a recovered density shape that is also distinguishably bimodal, though slightly less so than that produced by the lognormal expansion.

Figure 2 also plots two unimodal densities with nearly identical shapes. These are the densities associated with the volatility smile in equation (8) and the GLD call price formula in equation (6.a). Little visible difference exists between these two densities, except that the mode of the generalized lambda density lies slightly to the left of the mode of the volatility smile density.

CALIBRATING AN IMPLIED BINOMIAL TREE TO THE GENERALIZED LAMBDA DISTRIBUTION

An implied binomial tree infers a stochastic process from a security price distribution and is widely used to calculate prices for options on futures and other American-style options from non-standard security price distributions. Rubinstein (1994) points out that his proposed implied binomial tree algorithm can be used with almost any plausible distribution to obtain a discrete time stochastic process with a valid limiting diffusion process. Construction of an implied binomial tree begins by assigning to each of its terminal nodes a nodal value and a nodal probability. An ordered sequence of N nodal values and probabilities uniquely defines an implied tree with N-1 steps. This section describes a method to precisely calibrate the nodal values and probabilities of an implied tree to the generalized lambda percentile function in equation (4).

The method is straightforward. To calibrate an implied binomial tree with N-1 steps leading to N terminal nodes, first form an ascending sequence of N+1 strike prices from $k_0 \equiv 0$ through $k_N \equiv \infty$. The N intervals k_{j+1} - $k_j > 0$, $j = 0,1,2,..,N\!-\!1$ need not be equally spaced, but in practice at least a subset will normally be selected to match observed equally spaced strike prices. Precise calibration to the generalized lambda percentile function in equation (4) occurs when European option prices calculated on the tree for each of the first N strikes match exactly prices calculated using the corresponding GLD option pricing formula. This requires satisfying the N equalities stated immediately below, in which $C(k_j,N)$ denotes a call price calculated on an implied tree with N terminal nodes and $C(k_j, s, l_3, l_4)$ represents a call price calculated from equation (6.a):

$$C(k_j, s, l_3, l_4) \quad = \quad C\left(k_j, N\right) \qquad \forall \;\; j = 0,1,2,\ldots, N-1$$

The procedure for determining nodal values and probabilities needed to satisfy these equalities is described immediately below.

Following the notation used earlier, the probability of a security price realization less than or equal to the strike price k_j is denoted by $p(k_j)$. The probability of a security price realization in the interval k_{j+1} - k_j is therefore $p(k_{j+1})$ - $p(k_j)$, where $p(k_{N+1}) \equiv 1$ and $p(k_0) = 0$. The probability $p(k_{j+1})$ - $p(k_j)$ is assigned to node j.

In terms of the generalized lambda percentile function in equation (4), expected values of security price realizations within the interval k_{j+1} - k_j are specified in equation (10):

$$E_p\left[s \mid k_j < s \le k_{j+1}\right] \quad = \quad \frac{\int_{p\left(k_j\right)}^{p\left(k_{j+1}\right)} s(p)\,dp}{p\left(k_{j+1}\right) - p\left(k_j\right)} \quad = \tag{10}$$

$$s_0 e^{rt}\left[1 \;+\; \frac{\sqrt{e^{s^2 t}-1}}{l_2\left(l_3, l_4\right)}\left[\frac{1}{p\left(k_{j+1}\right) - p\left(k_j\right)} \times \left[\frac{p(k_{j+1})^{l_3+1} - p(k_j)^{l_3+1}}{l_3+1} - \frac{(1-p(k_{j+1}))^{l_4+1} - (1-p(k_j))^{l_4+1}}{l_4+1}\right]\right]\right]$$

The expected value $E_p\left[s \mid k_j < s \le k_{j+1}\right]$ is assigned to node *j* for *j* = 0,1,2,..,*N*–1.

These nodal values $E_p\left[s \mid k_j < s \le k_{j+1}\right]$ and probabilities $p(k_{j+1})$ - $p(k_j)$ precisely calibrate an implied binomial tree to the percentile function in equation (4). This is demonstrated by summing expected values of all call option payoffs above the j^{th} strike price to yield this call price equality:.

$$
\begin{aligned}
C\left(k_j, N\right) &= e^{-rt}\sum_{i=j}^{N}\left(E_p\left[s \mid k_i < s \le k_{i+1}\right] - k_j\right)\left(p\left(k_{i+1}\right) - p\left(k_i\right)\right) \\
&= e^{-rt}\left(E_p\left[s \mid k_j < s \le k_{N+1}\right] - k_j\right)\left(1 - p\left(k_j\right)\right) \\
&= e^{-rt}E_p \max\left(0, s(p) - k_j\right) \\
&= C(k_j, s, \lambda_3, \lambda_4)
\end{aligned}
\qquad (11)
$$

In particular, the strike $k_0 = 0$ yields the initial security price $C(k_0,N) = s_0$.

This calibration method might be implemented with other distributions. The principal advantage of the generalized lambda distribution is its adaptability to Monte Carlo applications. It also allows assigning any of a broad range of skewness and kurtosis values, while maintaining a unimodal, bell-shaped density. This contrasts with density expansion methods that yield a unimodal density within only a limited range of skewness and kurtosis values (Stuart and Ord (1987)). For example, Rubinstein (1998) shows that an expansion of the normal density must limit the absolute value of skewness to less than one to maintain a unimodal form. The next section provides a multivariate extension of the generalized lambda distribution.

MULTIVARIATE GENERALIZED LAMBDA DISTRIBUTION

This section begins by developing a multivariate version of the generalized lambda distribution that facilitates stochastic modeling of multiple, correlated security prices. Immediately subsequent, the multivariate generalized lambda distribution is extended to a multi-index security returns model.

Bradley and Fleisher (1994) suggest obtaining correlated generalized-lambda variables via a transformation from normal variables. Their method proceeds by first generating correlated normal variables using a Cholesky decomposition of the correlation matrix **R**. They then compute a univariate *p*-value for each generated normal variable. These univariate *p*-values are then transformed into generalized-lambda variables as, for example, described earlier in this paper. The correlation matrix **G** represents the resulting correlation structure among these generalized-lambda variables.

Bradley and Fleisher (1994) point out that the generalized-lambda correlations and normal correlations composing the matrices **G** and **R**, respectively, will differ somewhat. Obtaining a precise target configuration for the matrix **G** requires making incremental adjustments to the off-diagonal elements of **R** until the resulting correlation structure in **G** converges to the target configuration. Bradley and Fleisher suggest a simulation algorithm to compute the generalized-lambda correlations in **G** at each adjustment. However, the accuracy of the algorithm depends on the size of the simulation and therefore may be computationally burdensome. A direct approach involves computing generalized-lambda correlations via the following integral, where the left-side expression $\gamma(\lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24})$ denotes a correlation between two generalized-lambda variables with the indicated shape parameters:

$$\gamma(\lambda_{13}, \lambda_{14}, \lambda_{23}, \lambda_{24}) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \sigma\left[N(z_1); \lambda_{13}, \lambda_{14}\right] \sigma\left[N(z_2); \lambda_{23}, \lambda_{24}\right] \nu(z_1, z_2; \rho)\, dz_1 dz_2$$

In the right-side integrand, $\sigma\left[N(z); \lambda_3, \lambda_4\right]$ is a unit-variance generalized-lambda variable (a la equation (1)) generated by the normal probability $N(z)$ and shape parameters λ_3 and λ_4. $\nu(z_1, z_2; \rho)$ is a bivariate standard normal density with the correlation parameter ρ. The IMSL Fortran routine TWODAQ performs this numerical integration quickly and efficiently.

The panel immediately below compares target generalized-lambda correlations with the normal correlations required to obtain them for the four generalized-lambda densities presented in Figure 1. The first two columns state skewness and kurtosis values for these densities. As shown, the required normal correlations are somewhat larger in absolute value than the generalized-lambda correlations they produce.

Skewness	Kurtosis	Target Generalized-Lambda Correlations γ			Required Normal Correlations ρ		
0	3						
-1	6	.1			.104		
-3	30	.3	-.5		.340	-.583	
-6	2000	.5	-.3	-.1	.793	-.377	-.141

Development of the multivariate generalized lambda system in the context of a multi-index model proceeds as follows: Equation (12) immediately below defines the security return y_j as a linear function of the security-specific return x_j and the N orthogonal index returns and factor weights f_F and $\beta_{F,j}$, respectively.[2]

$$y_j = x_j + \sum_{F=1}^{N} b_{F,j} f_F \qquad (12)$$

Exact expressions for the variance, skewness and kurtosis of the security return y_j are stated as

$$Var(y_j) = Var(x_j) + \sum_{F=1}^{N} b_{F,j}^2 Var(f_F)$$

$$Skew(y_j) = \left[Skew(x_j)\, Var^{3/2}(x_j) + \sum_{F=1}^{N} b_{F,j}^3 Skew(f_F)\, Var^{3/2}(f_F) \right] / Var^{3/2}(y_j) \qquad (13)$$

$$Kurt(y_j) = \left[\begin{aligned} & Kurt(x_j) Var^2(x_j) + 6 Var(x_j) \sum_{F=1}^{N} b_{F,j}^2 Var(f_F) \\ & + \sum_{F=1}^{N} b_{F,j}^4 Kurt(f_F) Var^2(f_F) \\ & + 6 \sum_{F=1}^{N-1} \sum_{H=F+1}^{N} b_{F,j}^2 b_{H,j}^2 Var(f_F) Var(f_F) \end{aligned} \right] / Var^2(y_j)$$

This multivariate system lends itself to modeling the stochastic properties of portfolios. Portfolio variance, skewness and kurtosis based on the weights w_j, $j = 1,2,..,P$, for a portfolio of P securities are stated by these expressions:

---

[2] Elton and Gruber (1987) review multi-index models and provide an extensive bibliography.

$$Var\left(\sum_{j=1}^{P} w_j y_j\right) = \sum_{j=1}^{P} w_j^2 Var(x_j) + \sum_{F=1}^{N} Var(f_F)\left(\sum_{j=1}^{P} w_j b_{F,j}\right)^2$$

$$Skew\left(\sum_{j=1}^{N} w_j y_j\right) = \frac{\sum_{j=1}^{P} w_j^3 Skew(x_j) Var^{3/2}(x_j) + \sum_{F=1}^{N} Skew(f_F) Var^{3/2}(f_F)\left(\sum_{j=1}^{P} w_j b_{F,j}\right)^3}{Var^{3/2}\left(\sum_{j=1}^{N} w_j y_j\right)} \quad (14)$$

$$Kurt\left(\sum_{j=1}^{N} w_j y_j\right) = \frac{\left(\begin{aligned} &\sum_{j=1}^{P} w_j^4 Kurt(x_j)\, Var^2(x_j) + 6\sum_{j=1}^{P-1}\sum_{k=i+1}^{P} w_k^2 w_j^2 Var(x_k) Var(x_j) \\ &+ 3\sum_{j=1}^{P}\sum_{F=1}^{N} w_j^2 \left(\sum_{k=1}^{P} w_k b_{F,k}\right)^2 Var(x_j) Var(f_F) \\ &+ \sum_{F=1}^{N}\left(\sum_{j=1}^{P} w_j b_{F,j}\right)^4 Kurt(f_F)\, Var^2(f_F) \\ &+ 6\sum_{F=1}^{N-1}\sum_{H=F+1}^{N}\left(\sum_{j=1}^{P} w_j b_{F,j}\right)^2 Var(f_F)\left(\sum_{j=1}^{P} w_j b_{H,j}\right)^2 Var(f_H) \end{aligned}\right)}{Var^2\left(\sum_{j=1}^{N} w_j y_j\right)}$$

These portfolio variance, skewness and kurtosis expressions might be used in conjunction with the GLD option pricing formulas derived earlier to calculate accurate option values for security market indexes or other security portfolios.

SUMMARY AND CONCLUSION

This paper introduced the generalized lambda distribution as a flexible tool for modeling non-lognormal security price distributions. Major advantages of the generalized lambda distribution include: the flexibility to assign almost any combination of skewness and kurtosis values, ease of random number generation owing to the simple form of its percentile function, and ready construction of a multivariate distribution in which each marginal distribution may be assigned unique skewness and kurtosis values. These advantages suggest several financial applications, including estimation of option-implied state price densities and risk analyses based on Monte

Carlo simulations wherein varied distributional assumptions may be applied to portfolios of primary and derivative securities.

The generalized lambda distribution is readily applied to the problem of recovering an implied security price distribution from market-observed option prices. An example of such a recovery procedure revealed that while estimates based on the generalized lambda distribution are comparable to those obtained using other methods, generalized-lambda estimates ensure a unimodal, bell-shaped density. Complementing the recovery of an implied distribution, a method to calibrate exactly an implied binomial tree to a generalized-lambda density was demonstrated.

Finally, a multivariate version of the generalized lambda distribution was developed that allows an arbitrary correlation structure among generalized-lambda variables. This development was extended to a multi-index security returns model and skewness and kurtosis expressions for linear combinations of correlated generalized-lambda variables were provided. This multivariate system would be particularly useful in Monte Carlo simulations based on multiple correlated securities where the experimenter wishes to assign arbitrary skewness and kurtosis values to the target security price distributions.

APPENDIX: CLAMPING THE GENERALIZED LAMBDA DISTRIBUTION

In the case of negative shape parameters, i.e., $\lambda_3,\lambda_4 < 0$, the range of support for the generalized lambda distribution spans the real line, thereby assigning positive probabilities to negative security price values. Jackwerth and Rubinstein (1996) impose clamping to force a price density to zero in regions where a price realization is disallowed. This imposition is readily applied to the generalized lambda distribution. The resulting security price percentile function that disallows negative price realizations is this generalization of equation (1).

$$s(p) \quad = \quad \frac{s_0 e^{rt}}{1-p(0)}\left[1+\frac{\sqrt{e^{\sigma^2 t}-1}}{\lambda_2\left(p(0);\,\lambda_3,\,\lambda_4\right)}\left\{p^{\lambda_3}-(1-p)^{\lambda_4}+\frac{\left(1-p(0)\right)^{\lambda_4+1}}{\lambda_4+1}-\frac{1-p(0)^{\lambda_3+1}}{\lambda_3+1}\right\}\right] \qquad \text{(A.1)}$$

The scaling function $\lambda_2(p(0);\lambda_3,\lambda_4)$ defined in equation (A.2) below, in which $\beta_p\left(x,y\right)$ denotes the incomplete beta function, correctly realigns the variance of $s(p)$ in equation (A.1) on $\left(s_0 e^{rt}\right)^2\left(e^{\sigma^2 t}-1\right)$.

$$\begin{aligned}
\lambda_2(p(0);\lambda_3,\lambda_4) \quad &= \quad sign\left(\lambda_3\right)\times\sqrt{B-A^2} \\
A \quad &= \quad \frac{1-p(0)^{\lambda_3+1}}{\lambda_3+1}-\frac{\left(1-p(0)\right)^{\lambda_4+1}}{\lambda_4+1} \\
B \quad &= \quad \frac{1-p(0)^{2\lambda_3+1}}{2\lambda_3+1}+\frac{\left(1-p(0)\right)^{2\lambda_4+1}}{2\lambda_4+1}-2\left(\beta\left(\lambda_3+1,\,\lambda_4+1\right)-\beta_{p(0)}\left(\lambda_3+1,\,\lambda_4+1\right)\right)
\end{aligned} \qquad \text{(A.2)}$$

The resulting call price formula is this generalization of equation (6.a).

$$\begin{aligned}
Call \quad &= \quad s_0 G_1 \quad - \quad e^{-rt} k G_2 \\
G_1 \quad &= \quad \left(1-p(k)\right)/\left(1-p(0)\right) \\
&\quad + \quad \frac{\sqrt{e^{\sigma^2 t}-1}}{\lambda_2\left(1-p(0)\right)}\left(\begin{array}{l}\dfrac{p(k)-p(0)+p(0)^{\lambda_3+1}-p(k)^{\lambda_3+1}}{\lambda_3+1} \\ -\dfrac{p(k)-p(0)+(1-p(k))^{\lambda_4+1}-(1-p(0))^{\lambda_4+1}}{\lambda_4+1}\end{array}\right) \\
G_2 \quad &= \quad \left(1-p(k)\right)/\left(1-p(0)\right)
\end{aligned} \qquad \text{(A.3)}$$

Put-call parity yields the corresponding put price formula:

$$Put \quad = \quad e^{-rt} k\left(1-G_2\right) \quad - \quad s_0\left(1-G_1\right) \qquad \text{(A.4)}$$

In equations (A.3) and (A.4), the scaling parameter is stated simply as λ_2 but is taken from equation (A.2). With $p(0) = 0$, equations (A.1), (A.3) and (A.4) all collapse to equations (4), (6.a) and (6.b), respectively, and therefore are preferred as a more general implementation.

An implementation of clamping might also use these generalizations of the coefficients of skewness and kurtosis from equation (3), in which *A* and *B* are defined in equation (A.2) above:

$$Skewness \quad = \quad \frac{C-3AB+2A^3}{\left(B-A^2\right)^{3/2}} \qquad Kurtosis \quad = \quad \frac{D-4AC+6A^2B-3A^4}{\left(B-A^2\right)^2}$$

$$\begin{aligned} C \quad = \quad & \frac{1-p(0)^{3l_3+1}}{3l_3+1} - \frac{\left(1-p(0)\right)^{3l_4+1}}{3l_4+1} + 3\left(b\left(2l_4+1, l_3+1\right) - b_{p(0)}\left(2l_4+1, l_3+1\right)\right) \\ & - 3\left(b\left(2l_3+1, l_4+1\right) - b_{p(0)}\left(2l_3+1, l_4+1\right)\right) \\ D \quad = \quad & \frac{1-p(0)^{4l_3+1}}{4l_3+1} + \frac{\left(1-p(0)\right)^{4l_4+1}}{4l_4+1} - 4\left(b\left(3l_4+1, l_3+1\right) - b_{p(0)}\left(3l_4+1, l_3+1\right)\right) \\ & - 4\left(b\left(3l_4+1, l_3+1\right) - b_{p(0)}\left(3l_4+1, l_3+1\right)\right) \\ & + 6\left(b\left(2l_3+1, 2l_4+1\right) - b_{p(0)}\left(2l_3+1, 2l_4+1\right)\right) \end{aligned}$$

It is worth noting that with plausible parameter values, the probability $p(0)$ is typically trivial. For example, the generalized lambda distribution was applied to fit an implied security price distribution to the options price data listed in Table 1. Despite the strong negative skewness exhibited by this fitted distribution, the derived probability of a negative price realization is only $p(0) = 8.4\text{E-}5$. Extending the maturity from the original 164 days to 365 days, while retaining all other parameter values, yields the probability $p(0) = 9.0\text{E-}4$.

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TABLE I

S&P 500 Index Call Options Data

Strikes	*Call option prices* *Bid*	*Ask*	*Bid-ask price averages*	*Butterfly prices*
0	349.16	349.26	349.210	
250	109.47	109.71	109.590	
275	85.66	86.71	86.185	
300	63.00	64.04	63.520	
325	42.00	42.75	42.375	
330	37.97	38.60	38.285	.130
335	34.01	34.64	34.325	.050
340	30.10	30.73	30.415	.285
345	26.45	27.13	26.790	-.030
350	22.79	23.48	23.135	.120
355	19.32	19.88	19.600	.195
360	15.98	16.54	16.260	.520
365	13.13	13.75	13.440	.215
370	10.52	11.15	10.835	.515
375	8.37	9.12	8.745	.370
380	6.65	7.40	7.025	-.050
385	4.91	5.60	5.255	

Strike, bid and ask prices are reproduced from Figure 4 in Rubinstein (1994). A butterfly price represents a long butterfly position created by buying a call for each of two outer strikes and writing two calls for the inner strike. Butterfly prices are based on bid-ask price averages and listed only in cases where adjacent strikes are $5 apart.

TABLE II

Comparison of Option Price and Price Derivative Values

Strike	*B-S*	*GLD*	*B-S*	*GLD*
	Call price		*Delta*	
40	10.588	10.587	.933	.938
45	6.480	6.450	.782	.784
50	3.451	3.434	.553	.546
55	1.599	1.612	.326	.322
60	.652	.671	.163	.164
	Gamma		*Theta*	
40	.016	.014	-.095	-.101
45	.036	.038	-.169	-.178
50	.048	.049	-.207	-.212
55	.044	.043	-.182	-.178
60	.030	.029	-.122	-.117
	Rho		*Vega*	
40	.060	.060	.029	.029
45	.053	.055	.063	.064
50	.039	.040	.081	.082
55	.023	.024	.072	.071
60	.012	.013	.048	.048

B-S indicates Black-Scholes formula values. GLD indicates values based on the generalized lambda distribution formula in equation (6.a). Delta, theta, rho, vega values are based on derivatives with respect to security price, time, interest rate and volatility, respectively. Gamma values are based on a second derivative with respect to the security price. GLD derivative values are computed numerically. Theta values are calibrated to reflect a one-week time decay. Common parameter values are: security price $s_0 = 50$, volatility $\sigma = .4$. time $t = .167$ and interest rate $r = .05$.

TABLE III

S&P 500 Index Observed and Fitted Call Prices

		Fitted call option values				
Strikes	*Bid-ask average*	*Rubin-stein*	*Volatility smile*	*Log-normal*	*Normal*	*GLD*
250	109.590	109.71	109.59	109.20	109.11	109.84
275	86.185	86.70	86.22	85.83	85.91	86.47
300	63.520	63.90	63.53	63.72	63.82	63.69
325	42.375	42.09	42.13	42.50	42.46	42.14
330	38.285	37.99	38.09	38.36	38.31	38.07
335	34.325	34.01	34.17	34.30	34.26	34.11
340	30.415	30.16	30.36	30.36	30.32	30.28
345	26.790	26.45	26.70	26.56	26.54	26.59
350	23.135	22.90	23.20	22.95	22.95	23.07
355	19.600	19.55	19.87	19.56	19.59	19.73
360	16.260	16.45	16.74	16.43	16.47	16.61
365	13.440	13.65	13.84	13.58	13.62	13.73
370	10.835	11.15	11.18	11.03	11.06	11.11
375	8.745	8.96	8.79	8.79	8.80	8.78
380	7.025	7.07	6.69	6.85	6.83	6.75
385	5.255	5.45	4.89	5.21	5.15	5.04
	RMSD	.27	.24	.19	.21	.23

Bid-ask averages are based on bid and ask call option prices reported in Figure 4 of Rubinstein (1994). Fitted call option values in the third column are reproduced directly from Rubinstein. Fitted values in the fourth through seventh columns are computed using the volatility smile specified in equation (8), lognormal and normal density expansion formulas represented by equation (9) and the GLD call price formula in equation (6.a). Root mean squared deviations (RMSD) in third through seventh columns are based on deviations from corresponding bid-ask averages in second column.

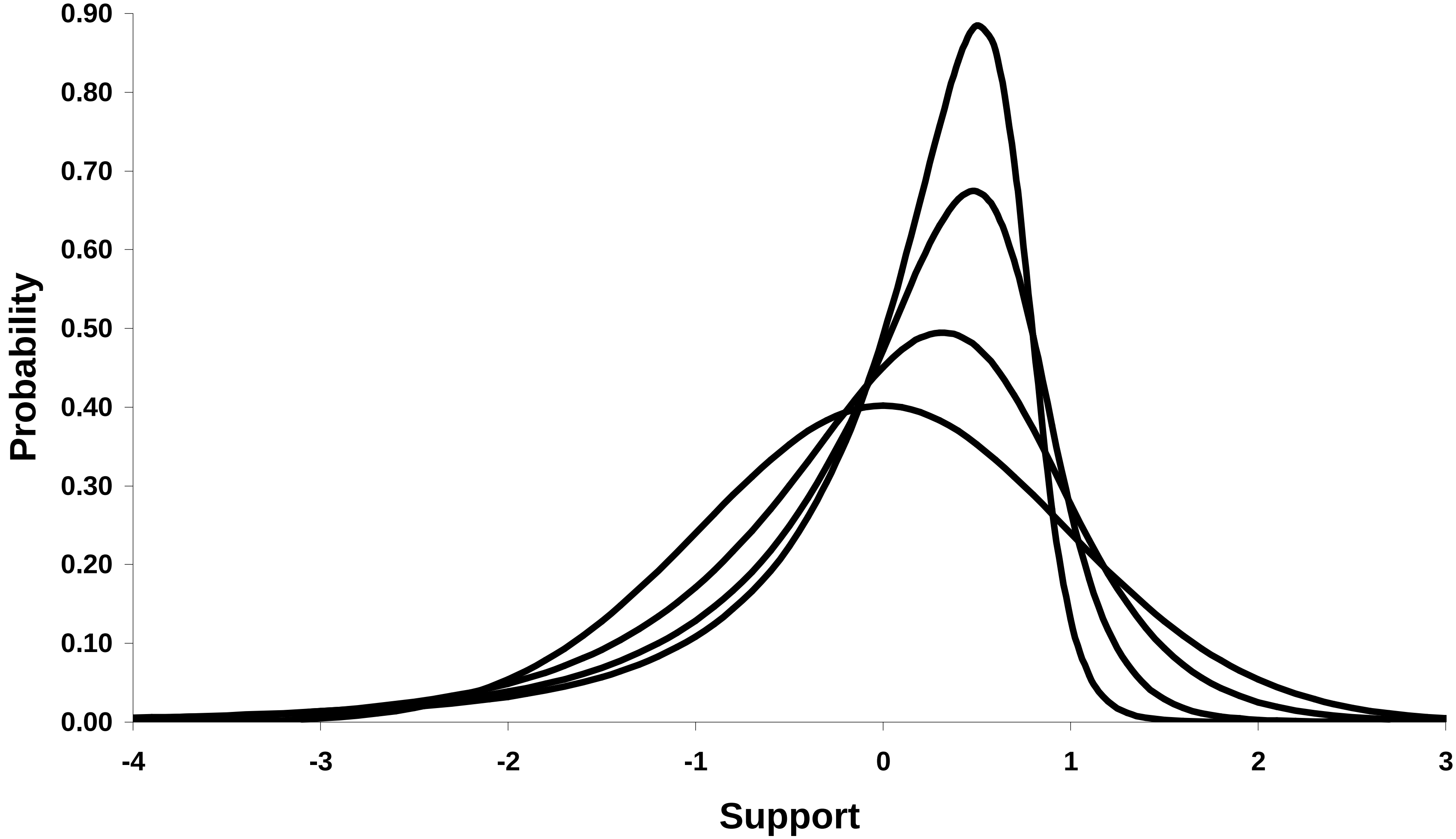

Figure 1

Generalized lambda density functions with unit variances.

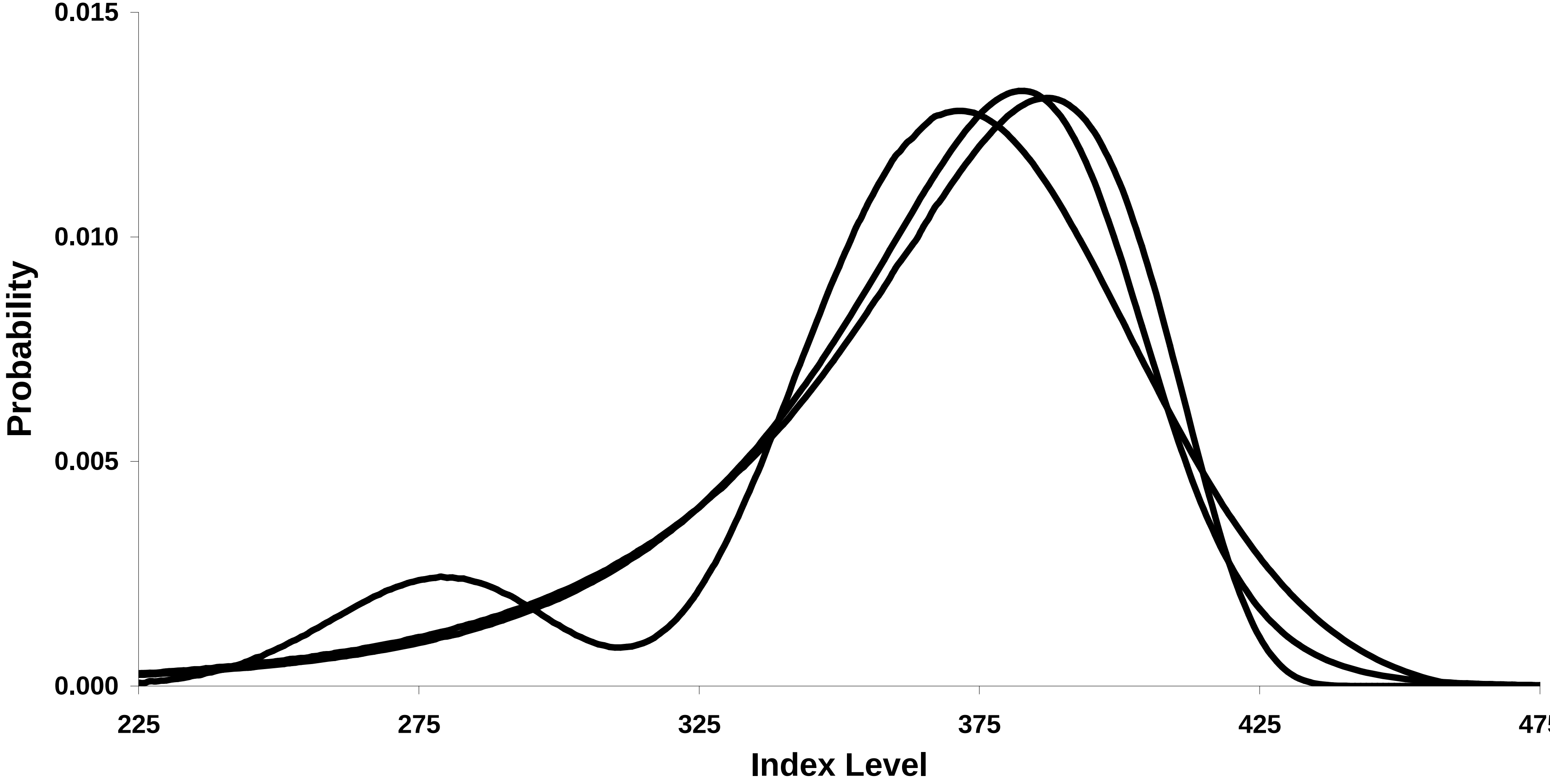

Figure 2

Option implied probability density functions.