

Pricing Barrier Option
Using
Finite Difference Method
and MonteCarlo Simulation

Yoon W. Kwon
CIMS¹, Math. Finance

Suzanne A. Lewis
CIMS, Math. Finance

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¹Courant Institute of Mathematical Science, New York University

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1 Introduction

A barrier option is the most important exotic option for structured products. Barrier options have special characteristics which distinguish them from vanilla options. The most popular standard barrier options are ‘knock-out’ and ‘knock-in’ options. These options are expired or exercisable automatically when the stock price hits the specified barrier level. Both out- and in-options are divided into down and up options by the level of current stock price compared to the barrier level, so there are eight kinds of basic standard barrier options. Sometimes a ‘knock-out’ option is called a ‘live-out’ option if it is significantly in-the-money when it knocks out. It is also possible that it has some payoff (called a ‘rebate’) when the barrier level is hit; capped call. For our project, we considered a basic standard knock out option that pays no rebate.

The closed form solution of the barrier option price can be obtained by solving the ‘heat equation’ (diffusion equation) with a boundary condition. It can also be obtained by numerical schemes. Here we calculated the **down-and-out call option** price using two different numerical schemes, a finite difference method and a Monte Carlo simulation. We then compared those solutions with prices calculated by the closed form solution.

2 Analytical Solution

To get the closed form solution, we have to solve heat equation reduced from Black Scholes partial differential equation. The only thing different for the barrier option is, as mentioned above, that the heat equation has a boundary (barrier level) condition at which the value of solution function is usually equal to zero. For some specific case, it can be solved using the solution without the boundary condition. We have already seen the case of ‘knock-out’ option with barrier level below the strike price¹.

$$V(S, t, B) = C(S, t) - \left[\left(\frac{S}{B} \right)^{-(k-1)} \cdot C\left(\frac{B^2}{S}, t \right) \right] \quad (1)$$

where $B < K$, $k = 2r/\sigma^2$

¹Paul Wilmott, The Mathematics of Financial Derivatives, 1997, Chap 12.

In formula (1), $C(S, t)$ represents the value function of vanilla call option whose strike price is K . But this closed form solution require that the barrier level is below the strike price K . The very complicated looking formula following is the closed form solution for more general cases².

$$\begin{aligned} V(S, t, B)_{B < K} &= S e^{-q(T-t)} (N(d_1) - b(1 - N(d_8))) \\ &\quad - K e^{-r(T-t)} (N(d_2) - b(1 - N(d_7))) \end{aligned} \quad (2)$$

$$\begin{aligned} V(S, t, B)_{B > K} &= S e^{-q(T-t)} (N(d_3) - b(1 - N(d_6))) \\ &\quad - K e^{-r(T-t)} (N(d_4) - b(1 - N(d_5))) \end{aligned} \quad (3)$$

$$\begin{aligned} d_1 &= \frac{\ln\left(\frac{S}{K}\right) + (r - q + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \\ d_2 &= \frac{\ln\left(\frac{S}{K}\right) + (r - q - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \\ d_3 &= \frac{\ln\left(\frac{S}{B}\right) + (r - q - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \\ d_4 &= \frac{\ln\left(\frac{S}{B}\right) + (r - q + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \\ d_5 &= \frac{\ln\left(\frac{S}{B}\right) - (r - q + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \\ d_6 &= \frac{\ln\left(\frac{S}{B}\right) - (r - q - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \\ d_7 &= \frac{\ln\left(\frac{SK}{B^2}\right) - (r - q - \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \\ d_8 &= \frac{\ln\left(\frac{SK}{B^2}\right) - (r - q + \frac{\sigma^2}{2})(T - t)}{\sigma\sqrt{T - t}} \end{aligned}$$

S stock price B barrier r risk-free rate
 K strike price T maturity q dividend yield

²Paul wilmott, Derivatives, 1999, Chap 14.

For the analytical solution, we used formula (2) and (3). Especially, the formula (2) is identical with (1) (when $B < K$).

3 Finite Difference Method

Finite difference methods are a means of obtaining numerical solutions of partial differential equations. They can be a very effective way, especially for diffusion equation (also heat equation), to find the solution. Formula (4) is diffusion equation for ‘down-and-out’ call option ($S > B$).

$$\frac{\partial u(x, \tau)}{\partial \tau} = \frac{\partial^2 u(x, \tau)}{\partial x^2} \quad (4)$$

$$\begin{aligned} u(b, \tau) &= 0 \\ u(x, 0) &= u_0(x) \\ &= \max\left(e^{\frac{(k+1)x}{2}} - e^{\frac{(k-1)x}{2}}, 0\right) \end{aligned}$$

where $b = \ln\left(\frac{B}{K}\right)$

Basically, there are two kinds of finite difference methods, explicit and implicit. We used an explicit finite difference method to find the solution of the diffusion equation with boundary condition. In this section, we will discuss some basic common technical points in the method and some specific points for the case of barrier option.

3.1 Explicit finite difference method

In the explicit finite difference method, forward difference approximation (5) is used for first derivative with respect to time variable and symmetric central finite difference approximation (6) is used for second derivative with respect to spatial variable.

$$\frac{\partial u(x, \tau)}{\partial \tau} = \frac{u(x, \tau + \delta\tau) - u(x, \tau)}{\delta\tau} + O(\delta\tau) \quad (5)$$

$$\frac{\partial^2 u(x, \tau)}{\partial x^2} = \frac{u(x + \delta x, \tau) - 2u(x, \tau) + u(x - \delta x, \tau)}{(\delta x)^2} + O((\delta x)^2) \quad (6)$$

Based on the finite difference approximations (5) and (6), diffusion equation (4) becomes equation (7)

$$\frac{u(x, \tau + \delta\tau) - u(x, \tau)}{\delta\tau} + O(\delta\tau) = \frac{u(x + \delta x, \tau) - 2u(x, \tau) + u(x - \delta x, \tau)}{(\delta x)^2} + O((\delta x)^2) \quad (7)$$

Now, we can determine the value of solution function at $\tau + \delta\tau$, using three points of solution functions at time τ ,

$$u(x, \tau + \delta\tau) \approx \lambda u(x + \delta x, \tau) - (2\lambda - 1)u(x, \tau) + \lambda u(x - \delta x, \tau) \quad (8)$$

where $\lambda = \delta\tau/(\delta x)^2$

Since we have a fixed time horizon τ , the mesh size $\delta\tau$ is usually determined by λ and $\delta\tau$ by the equation (9) ³.

$$\lambda = \sqrt{\frac{\delta\tau}{\lambda}} = \sigma \sqrt{\frac{T}{2M \cdot \lambda}} \quad (9)$$

Here M and T represent the number of iterations and the maturity of the option respectively. However, in the barrier option case, we have to consider another constraint that the barrier point should be always on the mesh node through the whole time horizon.

3.2 Putting the barrier on the mesh node

Unlike the usual case, we calculate spatial step δx first to put the barrier level on the mesh node (see (10)).

$$\delta x = \frac{\ln\left(\frac{S}{K}\right) - \ln\left(\frac{B}{K}\right)}{(\text{int})[\sqrt{\text{dummy}} \cdot (S - B)]} \quad (10)$$

³This is the case only for the plain vanilla option. We used a difference method to determine the mesh size.

After getting spatial time step, the number of iterations can be calculated by the formula (11) using the values of λ and $\delta\tau$.

$$M = (\text{int}) \left[\frac{\sigma^2 \cdot T}{2\lambda \cdot (\delta x)^2} \right] \quad (11)$$

In formula (10), we used dummy variable to control the number of iterations. Function (int) in (10) and (11) represents the function that transforms real number into integer.

3.3 Keeping zero boundary value

One more thing to consider in pricing barrier option is keeping the boundary value as zero always. Hence through given time horizon, we kept checking spatial variable. When spatial variable is less than the barrier level, zero was assigned for the value of solution function.

4 Monte Carlo Simulation

Monte Carlo Simulation provides a simple and flexible method for valuing financial instruments whose analytical solutions are impossible to get. It is especially useful because it can deal with one or more random factors such as random process of underlying asset price, stochastic volatility or random process of interest rate. Also, with Monte Carlo simulation one can consider realistic and complicated market conditions in relatively easy way (e.g. Asian option, look-back option, digital option etc.). However, Monte Carlo simulation has a critical flaw. It is computationally much inefficient in its basic form. So, most of the techniques in Monte Carlo simulation are developed to improve this inefficiency. Here we priced 'down-and-out call' option using Monte Carlo simulation with the antithetic variates method. And for stock price movement, we used one factor diffusion process (A.K.A Brownian Motion, (12) or (13)).

$$\frac{dS}{S} = \mu \cdot dt + \sigma \cdot dW \quad (12)$$

$$dS = S \cdot (\mu \cdot dt + \sigma \cdot dW) \quad (13)$$

4.1 Generating standard normal random numbers

A critical part of Monte Carlo simulation is the generation of the standard normal random variables. We programmed in C, which provides a uniform pseudo-random number generator. This generates random real numbers between zero and one from a uniform distribution. There are a couple of numerical methods to transform this uniformly distributed random numbers to standard normal random numbers. We used a simple method called the Box-Muller transformation (14) and (15). This is an exact transformation of a pair of standard uniform random numbers to a pair of standard normal random variables.

$$Z_1 = \cos(2\pi x_1) \sqrt{-2 \cdot \ln(x_2)} \quad (14)$$

$$Z_2 = \sin(2\pi x_1) \sqrt{-2 \cdot \ln(x_2)} \quad (15)$$

Here x_1 and x_2 represent standard uniform random numbers and we used ‘drand48’ library function in C to generate standard uniform random numbers.

4.2 Antithetic Variates for the reduction of variance

Suppose one have written an option on asset S_1 and at the same time, one is able to write an option on asset S_2 , which is perfectly negatively correlated with S_1 and currently has the precisely same price as S_1 . Then S_1 and S_2 satisfy the stochastic differential equations.

$$dS_{1,t} = S_{1,t} \cdot (r \cdot dt + \sigma \cdot dW) \quad (16)$$

$$dS_{2,t} = S_{2,t} \cdot (r \cdot dt - \sigma \cdot dW) \quad (17)$$

The values of these two options should be identical since the price and volatility of two assets are identical. However, the variance of the payoff of a portfolio consisting of the two options is much less than that of the payoff of each individual option. This technique of creating a hypothetical asset that is perfectly negatively correlated with the original asset is called antithetic variance reduction and the created asset is called an antithetic variate. To implement this technique, we created another sample path of

asset process that is moving exactly opposite way to the first path. In fact, since we have generated a pair of standard normal random numbers, four paths are moving along at the same time (see (18) - (21)).

$$dS_{1,t} \approx S_{1,t} \cdot (r \cdot dt + \sigma\sqrt{dt} \cdot z_1) \quad (18)$$

$$dS_{2,t} \approx S_{2,t} \cdot (r \cdot dt - \sigma\sqrt{dt} \cdot z_1) \quad (19)$$

$$dS_{3,t} \approx S_{3,t} \cdot (r \cdot dt + \sigma\sqrt{dt} \cdot z_2) \quad (20)$$

$$dS_{4,t} \approx S_{4,t} \cdot (r \cdot dt - \sigma\sqrt{dt} \cdot z_2) \quad (21)$$

5 Results

We chose three cases by the money-ness of current stock prices (80, 100, 120 with strike 100). For each case, we calculated prices with three different levels of barrier. Also in order to see the convergence, we calculated with three different numbers of iterations. Table 1 is the list of parameters we used. λ has been chosen as 1/6 to remove second order error term with respect to spatial variable.

Table 1: Parameters

lambda(λ)	0.167	risk-free rate(r)	0.05
strike(K)	100.0	dividend yld. (q)	0.01
maturity(T)	1.00	volatility(σ)	0.25

5.1 Results of Finite Difference Method

As expected, we had relatively bigger errors for the in-the-money case (Table 4). The reason for bigger error is discontinuity of at value function at initial time. When the barrier is above the strike price initial value function at the barrier is not equal to zero but should be zero at the same time by boundary condition (i.e. $u_0(b) \neq 0 = u(b, \tau)$). However, In the case that the barrier level is below the strike price, we don't have any discontinuity (i.e. $u_0(b) = 0 = u(b, \tau)$). Also, in any case, we can see the convergence of approximation solution to analytical solution as iteration number increases.

The following three tables are the price and error results of finite difference method.

Table 2: Out-of-the-money case with stock price 80 and strike 100

Barrier	Iteration Number	Ana. price	FDM price	Error (in %)
50	764	2.91988	2.92058	0.024
	1570		2.92192	0.070
	2296		2.92168	0.062
60	907	2.91484	2.91925	0.152
	1906		2.91669	0.064
	2776		2.91571	0.030
70	1052	2.66087	2.66255	0.063
	2367		2.66191	0.039
	3408		2.91571	0.027

Table 3: At-the-money case with stock price 100 and strike 100

Barrier	Iteration Number	Ana. price	FDM price	Error (in %)
70	1327	11.70273	11.70378	0.009
	2726		11.70404	0.011
	3986		11.70336	0.005
80	1507	11.32229	11.32411	0.014
	3167		11.32196	0.003
	4613		11.32208	0.001
90	1690	8.61503	8.61625	0.014
	3801		8.61532	0.003
	5473		8.61514	0.001

Table4: In-the-money case with stock price 80 and strike 120

Barrier	Iteration Number	Ana. price	FDM price	Error (in %)
90	2040	25.73921	25.74163	0.009
	4190		25.74064	0.006
	6127		25.73998	0.003
100	2257	22.60767	22.67813	0.312
	4744		22.67728	0.308
	6910		22.67723	0.308
110	2477	14.39291	14.51313	0.835
	5573		14.51325	0.836
	8025		14.51328	0.836

Ana. Price : Analytical Solution
FDM Price : Finite Difference Method Approximation

5.2 Results of Monte Carlo Simulation

The following tables are the price and error results of Monte Carlo Simulation.

Table 5. Out-of-the-money case with stock price 80 and strike 100

Barrier	Iteration Number	Ana. price	MCS price	Error (in %)
50	10,000	2.91988	2.89206	0.953
	20,000		2.96751	1.631
	40,000		2.95696	1.270
60	10,000	2.91484	2.88766	1.619
	20,000		2.88803	0.920
	40,000		2.91986	0.172
70	10,000	2.66087	2.71119	1.891
	20,000		2.67315	0.461
	40,000		2.69583	1.314

Table 6. At-the-money case with stock price 100 and strike 100

Barrier	Iteration Number	Ana. price	MCS price	Error (in %)
70	10,000	11.70273	11.77419	0.617
	20,000		11.64066	0.530
	40,000		11.74976	0.401
80	10,000	11.32229	11.28537	0.327
	20,000		11.33218	0.087
	40,000		11.40166	0.701
90	10,000	8.61503	9.04559	3.457
	20,000		9.09942	2.330
	40,000		9.16626	2.419

Table 7. In-the-money case with stock price 80 and strike 120

Barrier	Iteration Number	Ana. price	MCS price	Error (in %)
90	10,000	25.73921	25.79725	0.225
	20,000		25.73844	0.003
	40,000		25.80913	1.373
100	10,000	22.60767	23.10899	2.218
	20,000		22.91807	1.373
	40,000		22.93958	1.468
110	10,000	14.39291	14.93807	3.788
	20,000		14.96592	3.981
	40,000		15.06374	4.661

Ana. Price : Analytical Solution
MCS Price : Monte Carlo Simulation Approximation

As shown in the above tables, for most of the cases we had larger errors than the comparable finite difference method cases. This is especially true when the barrier level is close to the current stock price. Also, in some cases, even if we increase the number of sample paths, we could not get convergence of the errors. We used 10,000 time steps for each path of the simulation.

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