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MGMT 657

CURRENCY OPTION PRICING II

- Calibrating the Binomial Tree to Volatility
- Black-Scholes Model for Currency Options
  - Properties of the BS Model
- Option Sensitivity Analysis
  - Delta
  - Gamma
  - Vega
  - Theta
  - Rho

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Calibrating the Binomial Tree

Instead of u and d, you will usually obtain the volatility, σ, either as

- **Historical volatility**
  - Compute the standard deviation of daily log return ($\ln(S_t/ S_{t-1})$)
  - Multiply the daily volatility by $\sqrt{252}$ to get annual volatility
- **Implied volatility**

➢ In reality, both types of volatility are available from, e.g., Reuters.

➢ In fact, OTC options are **quoted by implied volatility**.

Euro options quotes, Reuters, 11/27/2003

EURVOL EUR FX VOL LINKED DISPLAYS MONEY

EUR DEALING

SW	10.05	10.90	RBS	LON	RBSO	18:34
1M	9.90	10.40	TOKYO MITSUB	TOK	TMOT	20:08
2M	10.30	10.80	TOKYO MITSUB	TOK	TMOT	20:08
3M	10.60	10.80	RBS	LON	RBSO	20:07
6M	10.80	11.00	RBS	LON	RBSO	20:07
9M	10.85	11.00	RBS	LON	RBSO	20:07
1Y	10.90	11.05	RBS	LON	RBSO	18:06

➢ To construct a binomial tree that is consistent with a given volatility, set

$$u = e^{\sigma\sqrt{dt}}, \qquad d = e^{-\sigma\sqrt{dt}},$$

where $dt = \tau/n$ is the length of a time step (a subtree) and n is the number of time steps (subtrees) until maturity.

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Example. Let us construct a one-period tree that is consistent with a 10% volatility. Using this tree, we will price call and put options.

- Spot S = 1.15\$/€
- Strike K = 1.15\$/€
- Domestic interest rate $i^{\$}$ = 1.2% (continuously compounded)
- Foreign interest rate $i^{€}$ = 2.2% (continuously compounded)
- Volatility σ = 10% (annualized)
- Time to maturity = 6 months (τ = 0.5)

$u = \exp(.1\times\sqrt{.5}) = 1.0773$

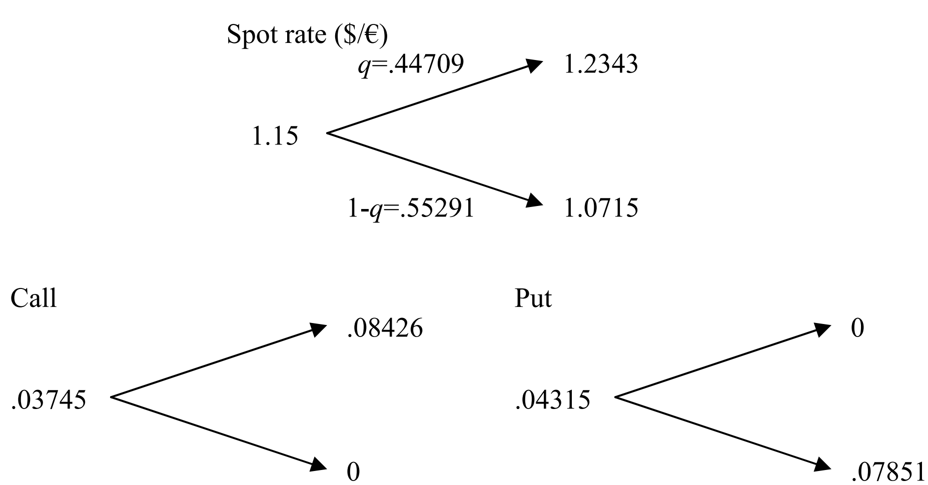
$d = \exp(-.1\times\sqrt{.5}) = 0.93173$

$q = [\exp((.012 - .022)\times.5) - d]/(u - d) = 0.44709$

Risk neutral pricing gives

$C = [.08426\times q + 0\times(1 - q)]\times\exp(-.012\times.5) = .03745$

$P = [0\times q + .07851\times(1 - q)]\times\exp(-.012\times.5) = .04315$



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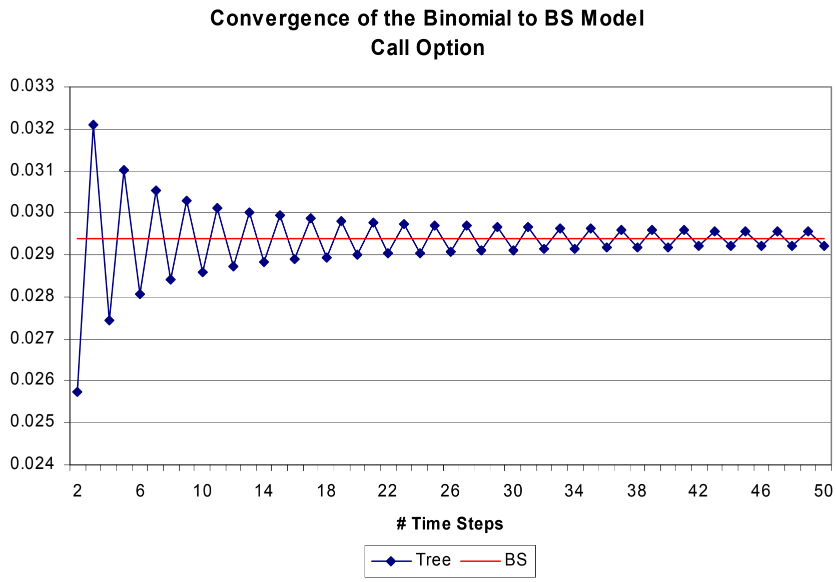
- As the number of time steps increases ($n \to \infty$ and $dt \to 0$), the binomial model price **converges to the Black-Scholes price**.

Time step *n*	1	5	10	50	100	500	BS
Call	0.0374	0.0310	0.0286	0.0292	0.0293	0.0294	0.0294
Put	0.0431	0.0367	0.0343	0.0349	0.0350	0.0351	0.0351

- Graphically:



Black-Scholes Model for Currency Options

To price currency options, you can use the Black-Scholes formula on a dividend-paying stock with the **dividend yield** replaced by the **foreign interest rate**.[1]

Notation

Today is date t, the maturity of the option is on a future date T.
$S_t^{d/f}$ or S: Spot rate on date t, value of currency f in currency d
F : Forward rate
K : Strike price
i^d : Interest rate on currency d (continuously compounded)
i^f : Interest rate on currency f (continuously compounded)
$\tau = T - t$: Time to maturity
σ : Volatility of the spot rate (annualized)
C : Call
P : Put

$$C = Se^{-i^f \tau} N(d_1) - Ke^{-i^d \tau} N(d_2), \qquad P = Ke^{-i^d \tau} N(-d_2) - Se^{-i^f \tau} N(-d_1) \quad (1)$$

where

$$d_1 \equiv \frac{\ln(S/K) + \left(i^d - i^f + \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}, \qquad d_2 \equiv \frac{\ln(S/K) + \left(i^d - i^f - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} \quad (2)$$

[1] This is known as the Garman-Kohlhagen model

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Note that, in the FX context, you can write the formula in terms of the **forward rate** so that the foreign interest rate (or even the spot rate!) does not appear.

Since $F = Se^{(i^d - i^f)\tau}$,

$$C = e^{-i^d \tau}[FN(d_1) - KN(d_2)], \qquad P = e^{-i^d \tau}[KN(-d_2) - FN(-d_1)] \tag{3}$$

where

$$d_1 \equiv \frac{\ln(F/K) + \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}}, \qquad d_2 \equiv \frac{\ln(F/K) - \frac{1}{2}\sigma^2 \tau}{\sigma\sqrt{\tau}} = d_1 - \sigma\sqrt{\tau} \tag{4}$$

- Discounting by the risk-free rate in Equation (3) indicates that the terms in the square brackets are **certainty equivalent** of the option payoff at maturity.
- Note: you do have to use the forward rate that **corresponds to the maturity** of the option.

Example.

- Spot S = 1.15\$/€
- Strike K = 1.15\$/€
- Domestic interest rate $i^{\$}$ = 1.2% (continuously compounded)
- Foreign interest rate $i^{€}$ = 2.2% (continuously compounded)

(Or you might observe the forward rate, F = 1.1443\$/€. Then use (3)-(4))

- Volatility σ = 10%
- Time to maturity = 6 months

$d_1 = [\ln(1.15/1.15) + (.012 - .022 + .1^2/2)\times.5]/(.1\times\sqrt{.5}) = -.035355$

$d_2 = d_1 - .1\times\sqrt{.5} = -.10607$

$N(d_1) = .48590$, $N(-d_1) = 1 - N(d_1) = .51410$

$N(d_2) = .44776$, $N(-d_2) = 1 - N(d_2) = .54224$

$C = 1.15\times e^{-.22\times.5}\times.48590 - 1.15\times e^{-.12\times.5}\times.44776 = .02939$

$P = 1.15\times e^{-.12\times.5}\times.54224 - 1.15\times e^{-.22\times.5}\times.51410 = .03509$

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Properties of the Black-Scholes Model for Currency Options

These are also the properties of the BS model on a dividend-paying stock.

1. The B-S model assumes the future spot rate is distributed **lognormally**. Without this strong assumption we can still put a **lower bound** on a European call option:

$$C \geq \text{Max}(S\ \exp(-i^{\text{f}}\tau) - K\ \exp(-i^{\text{d}}\tau), 0)$$

- To see this, let us first derive the following important result:

Result 1

The present value of receiving S_T at maturity is $S_t\exp(-i^{\text{f}}\tau)$.

- Note: this is **NOT** S_t. The value of foreign interest is subtracted.

This can be seen by considering the following investment strategy (take currency d = \$, currency f = €):

- Now: invest $\$S_t\exp(-i^{\text{f}}\tau)$ = €$\exp(-i^{\text{f}}\tau)$ in a euro deposit. Reinvest the interest continuously in the deposit itself.
- At maturity: you will receive €1 (=$\exp(-i^{\text{f}}\tau)\times\exp(i^{\text{f}}\tau)$) or equivalently $\$S_T$.

- The payoff of a call option at maturity is the larger of $S_T - K$ or 0.
- The PV of receiving $S_T - K$ **at maturity** is, from the above result,

$$S_t\exp(-i^{\text{f}}\tau) - K\exp(-i^{\text{d}}\tau).$$

- Since the value of a call option is never negative, we have the above inequality.
- Graphically:

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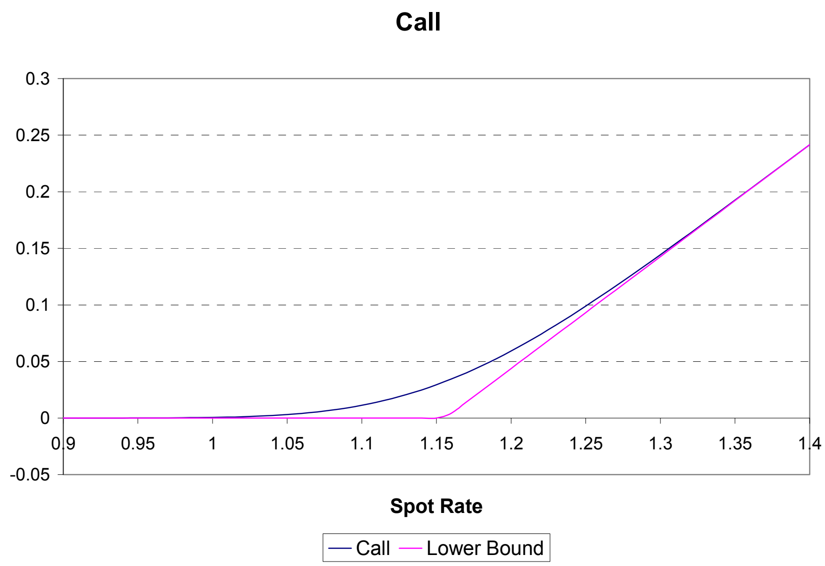
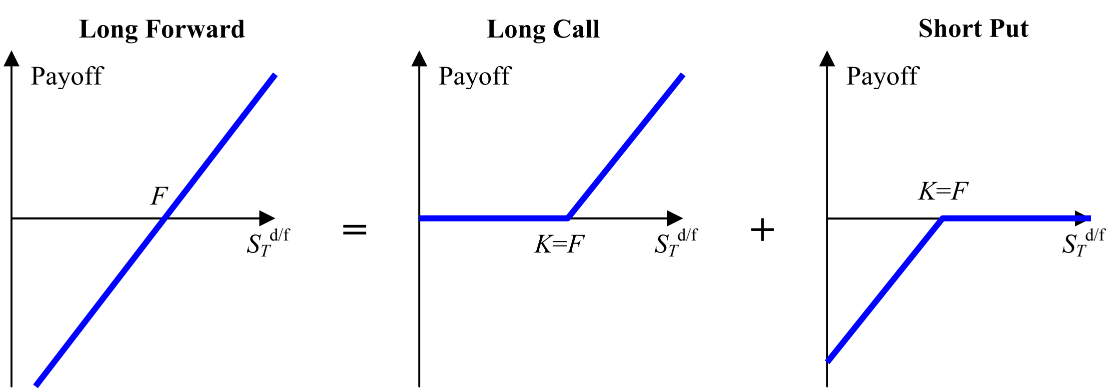


2. The prices of **forward ATM call** and **put** options are the **same**.
   - Graphically, recall that we can create a **synthetic forward** by a call and a put.
     - Since the forward costs nothing, the price of the call and the put must balance.



   - Mathematically, since we always have

$$N(-d_2) - N(-d_1) = 1 - N(d_2) - [1 - N(d_1)] = N(d_1) - N(d_2),$$

if $K = F$ in (3),

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$$C = e^{-i^d \tau} F[N(d_1) - N(d_2)], \qquad P = e^{-i^d \tau} F[N(-d_2) - N(-d_1)]$$
$$= e^{-i^d \tau} F[N(d_1) - N(d_2)] = C.$$

- Note: Equation (4) also becomes very simple:

$$d_1 = \sigma\sqrt{\tau} / 2, \qquad d_2 = -\sigma\sqrt{\tau} / 2 = -d_1.$$

Thus,

$$N(d_2) = 1 - N(-d_2) = 1 - N(d_1)$$

and therefore we can further write

$$C = P = e^{-i^d \tau} F[2N(d_1) - 1].$$

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3. **Digital (Binary) Options**.

- **Asset-or-nothing** (AON) and **cash-or-nothing** (CON) options are not really exotic. They are the **basic building blocks** of the Black-Scholes Model. By definition,

$$C = \text{AON}(S_T > K) - K\cdot\text{CON}(S_T > K)$$
$$P = K\cdot\text{CON}(S_T < K) - \text{AON}(S_T < K)$$

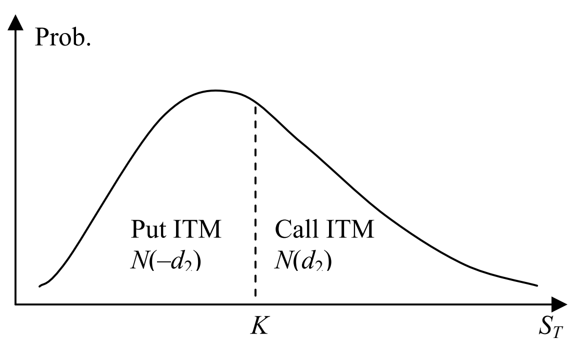
Compare with (1). We obtain:

$$AON(S_T > K) = Se^{-i^f\tau}N(d_1), \qquad CON(S_T > K) = e^{-i^d\tau}N(d_2)$$
$$AON(S_T < K) = Se^{-i^f\tau}N(-d_1), \qquad CON(S_T < K) = e^{-i^d\tau}N(-d_2)$$

From the expressions for CON, we know that the probability of

- the **call** ending up **in the money** ($S_T > K$) is $N(d_2)$,
- the **put** ending up **in the money** ($S_T < K$) is $N(-d_2)$.

Risk-neutral probabilities of the call and the put ending up ITM



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Option Sensitivity Analysis

Delta

The **delta** of an option (or a portfolio), Δ, is the rate of change in the price of the option (or portfolio) with respect to the spot rate.

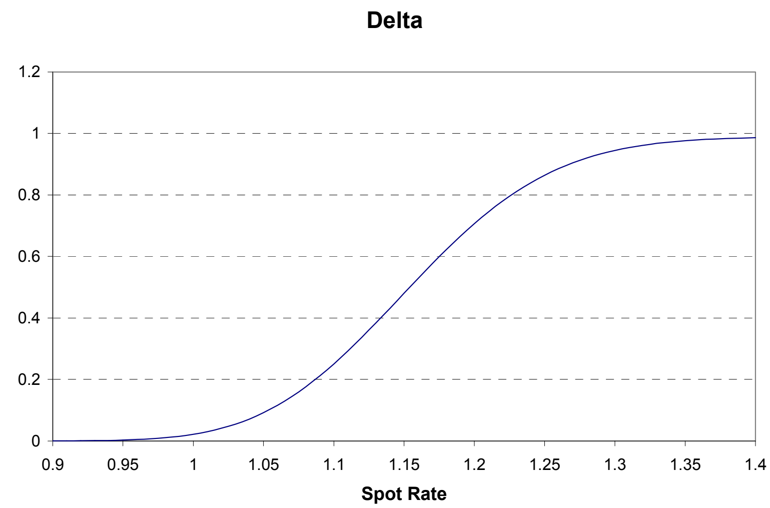
- Delta of a **European call**: $\Delta_{\text{Call}} = \dfrac{\partial C}{\partial S} = e^{-i^f \tau} N(d_1)$
  - Recall that $0 < \Delta_{\text{Call}} < 1$. In the B-S world, a tighter bound obtains:

    $$0 < \Delta_{\text{Call}} < \exp(-i^f \tau) < 1.$$

- Delta of a **European put**: $\Delta_{\text{Put}} = \dfrac{\partial P}{\partial S} = -e^{-i^f \tau} N(-d_1)$

  $$-1 < -\exp(-i^f \tau) < \Delta_{\text{Put}} < 0.$$

Delta of the European call option in the Example



As the figure shows,

- $\Delta_{\text{Call}} \rightarrow \exp(-i^f \tau)$ (closer to 1) as $S_t \rightarrow \infty$ (ITM).
- $\Delta_{\text{Call}} \rightarrow 0$ as $S_t \rightarrow 0$ (OTM).

❖ **Q1**. What is the limit of the delta of a put as $S_t \rightarrow \infty$ or $S_t \rightarrow 0$?

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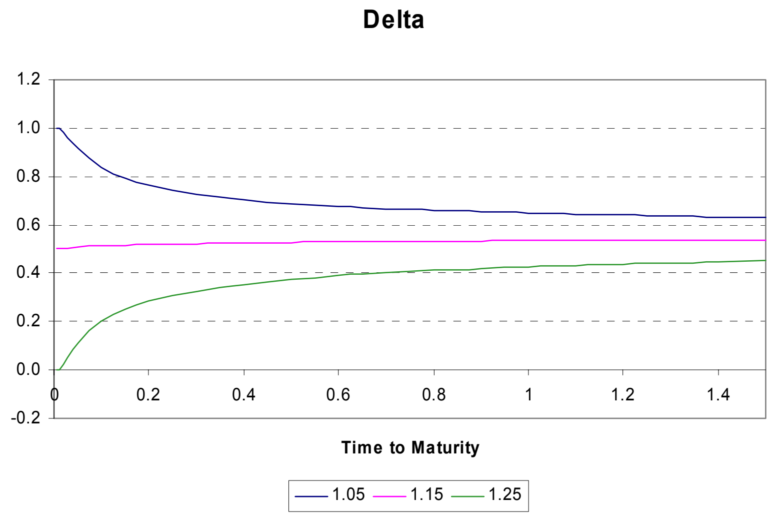
Delta of ITM, ATM, and OTM calls plotted against time to maturity



- **Q2**. In the above example, recall that $S = 1.15$ and $F = 1.1443$.
  - Why does the delta of the 1.25 call converge to 0?
  - Why does the delta of the 1.05 call become closer to 1?

- The delta of a **spot** contract is 1 (confirm this).
- The delta of a **forward** contract is $\exp(-i^f\tau)$.

- **Q3**. Show this by first writing down the value of a long forward contract (this is **different** from the formula for the forward **rate**) and then differentiating it with respect to the spot rate.

- The delta of a **futures** contract is $\exp((i^d - i^f)\tau)$. This is slightly different from the forward contract because of the mark-to-market.
  - The mark-to-market enables you to realize the gain or loss caused by the change in the spot rate immediately (daily).
- The delta of a **portfolio** is the **sum** of the deltas of its component assets.

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Example, continued. The delta of the above call option is

$$\exp(-i^f\tau)N(d_1) = \exp(-2.2\%\times0.5)\times0.48590 = 0.4806.$$

The delta of the put option is

$$-\exp(-i^f\tau)N(-d_1) = -\exp(-2.2\%\times0.5)\times0.51410 = -0.5085.$$

❖ **Q4**. A bank has sold the above put option on €1 million. How can the bank make its position delta neutral using the CME futures contract? The size of the CME euro futures contract is €125,000. What cares should be taken about this delta hedge?

Gamma

- The **gamma** of an option/portfolio is the rate of change of the option/portfolio's delta with respect to the spot rate.
- It is the **second partial derivative** of the option/portfolio price with respect to the spot rate.
- It measures the **curvature** of the relation between the option/portfolio price and the spot rate.
- The gamma of a European **call** option and a **put** option with the same strike price turns out to be the **same**:

$$\Gamma = \frac{\partial^2 C}{\partial S^2} = \frac{\partial^2 P}{\partial S^2} = \frac{e^{-i^f\tau} n(d_1)}{S\sigma\sqrt{\tau}}$$

where $n(\cdot)$ is the standard normal density function.

❖ **Q5**. Show this equivalence (not the formula) by the put-call parity.

- The gammas of a spot, forward, and futures contract are all zero.

❖ You should be able to derive this.

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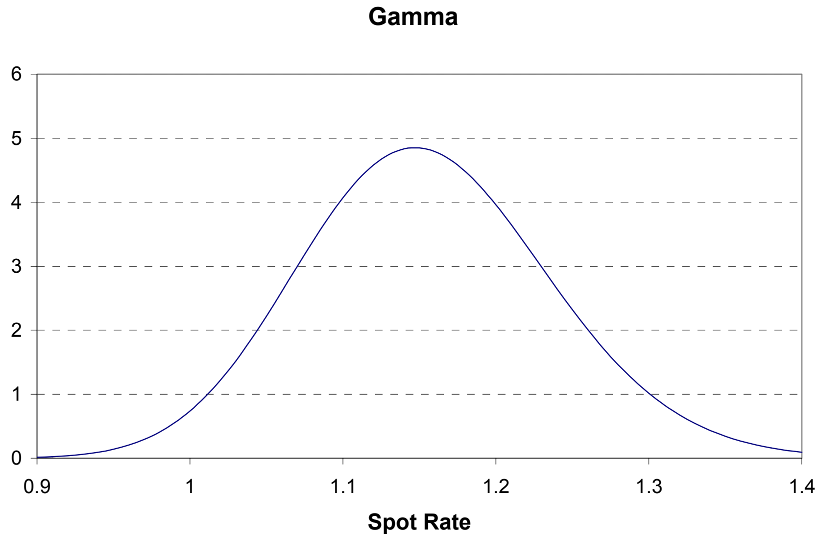
Gamma of the European call in the example



❖ Graphically, this is the slope of the graph of the delta. Confirm this.

- **Delta neutrality** provides protection against relatively **small** spot rate movements.
- **Delta-Gamma neutrality** provides protection against **larger** spot price movements.
  - Traders typically make their position **delta-neutral** at least once a day.
  - Gamma and vega (see below) are **more difficult** to zero away, because they **cannot** be altered by trading the spot, forward, or futures contract.
  - Fortunately, as time elapses, options tend to become **deep OTM or ITM**. Such options have **negligible** gamma and vegas.

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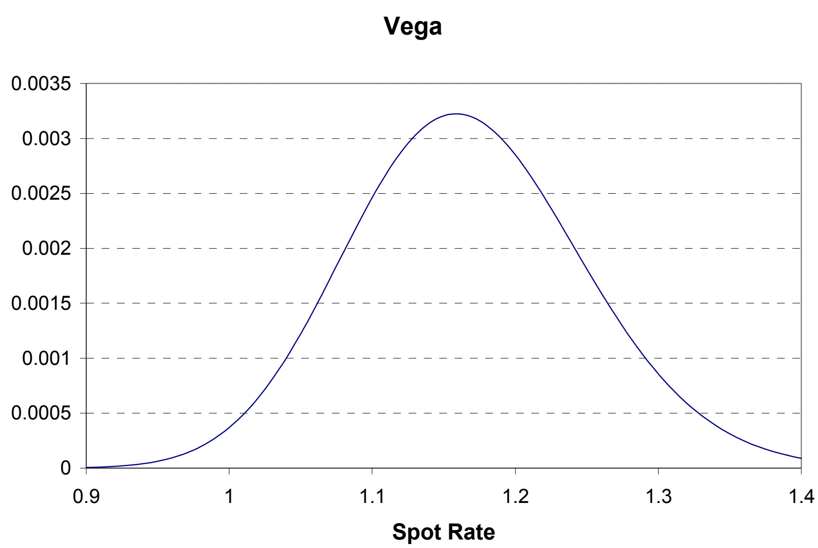
Vega

The **vega** of an option/portfolio, υ, is the rate of change of the option/portfolio value with respect to the volatility of the spot rate.

- The vega of a **regular** European or American option is **always positive**.
    - Intuitively, the insurance (time) value of an option increases with volatility.
- The vegas of a European **call** and a **put** are the **same**.

❖ **Q6**. Show this using put-call parity.

Vega of the European call in the example



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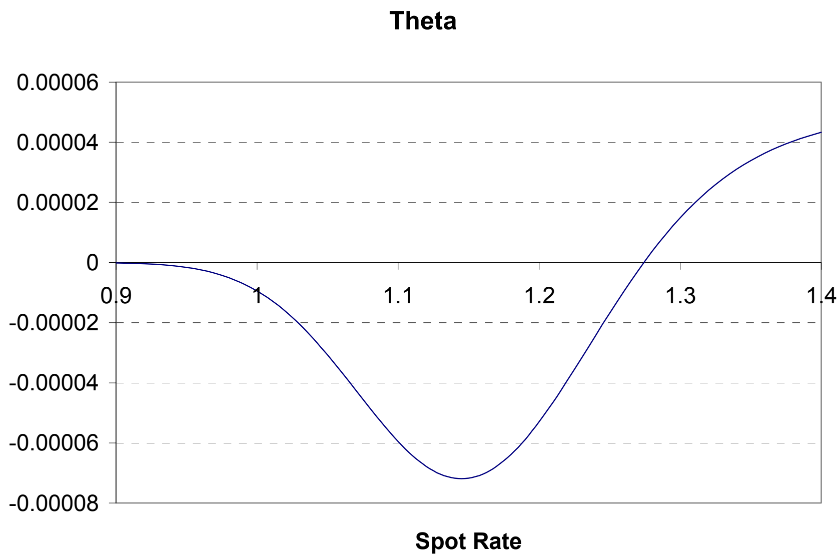
Theta

The **theta** of an option/portfolio, Θ, is the rate of change of the option/portfolio value with respect to the passage of time.

- Theta is **usually negative** for an option, because the passage of time **decreases the time value**.
- For an **ITM call** option on a currency with a relatively **high interest rate**, theta can be **positive**, because the passage of time shortens the time until the receipt of the foreign interest if exercised in the money (see the picture below).
- Theta is not the same type of hedge measure as others Greeks, because there is **no uncertainty** about the passage of time.

Theta of the European call in the example



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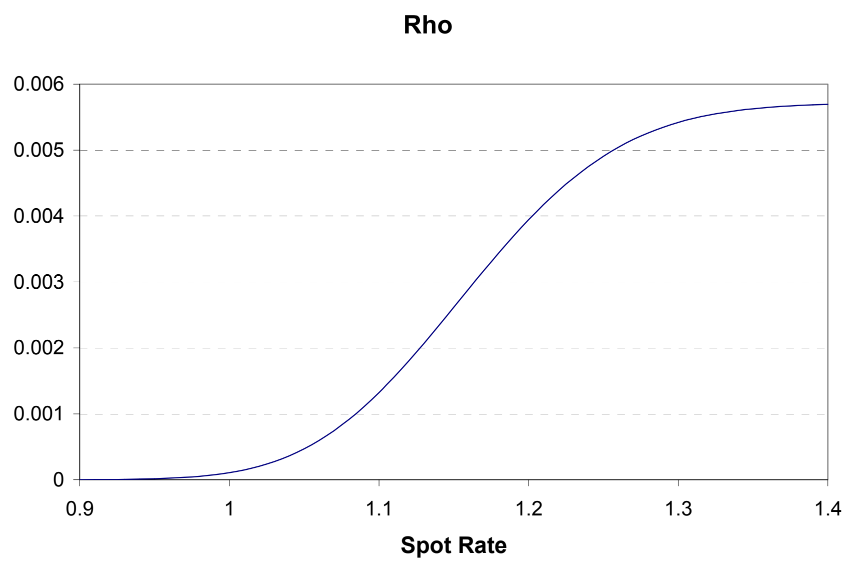
Rho

The **rho** of an option/portfolio is the rate of change of the option/portoflio value with respect to the interest rate.

- For **currency** options, there are **two** rhos, one corresponding to the change in the **domestic** interest rate and the other corresponding to the **foreign** interest rate.
- The rho of a European **call** option with respect to the **domestic** interest rate is **positive**. It is **negative** for a **put**.
- The rho of a European **call** option with respect to the **foreign** interest rate is **negative**. It is **positive** for a **put**.

❖ **Q7**. Explain the above two points.

Rho of the European call w.r.t. the domestic interest rate in the example



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Suggested solutions to questions

Q1. $\Delta_{Put} \rightarrow 0$ as $S_t \rightarrow \infty$ (OTM).

$\Delta_{Put} \rightarrow -\exp(-i^{f}\tau)$ (closer to -1) as $S_t \rightarrow 0$ (ITM).

Q2. The delta of the 1.25 call converges to 0 because, as $\tau \rightarrow 0$, it becomes progressively sure that the option will end up OTM. On the other hand, the delta of the 1.05 call becomes closer to 1 because it becomes very likely that the option will end up ITM.

Q3. The value of a forward contract is

$$f = S\exp(-i^{f}\tau) - K\exp(-i^{d}\tau)$$

Thus, the delta of a forward contract is $\partial f/\partial S = \exp(-i^{f}\tau)$.

Q4. The delta of the futures contract is

$$\exp((1.2\% - 2.2\%)\times 0.5) = 0.9950.$$

Let x be the amount of futures contracts that the bank should hold long. We set

$$-0.5085\times(-1 \text{ million}) + 0.9950\,x = 0.$$

$$x = -0.5111 \text{ million} \qquad \Rightarrow \qquad x / 125{,}000 = 4.088$$

Thus, the bank should sell four CME euro futures contracts. As in the Dozier case, this is **not** a perfect hedge.

- The bank must **rebalance dynamically** to remain delta-hedged.
- There is a **size mismatch**.

Q5. By the put-call parity, we have

$$C - P = S\exp(-i^{f}\tau) - K\exp(-i^{d}\tau). \qquad (*)$$

Differentiating with respect to S gives

$$\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = e^{-i^{f}\tau}.$$

That is, the difference between the deltas of the call and the put equals the foreign discount factor. Further differentiation yields

$$\frac{\partial^2 C}{\partial S^2} - \frac{\partial^2 P}{\partial S^2} = 0.$$

This shows the equivalence of the two deltas.

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Q6. The put-call parity relation (*) above does not involve σ. Thus, differentiating with respect to σ gives $\frac{\partial C}{\partial \sigma} - \frac{\partial P}{\partial \sigma} = 0$.

Q7. Equations (3) and (4) can be considered a B-S model "on a forward contract." A higher domestic interest rate or a lower foreign interest rate will increase the forward rate and therefore makes the call option more ITM, and the put more OTM.